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Vanishing of the local cohomologies of D-modules associated to A-hypergeometric systems

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Abstract

Given a finite set A of integral vectors and a parameter vector, Gel'fand, Kapranov and Zelevinsky defined a system of differential equiations, called an *A-hypergeometric* (or a *GKZ hypergeometric*) *system*.
Throughout this paper, we consider a finite set A fixed. Saito [Iso] introduced a finite set $E_\tau(\beta)$ associated to a parameter β and a face τ of the cone generated by A. The set $E_\tau(\beta)$ is important to classify the parameters according to the D-isomorphism classes of their corresponding A-hypergeometric systems.
The purpose of this paper is to relate the set $E_\tau(\beta)$ to the algebraic local cohomologies of a D-module associated to the A-hypergeometric system.

Contents

1 Introduction

Given a finite set A of integral vectors and a parameter vector, Gel'fand, Kapranov and Zelevinsky defined a system of differential equiations, called an *A-hypergeometric* (or a *GKZ hypergeometric*) *system*. In the theory of D-modules, there are several notions: characteristic variety, tensor product, restriction, localization, de Rahm cohomology, algebraic local cohomology, and others. In calculating them, this system is an important example.
Throughout this paper, we consider a finite set A fixed. Saito [Iso] introduced a finite set $E_\tau(\boldsymbol{\beta})$ associated to a parameter $\boldsymbol{\beta}$ and a face τ of the cone generated by A. The set $E_\tau(\boldsymbol{\beta})$ is important to classify the parameters according to the D-isomorphism classes of their corresponding A-hypergeometric systems.
The purpose of this paper is to relate the set $E_\tau(\boldsymbol{\beta})$ to the algebraic local cohomologies of a D-module associated to the A-hypergeometric system. In this paper, we give a result about the relation between the condition of the vanishing of the local cohomologies and that of $E_\tau(\boldsymbol{\beta})$.
In Section 2, we will prepare some notions and introduce some facts concerned with them: A-hypergeometric system, the set $E_\tau(\boldsymbol{\beta})$, orbits of the canonical action of the algebraic torus on the toric variety determined by the set A. In Section3, we will state the main theorem (Theorem 3.1) in this paper and compute the set of parameters satisfying the vanishing conditions of the algebraic local cohomologies in some easy examples. We will give the proof of the main theorem in detail in Section 4.

2 Preliminaries

2.1 A-hypergeometric systems

Let $A = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$-integer matrix of rank d. We suppose that all $\boldsymbol{a}_j$ belong to one hyperplane off the origin in $\mathbf{R}^d$. We denote by I_A the *toric ideal* in $\mathbf{C}[\boldsymbol{\partial}] := \mathbf{C}[\partial_1, \ldots, \partial_n]$, that is

$$I_A = \langle \boldsymbol{\partial}^{\boldsymbol{u}} - \boldsymbol{\partial}^{\boldsymbol{v}} \mid A\boldsymbol{u} = A\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{v} \in \mathbf{N}^n \rangle \subset \mathbf{C}[\boldsymbol{\partial}].$$

For a column vector $\boldsymbol{\beta} = {}^t(\beta_1, \ldots, \beta_d) \in \mathbf{C}^d$, we denote by $H_A(\boldsymbol{\beta})$ the left ideal of the *Weyl algebra*

$$D = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by I_A and $\sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ $(i = 1, \ldots, d)$. The quotient module $M_A(\boldsymbol{\beta}) = D/H_A(\boldsymbol{\beta})$ is called the *A-hypergeometric system with parameter* $\boldsymbol{\beta}$. In this paper, we consider not $M_A(\boldsymbol{\beta})$ itself but its *Fourier transform* $\widehat{M_A(\boldsymbol{\beta})}$ defined as follows:

$$\widehat{M_A(\boldsymbol{\beta})} := D/\widehat{H_A(\boldsymbol{\beta})},$$

where $\widehat{H_A(\beta)} = D \cdot \{\sum_{j=1}^{n} a_{ij}\partial_j x_j + \beta_i \mid i = 1, \dots, d\} + DI_A(\boldsymbol{x}),\ I_A(\boldsymbol{x})$: the toric ideal in $\mathbf{C}[\boldsymbol{x}] = \mathbf{C}[x_1, \dots, x_n]$.

2.2 The set $E_\tau(\beta)$ and orbits of the canonical action of the algebraic torus on the toric variety $V(I_A(\boldsymbol{x}))$

We denote by A the set $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ as well, and by $\mathbf{R}_{\geq 0}A$ the cone

$$\left\{ \sum_{j=1}^{n} c_j \boldsymbol{a}_j \,\middle|\, c_j \in \mathbf{R}_{\geq 0} \right\}.$$

We denote the set of all faces of $\mathbf{R}_{\geq 0}A$ by $\mathcal{S}(A)$. For a face $\tau \in \mathcal{S}(A)$, we denote

- by $\mathbf{Z}(A \cap \tau)$ the $\mathbf{Z}$-submodule of $\mathbf{Z}^d$ generated by $A \cap \tau$,
- by $\mathbf{C}(A \cap \tau)$ the $\mathbf{C}$-subspace of $\mathbf{C}^d$ generated by $A \cap \tau$,
- by $\mathbf{N}A$ the monoid generated by A.

We agree that $\mathbf{Z}(A \cap \tau) = \mathbf{C}(A \cap \tau) = (0)$ when $\tau = \{0\}$. For a parameter $\beta \in \mathbf{C}^d$, we define the set $E_\tau(\beta)$ as follows:

$$E_\tau(\beta) := \{\boldsymbol{\lambda} \in \mathbf{C}(A \cap \tau)/\mathbf{Z}(A \cap \tau) \mid \beta - \boldsymbol{\lambda} \in \mathbf{N}A + \mathbf{Z}(A \cap \tau)\}.$$

According to the paper [Iso], the following facts hold.

Proposition 2.1 *Let $\tau \in \mathcal{S}(A)$. Then we have the following.*

1. *If $\sigma \in \mathcal{S}(A)$, and $\tau \prec \sigma$, then there exists a natural map from $E_\tau(\beta)$ to $E_\sigma(\beta)$. In particular, if $E_\tau(\beta) \neq \varnothing$, then $E_\sigma(\beta) \neq \varnothing$.*
2. *For any $\chi \in \mathbf{N}A$, there exists a natural inclusion from $E_\tau(\beta)$ to $E_\tau(\beta + \chi)$*

Theorem 2.2 *The A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta')$ are isomorphic as D-modules if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces $\tau \in \mathcal{S}(A)$.*

Evidently, $M_A(\beta) \simeq M_A(\beta')$ as D-modules if and only if $\widehat{M_A(\beta)} \simeq \widehat{M_A(\beta')}$ as D-modules. Thus we obtain the following.

Corollary 2.3 *The A-hypergeometric systems $\widehat{M_A(\beta)}$ and $\widehat{M_A(\beta')}$ are isomorphic as D-modules if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces $\tau \in \mathcal{S}(A)$*

Next, we will consider *'orbits'*. It is well-known that the algebraic torus $(\mathbf{C}^\times)^d$ canonically acts on the toric variety $V(I_A(\boldsymbol{x}))$. For $\tau \in \mathcal{S}(A)$, we define a subset X_τ in $\mathbf{C}^n$ by

$$X_\tau := \{(x_1, \ldots, x_n) \in \mathbf{C}^n \mid x_j = 0 \text{ (if } \boldsymbol{a}_j \notin \sigma), x_j \neq 0 \text{ (if } \boldsymbol{a}_j \in \sigma)\}.$$

In fact, X_σ is the orbit of the action, that is

$$V(I_A(\boldsymbol{x})) = \coprod_{\sigma \in \mathcal{S}(A)} X_\sigma : \text{disjoint union.}$$

3 Main theorem and some easy examples

3.1 Main theorem

Let $\mathbf{R}\Gamma_{[Z]}(\ \cdot\ )$ be the *algebraic local cohomology functor* with respect to $Z \subset \mathbf{C}^n$ in $\mathrm{Mod}(D)$. The following is the main theorem in this paper. We will prove this theorem in detail in the next section.

Theorem 3.1 *Fix a parameter $\boldsymbol{\beta}$ and $\boldsymbol{a}_k$. Then we have the following.*

1. *If $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} + m\boldsymbol{a}_k)$ for all $m \in \mathbf{N}$ and all faces $\tau \in \mathcal{S}(A)$, then $\mathbf{R}\Gamma_{[X_\tau]}(\widehat{M_A(\boldsymbol{\beta})}) = 0$ for all faces $\tau \in \mathcal{S}(A)$ with $\boldsymbol{a}_k \notin \tau$.*
2. *If $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} - m\boldsymbol{a}_k)$ for all $m \in \mathbf{N}$ and all faces $\tau \in \mathcal{S}(A)$, then $\mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0$ for all faces $\tau \in \mathcal{S}(A)$ with $\boldsymbol{a}_k \notin \tau$, where $\mathbb{D}(\ \cdot\ )$ is the dual functor in* $\mathrm{Mod}(D)$.

By the definition of $E_\tau(\boldsymbol{\beta})$, we can easily prove that $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} - m\boldsymbol{a}_k)$ for all $m \in \mathbf{N}$ and all faces $\tau \in \mathcal{S}(A)$ if and only if $E_\tau(\boldsymbol{\beta}) = \varnothing$ for all *facets* $\tau \in \mathcal{S}(A)$ with $\boldsymbol{a}_k \notin \tau$. Hence, we obtain the following.

Corollary 3.2 *If $E_\tau(\boldsymbol{\beta}) = \varnothing$ for all facets $\tau \in \mathcal{S}(A)$ with $\boldsymbol{a}_k \notin \tau$, then $\mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0$ for all faces $\tau \in \mathcal{S}(A)$ with $\boldsymbol{a}_k \notin \tau$.*

3.2 Some examples

In this section, we consider some easy cases.

Case 1 $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

In this case, we have $\mathcal{S}(A) = \{\mathbf{R}_{\geq 0}A, \sigma_1, \sigma_2, \{(0,0)\}\}$, where

$$\mathbf{R}_{\geq 0}A = \mathbf{R}_{\geq 0}{}^t(1,0) + \mathbf{R}_{\geq 0}{}^t(1,2), \sigma_1 = \mathbf{R}_{\geq 0}{}^t(1,0), \sigma_2 = \mathbf{R}_{\geq 0}{}^t(1,2)$$

Computing the sets $E_\tau(\boldsymbol{\beta})$ $(\tau \in \mathcal{S}(A))$, we have

$$
\begin{aligned}
E_{\mathbf{R}_{\geq 0}A}(\boldsymbol{\beta}) &= \{\boldsymbol{\beta} \text{ mod } \mathbf{Z}A\},\\
E_{\sigma_1}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 0) \text{ mod } \mathbf{Z}^t(1,0)\} & (\text{if } \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_2 \notin \mathbf{N}), \end{cases}\\
E_{\sigma_2}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 2\beta_1) \text{ mod } \mathbf{Z}^t(1,2)\} & (\text{if } 2\beta_1 - \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } 2\beta_1 - \beta_2 \notin \mathbf{N}), \end{cases}\\
E_{\{(0,0)\}}(\boldsymbol{\beta}) &= \begin{cases} \{(0,0)\} & (\text{if } \boldsymbol{\beta} \in \mathbf{N}A) \\ \varnothing & (\text{if } \boldsymbol{\beta} \notin \mathbf{N}A), \end{cases}
\end{aligned}
$$

Therefore, by Corollary 3.2,

$$2\beta_1 - \beta_2 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_{\sigma_2}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0, \mathbf{R}\Gamma_{[X_{\{(0,0)\}}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0.$$

Similarly, we obtain

$$\beta_2 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_{\sigma_1}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0, \mathbf{R}\Gamma_{[X_{\{(0,0)\}}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0.$$

Case 2 $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$: not *Cohen-Macauley* case.
In this case, we have $\mathcal{S}(A) = \{\mathbf{R}_{\geq 0}A, \sigma_1, \sigma_2, \{(0,0)\}\}$, where

$$\mathbf{R}_{\geq 0}A = \mathbf{R}_{\geq 0}{}^t(1,0) + \mathbf{R}_{\geq 0}{}^t(1,4), \sigma_1 = \mathbf{R}_{\geq 0}{}^t(1,0), \sigma_2 = \mathbf{R}_{\geq 0}{}^t(1,4)$$

Computing the sets $E_\tau(\boldsymbol{\beta})$ $(\tau \in \mathcal{S}(A))$, we have

$$
\begin{aligned}
E_{\mathbf{R}_{\geq 0}A}(\boldsymbol{\beta}) &= \{\boldsymbol{\beta} \text{ mod } \mathbf{Z}A\},\\
E_{\sigma_1}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 0) \text{ mod } \mathbf{Z}^t(1,0)\} & (\text{if } \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_2 \notin \mathbf{N}), \end{cases}\\
E_{\sigma_2}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 4\beta_1) \text{ mod } \mathbf{Z}^t(1,4)\} & (\text{if } 4\beta_1 - \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } 4\beta_1 - \beta_2 \notin \mathbf{N}), \end{cases}\\
E_{\{(0,0)\}}(\boldsymbol{\beta}) &= \begin{cases} \{(0,0)\} & (\text{if } \boldsymbol{\beta} \in \mathbf{N}A) \\ \varnothing & (\text{if } \boldsymbol{\beta} \notin \mathbf{N}A). \end{cases}
\end{aligned}
$$

Therefore, by Corollary 3.2,

$$4\beta_1 - \beta_2 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_{\sigma_2}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0, \mathbf{R}\Gamma_{[X_{\{(0,0)\}}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0.$$

Similarly, we obtain

$$\beta_2 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_{\sigma_1}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0, \mathbf{R}\Gamma_{[X_{\{(0,0)\}}]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0.$$

Case 3 $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$: normal case with $d = 3$.

In this case, we have $\mathcal{S}(A) = \{\mathbf{R}_{\geq 0}A, \tau_1, \tau_2, \tau_3, \tau_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \{(0,0)\}\}$, where

$$\begin{aligned} \mathbf{R}_{\geq 0}A &= \mathbf{R}_{\geq 0}{}^t(1,0,0) + \mathbf{R}_{\geq 0}{}^t(0,1,0) + \mathbf{R}_{\geq 0}{}^t(0,0,1) + \mathbf{R}_{\geq 0}{}^t(-1,1,1), \\ \tau_1 &= \mathbf{R}_{\geq 0}{}^t(1,0,0) + \mathbf{R}_{\geq 0}{}^t(0,1,0), \tau_2 = \mathbf{R}_{\geq 0}{}^t(0,1,0) + \mathbf{R}_{\geq 0}{}^t(-1,1,1), \\ \tau_3 &= \mathbf{R}_{\geq 0}{}^t(0,0,1) + \mathbf{R}_{\geq 0}{}^t(-1,1,1), \tau_4 = \mathbf{R}_{\geq 0}{}^t(1,0,0) + \mathbf{R}_{\geq 0}{}^t(0,0,1), \\ \sigma_1 &= \mathbf{R}_{\geq 0}{}^t(1,0,0), \sigma_2 = \mathbf{R}_{\geq 0}{}^t(0,1,0), \sigma_3 = \mathbf{R}_{\geq 0}{}^t(0,0,1), \sigma_4 = \mathbf{R}_{\geq 0}{}^t(-1,1,1). \end{aligned}$$

Computing the sets $E_\tau(\boldsymbol{\beta})$ $(\tau \in \mathcal{S}(A))$, we have

$$\begin{aligned} E_{\mathbf{R}_{\geq 0}A}(\boldsymbol{\beta}) &= \{\boldsymbol{\beta} \bmod \mathbf{Z}A\}, \\ E_{\tau_1}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, \beta_2, 0) \bmod \mathbf{Z}^t(1,0,0) + \mathbf{Z}^t(0,1,0)\} & (\text{if } \beta_3 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_3 \notin \mathbf{N}), \end{cases} \\ E_{\tau_2}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, \beta_2, -\beta_1) \bmod \mathbf{Z}^t(0,1,0) + \mathbf{Z}^t(-1,1,1)\} & (\text{if } \beta_1 + \beta_3 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_1 + \beta_3 \notin \mathbf{N}), \end{cases} \\ E_{\tau_3}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, -\beta_1, \beta_3) \bmod \mathbf{Z}^t(0,0,1) + \mathbf{Z}^t(-1,1,1)\} & (\text{if } \beta_1 + \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_1 + \beta_2 \notin \mathbf{N}), \end{cases} \\ E_{\tau_4}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 0, \beta_3) \bmod \mathbf{Z}^t(1,0,0) + \mathbf{Z}^t(0,0,1)\} & (\text{if } \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_2 \notin \mathbf{N}), \end{cases} \\ E_{\sigma_1}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(\beta_1, 0, 0) \bmod \mathbf{Z}^t(1,0,0)\} & (\text{if } \beta_2 \in \mathbf{N} \text{ and } \beta_3 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_2 \notin \mathbf{N} \text{ or } \beta_3 \notin \mathbf{N}), \end{cases} \\ E_{\sigma_2}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(0, \beta_2, 0) \bmod \mathbf{Z}^t(0,1,0)\} & (\text{if } \beta_1 + \beta_3 \in \mathbf{N} \text{ and } \beta_3 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_1 + \beta_3 \notin \mathbf{N} \text{ or } \beta_3 \notin \mathbf{N}), \end{cases} \\ E_{\sigma_3}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(0, 0, \beta_3) \bmod \mathbf{Z}^t(0,0,1)\} & (\text{if } \beta_1 + \beta_2 \in \mathbf{N} \text{ and } \beta_2 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_1 + \beta_2 \notin \mathbf{N} \text{ or } \beta_2 \notin \mathbf{N}), \end{cases} \\ E_{\sigma_4}(\boldsymbol{\beta}) &= \begin{cases} \{{}^t(-\beta_1, \beta_1, \beta_1) \bmod \mathbf{Z}^t(-1,1,1)\} & (\text{if } \beta_1 + \beta_2 \in \mathbf{N} \text{ and } \beta_1 + \beta_3 \in \mathbf{N}) \\ \varnothing & (\text{if } \beta_1 + \beta_2 \notin \mathbf{N} \text{ or } \beta_1 + \beta_3 \notin \mathbf{N}), \end{cases} \\ E_{\{(0,0,0)\}}(\boldsymbol{\beta}) &= \begin{cases} \{(0,0,0)\} & (\text{if } \boldsymbol{\beta} \in \mathbf{N}A) \\ \varnothing & (\text{if } \boldsymbol{\beta} \notin \mathbf{N}A). \end{cases} \end{aligned}$$

Therefore, by Corollary 3.2,

$$\boldsymbol{\beta} \notin \mathbf{N}A, \beta_1 + \beta_2 \notin \mathbf{N}, \beta_1 + \beta_3 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\boldsymbol{\beta})})) = 0 \ (\tau = \tau_2, \tau_3, \sigma_2, \sigma_3, \sigma_4, \{\mathbf{0}$$

Similarly, we obtain

$$\beta \notin \mathbf{N}A, \beta_1+\beta_2 \notin \mathbf{N}, \beta_2 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\beta)})) = 0 \ (\tau = \tau_3, \tau_4, \sigma_1, \sigma_3, \sigma_4, \{\mathbf{0}\}),$$
$$\beta \notin \mathbf{N}A, \beta_3 \notin \mathbf{N}, \beta_1+\beta_3 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\beta)})) = 0 \ (\tau = \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_4, \{\mathbf{0}\}),$$
$$\beta \notin \mathbf{N}A, \beta_2 \notin \mathbf{N}, \beta_3 \notin \mathbf{N} \Rightarrow \mathbf{R}\Gamma_{[X_\tau]}(\mathbb{D}(\widehat{M_A(\beta)})) = 0 \ (\tau = \tau_1, \tau_4, \sigma_1, \sigma_2, \sigma_3, \{\mathbf{0}\}).$$

4 Proof of the main theorem

In this section, we will prove the main theorem. In order to prove it, we will prepare some facts. For a face $\tau \in \mathcal{S}(A)$ and $1 \le j \le n$, we denote

$$S_{\tau,j} := \begin{cases} \{x_j = 0\} & (\text{if } \boldsymbol{a}_j \notin \tau) \\ \{x_j \neq 0\} & (\text{if } \boldsymbol{a}_j \in \tau). \end{cases}$$

Then we have $X_\tau = V(I_A(\boldsymbol{x})) \cap \bigcap_{j=1}^n S_{\tau,j}$. Moreover, according to the theory of D-modules, we can obtain

$$\mathbf{R}\Gamma_{[X_\tau]} \simeq \mathbf{R}\Gamma_{[V(I_A(\boldsymbol{x}))]}\mathbf{R}\Gamma_{[S_{\tau,1}]}\cdots\mathbf{R}\Gamma_{[S_{\tau,n}]}.$$

Since $\mathrm{Supp}(\widehat{M_A(\beta)}) \subset V(I_A(\boldsymbol{x}))$ (resp. $\mathrm{Supp}(\mathbb{D}(\widehat{M_A(\beta)})) \subset V(I_A(\boldsymbol{x}))$), we can easily proove that

$$\mathbf{R}\Gamma_{[V(I_A(\boldsymbol{x}))]}(\widehat{M_A(\beta)}) \simeq \widehat{M_A(\beta)} \ (\text{resp. } \mathbf{R}\Gamma_{[V(I_A(\boldsymbol{x}))]}(\mathbb{D}(\widehat{M_A(\beta)})) \simeq \mathbb{D}(\widehat{M_A(\beta)})),$$

and

$$\mathbf{R}\Gamma_{[\{x_k=0\}]} = 0 \iff \mathbf{R}\Gamma_{[X_\tau]} = 0 \ (\text{for all } \tau \in \mathcal{S}(A) \text{ with } \boldsymbol{a}_k \notin \tau).$$

Therefore, it is sufficient to focus on the functor $\mathbf{R}\Gamma_{[S_{\tau,k}]}$. According to [Kas], the following theorems hold.

Theorem 4.1 *Let M be a holonomic D-module, then the following conditions are equivalent.*

1. $\mathbf{R}\Gamma_{[\{x_k=0\}]}(M) = 0$.
2. (a) *the module M has no nonzero coherent* submodules *supported in $\{x_k = 0\}$.*
   (b) *Let N be a holonomic D-module and $f : M \to N$ be an* injective *D-homomorphism. If the restriction of f on $\{x_k \neq 0\}$ is an isomorphism and N has no nonzero coherent* submodules *supported in $\{x_k = 0\}$, then f is an isomorphism.*

Remark The conditions 1. and 2. are equivalent to 2'.

2'. $M \simeq \mathbf{C}[\boldsymbol{x}]_{x_k} \otimes_{\mathbf{C}[\boldsymbol{x}]} M$, where $\mathbf{C}[\boldsymbol{x}]_{x_k}$ is the localization of $\mathbf{C}[\boldsymbol{x}]$ with respect to the multiplicatively closed set $\{1, x_k, x_k^2, \cdots\}$.

On the other hand, considering the dual theorem of this, we obtain the following.

Theorem 4.2 *Let M be a holonomic D-module, then the following conditions are equivalent.*

1. $\mathbf{R}\Gamma_{[\{x_k=0\}]}(\mathbb{D}(M)) = 0$.
2. (a) *the module M has no nonzero coherent* **quotient modules** *supported in $\{x_k = 0\}$.*

   (b) *Let L be a holonomic D-module and $g : L \to M$ be a* **surjective** *D-homomorphism. If the restriction of f on $\{x_k \neq 0\}$ is an isomorphism and L has no nonzero coherent* **quotient modules** *supported in $\{x_k = 0\}$, then g is an isomorphism.*

Before proving the main theorem, we need to show the following proposition.

Proposition 4.3 *Fix a prameter $\boldsymbol{\beta}$ and an index k. Then we obtain the following.*

1. *If $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} + m\boldsymbol{a}_k)$ for all $m \in \mathbf{N}$ and all faces $\tau \in \mathcal{S}(A)$, then the module $\widehat{M_A(\boldsymbol{\beta})}$ satisfies the condition (2) of Theorem 4.1.*
2. *If $E_\tau(\boldsymbol{\beta}) = E_\tau(\boldsymbol{\beta} - m\boldsymbol{a}_k)$ for all $m \in \mathbf{N}$ and all faces $\tau \in \mathcal{S}(A)$, then the module $\widehat{M_A(\boldsymbol{\beta})}$ satisfies the condition (2) of Theorem 4.2.*

Proof of 1. By the remark of Theorem 4.1, it is sufficient to show that $\widehat{M_A(\boldsymbol{\beta})} \overset{\varphi}{\simeq} \widehat{M_A(\boldsymbol{\beta})}_{x_k}$.

First, we will check the injectivity of φ. Let P mod $\widehat{H_A(\boldsymbol{\beta})} \in \operatorname{Ker} \varphi$ ($P \in D$). Then there exists $l \in \mathbf{N}$ such that $x_k^l P$ mod $\widehat{M_A(\boldsymbol{\beta})} = 0$ in $\widehat{M_A(\boldsymbol{\beta})}$. Therefore, we can write

$$x_k^l P = \sum_{i=1}^{d} Q_i \left(\sum_{j=1}^{n} a_{ij} \partial_j x_j + \beta_i\right) + \sum_{\alpha} R_\alpha c_\alpha,$$

where $Q_i, R_\alpha \in D, c_\alpha \in I_A(\boldsymbol{x})$. Multiply the both sides by x_k^m, we obtain

$$\begin{aligned} x_k^l P x_k^m &= \sum_{i=1}^{d} Q_i \left(\sum_{j=1}^{n} a_{ij} \partial_j x_j + \beta_i\right) x_k^m + \sum_{\alpha} R_\alpha c_\alpha x_k^m \\ &= \sum_{i=1}^{d} Q_i x_k^m \left(\sum_{j=1}^{n} a_{ij} \partial_j x_j + (\beta_i + m a_{ik})\right) + \sum_{\alpha} R_\alpha x_k^m c_\alpha. \end{aligned}$$

If $m \in \mathbf{N}$ is sufficiently large, then we have $Q_i x_k^m, R_\alpha x_k^m \in x_k^l D$. Hence we obtain $x_k^l P x_k^m \in x_k^l H_A(\widehat{\beta + m a_k})$. Since the element x_k^l is not a zerodivisor in D, thus we have $P x_k^m \in H_A(\widehat{\beta + m a_k})$.
On the othe hand, by the assumption and Corollary 2.3, we obtain

$$\widehat{M_A(\beta)} \overset{\cdot x_k^m}{\simeq} M_A(\widehat{\beta + m a_k}).$$

This implies $P \in \widehat{H_A(\beta)}$ and φ is injective.
Second, we will check the surjectivity of φ. By the assumption and Corollary 3.2, for any $m \in \mathbf{N}$, there exists $Q_m \in D$ such that

$$1 \bmod \widehat{H_A(\beta)} = x_k^m Q_m \bmod \widehat{H_A(\beta)}.$$

Hence, we immidiately obtain $\mathbf{C}[\boldsymbol{x}]_{x_k} \otimes (1 \bmod \widehat{H_A(\beta)}) \subset \mathrm{Im}\ \varphi$.
Since $D(\mathbf{C}[\boldsymbol{x}]_{x_k} \otimes (1 \bmod \widehat{H_A(\beta)})) = \mathbf{C}[\boldsymbol{x}]_{x_k} \otimes_{\mathbf{C}[\boldsymbol{x}]} \widehat{M_A(\beta)}$ and φ is a D-morphism, finally we obtain $\mathrm{Im}\ \varphi = \mathbf{C}[\boldsymbol{x}]_{x_k} \otimes_{\mathbf{C}[\boldsymbol{x}]} \widehat{M_A(\beta)}$ and φ is surjective.
Proof of 2. (*the condition (a)*)
We consider the following exact sequence:

$$0 \longrightarrow F \longrightarrow \widehat{M_A(\beta)} \xrightarrow{\psi} G \longrightarrow 0,$$

where $\mathrm{Supp}(G) \subset \{x_k = 0\}$. For a sufficiently large $m \in \mathbf{N}$, we have

$$\psi(x_k^m \bmod \widehat{H_A(\beta)}) = x_k^m \psi(1 \bmod \widehat{H_A(\beta)}) = 0 \text{ in } G.$$

Hence, $x_k^m \bmod \widehat{H_A(\beta)} \in F(= \mathrm{Ker}\ \psi)$. By the assumption and Corollary 2.3, we obtain

$$M_A(\widehat{\beta - m a_k}) \overset{\cdot x_k}{\simeq} \widehat{M_A(\beta)}\ (1 \bmod H_A(\widehat{\beta - m a_k}) \mapsto x_k^m \bmod \widehat{H_A(\beta)}).$$

The module K contains the image of the morphism $\cdot x_k^m$, therefore $K = \widehat{M_A(\beta)}$. This implies $G = 0$.
(*the condition (b)*)
Suppose that a morphism $g : L \to \widehat{M_A(\beta)}$ satisfies the condition of the proposition. We will show that the following exact sequence is split:

$$0 \longrightarrow \mathrm{Ker}\ g \longrightarrow L \xrightarrow{g} \widehat{M_A(\beta)} \longrightarrow 0. \tag{1}$$

Since g is surjective, there exists $u \in L$ such that $g(u) = 1 \bmod \widehat{H_A(\beta)}$. we define $D[s] := D \otimes_{\mathbf{C}} \mathbf{C}[s]$ $(s = (s_1, \dots, s_d))$, and

$$\widehat{H_A[s]} := D[s] I_A(\boldsymbol{x}) + \sum_{i=1}^{d} D[s] \cdot (\sum_{j=1}^{n} a_{ij} \partial_j x_j + s_i).$$

It is easy to check that for any $P(s) \in D[s]$ there exist $Q(s) \in D[s]$ and $c \in \mathbf{N}$ such that $P(\beta - m\mathbf{a}_k)x_k^m = x_k^{m-c}Q(\beta - m\mathbf{a}_k)$ in D for any $m \geq c$. In particular, $P(s) \in \widehat{H_A[s]}$ implies

$$\begin{aligned} x_k^{m-c}Q(\beta - m\mathbf{a}_k) \bmod \widehat{H_A(\beta)} &= P(\beta - m\mathbf{a}_k)x_k^m \bmod \widehat{H_A(\beta)} \\ &= 0. \end{aligned}$$

Recall that the restriction of g on $\{x_k \neq 0\}$ is an isomorphism, thus we have $Q(\beta - m\mathbf{a}_k) \bmod \widehat{H_A(\beta)} = 0$ on $\{x_k \neq 0\}$. This implies $\mathrm{supp}(Q(\beta - m\mathbf{a}_k)u) \subset \{x_k = 0\}$.
Hence, for a suffciently large $l \in \mathbf{N}$, $x_k^l Q(\beta - m\mathbf{a}_k)u = 0$ in L for any $m \geq c$. Thus, for all $m \in \mathbf{N}$, $m \geq c + l$ implies $P(\beta - m\mathbf{a}_k)x_k^m u = 0$. In summary, for any $P(s) \in \widehat{H_A[s]}$, there exists $c' \in \mathbf{N}$ such that $P(\beta - m\mathbf{a}_k)x_k^m u = 0$ for $m \geq c'$.
Furthermore, since the left ideal $\widehat{H_A[s]}$ is finitely generated as a $D[s]$-module, we can choose c' independently of $P(s)$. Therefore, we will define a D-morphism $\xi : M_A\widehat{(\beta - c'\mathbf{a}_k)} \to L$ by

$$\xi(1 \bmod H_A\widehat{(\beta - c'\mathbf{a}_k)}) := x_k^{c'}u.$$

By the assumption and Corollary 2.3, we have

$$M_A\widehat{(\beta - c'\mathbf{a}_k)} \overset{\cdot x_k^{c'}}{\simeq} \widehat{M_A(\beta)}.$$

Considering the composite mapping of ξ and the inverse of $\cdot x_k^{c'}$, we define a morphism $\tilde{\xi} : \widehat{M_A(\beta)} \to L$. Obviously, $g \circ \tilde{\xi} = \mathrm{id}_{\widehat{M_A(\beta)}}$. This implies the exact sequence (1) is split. Therefore Ker g is a quotient module of L. Finally, since Supp(Ker g) $\subset \{x_k = 0\}$, by the assumption of g, we obtain Ker $g = 0$ and g is an isomorphism.

Finally, the statement 1. (resp. 2.) of the main theorem immidiately results from Theorem 4.1 and Proposition 4.3.1 (resp. Theorem 4.2 and Proposition 4.3.2).

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