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Painlevé 方程式の特殊解

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1 Introduction

In this paper we will study the third Painlevé equations $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{1}$$

for $\gamma = 0$ and $\alpha\delta \neq 0$.

The values of complex parameters $\alpha, \beta, \gamma, \delta$ of the third Painlevé equations satisfy one of four cases:

(Q) $\alpha = 0, \gamma = 0$ (or $\beta = 0, \delta = 0$),

(D_8) $\gamma = 0,\ \delta = 0\ \ \alpha\beta \neq 0$

(D_7) $\gamma = 0,\ \alpha\delta \neq 0$ (or $\delta = 0, \beta\gamma \neq 0$),

(D_6) $\gamma\delta \neq 0$.

In the case (Q), P_{III} are sovable by quadraturs ([11], [23]). Since all of solutions of (Q) are classical in Umemura's meaning ([27]), we do not include the equation (Q) in the Painlevé equations. Gromak ([4]) also excluded the case (Q). The type D_6, which is generic case, is studied in many articles ([23], [18], [28]). The equations of type D_7 and D_8 are missed in most study of the third Painlevé equation so far. Gromak studied the type D_7 in [2].

Recently Sakai pointed out the significance of the type D_7 and D_8 in [24]. He showed that the spaces of the initial values for the type D_6, D_7 and D_8 are different from each other. The vertical leaves become a sum of rational curves, whose intersection diagrams are the root lattices D_6, D_7 and D_8 in each case. This is the origin of the name of each type. Moreover the Bäcklund transformation groups of the equation of type D_6, D_7 and D_8 are $W(A_1 \oplus A_1)$, $W(A_1)$ and $\mathbb{Z}_2$, respectively. From Sakai's viewpoint, we should study eight (not six) type of Painlevé equations.

This papar is a supplement of Okamoto's series of four papers "Studies on the Painlevé equations" ([20], [21], [22], [23]) published in 1980s. We will study Hamitonian structures, a transformation group, τ-functions and special solutions of type D_7. The type D_8 reduced to special case of D_6 by a quadratic transformation. We will comment about the type D_8 in Theorem 3.1.

We will study the equation $P'_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$

$$q'' = \frac{q'^2}{q} - \frac{q'}{t} + \frac{\alpha q^2}{4t^2} + \frac{\beta}{4t} + \frac{\gamma q^3}{4t^2} + \frac{\delta}{4q}, \tag{2}$$

which is equivalent to P_{III} by

$$t = x^2, \quad y = xq. \tag{3}$$

We will consider P'_{III} instead of P_{III} ([23]), because the action of a transformaion group on P'_{III} is simpler than P_{III} .

By change of variables

$$x = \lambda x_1, \quad y = \mu y_1 \qquad (\lambda\mu \neq 0), \tag{4}$$

we can normalize the parameter $(\alpha, \beta, \gamma, \delta)$. Essentially, the equations of type D_6 have two comlex parameters, the equations of type D_7 have one comlex parameter and the equations of type D_8 have no comlex parameters. For the type D_7, we can take standard form

$$q'' = \frac{q'^2}{q} - \frac{q'}{t} - \frac{q^2}{t^2} + \frac{a}{t} - \frac{1}{q}. \tag{5}$$

This equation is the main object in this paper.

An algebraic solution of the third Painlevé equations are found by Lukashevich ([11], [12]). Gromak classified all algebraic solutions of the third Painlevé equations not only for type D_6 but also for type D_7 ([3], [4]). If a is an integer, (5) has one and only one algebraic solution. If a is not an integer, (5) has no rational solution. The equation of type D_8 has two rational solutions.

The algebraic solutions of Painlevé equations are studied by many authors. There are many works by Belorussian school (see [5]). After Okamoto showed that the transformation groups of Painlevé equations are isomorphic to affine Wyle groups, it is easy to understand their works. Murata gives classification of algebraic solutions of the second, third and fourth Painlevé equations in terms of affine Wyle groups ([17], [18]). Kitaev-Law-McLeod [9] classified rational solutions of the fifth Painlevé equations. Mazzocco [15] classified rational solutions of the sixth Painlevé equations. Any algebraic solutions become rational for the fifth Painlevé equations (announced in [30]). Algebraic solutions for the sixth Painlevé equations are very interesiting ([6], [7], [1], [14], [26], [19], [10]) and they are not classified yet. From second to fifth Painlevé equations, all algebraic solutions turn to be rational except for type D_7.

All of algebraic solutions of (5) are transformed to each other by the affine Wyle group $W(A_1)$. But it is difficult to calculate all algebraic solutions by the direct action of the affine Wyle group. If we consider τ-functions, the action becomes very simple. The action of the affine Wyle group reduces to the Toda equation on τ-functions. For the second Painlevé equations, the Yablonskii-Vorob'ev polynomials give transformations of τ-functions ([29], [22]). There are similar polynomials for other Painlevé equations ([22] for Painlevé IV, [26] and [19] for type D_6, V and VI). Although the solution of the third Painlevé equations of type D_7 is algebraic, the action of the affine Wyle group is given by polynomials. This is an analogue of Umemura's polynomials for the sixth Painlevé equations ([26]).

Yablonskii-Vorob'ev polynomials or Umemura's polynomials are related to Shur polynomials ([8], [25]). It is an open problem to represent our new polynomials by Shur polynomials.

In section four, we will study transcendental classical functions of the equations of type D_7. The third Painlevé equation of type D_7 have o like the second

Painleve equation. The second Painleve equation has also one parameter and the Bäcklund transformation group is the affine Wyle group $W(A_1)$. The second Painleve equation has transcendental classical functions which are reduced to Airy functions ([27]). On the contrary, the third Painlevé equation of type D_7 does not have transcendental classical functions, following the idea of Umemura and Watanabe. The third Painlevé equation of type D_6 has transcendental classical functions which are reduced to Bessel functions ([28]).

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2 Third Painlevé equation

In this section we will review basic facts on the third Painlevé equations ([23]). We will write (1) as $P_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ and (2) as $P'_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$.

2.1 Fundamental transform

Although the third Painlevé equations have four complex parameters, essensially they have two complex parameters by simple transformation.

Theorem 2.1. *(1) By the change of variables*

$$t = x^2, \quad y = xq, \tag{6}$$

$P_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ *and* $P'_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ *are equivaliant.*
(2) By the change of variables

$$x = x_1, \quad y = \frac{x}{y_1}, \tag{7}$$

$P'_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ *changes to* $P'_{\mathrm{III}}(-\beta,-\alpha,-\delta,-\gamma)$.
(3) By the change of variables

$$x = x_1^2, \quad q = y_1^2, \tag{8}$$

$P'_{\mathrm{III}}(\alpha,\beta,0,0)$ *changes to* $P'_{\mathrm{III}}(0,0,2\alpha,2\beta)$.
(4) By the change of variables

$$t = \lambda t_1, \quad q = \mu q_1 \tag{9}$$

$P'_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ *changes to* $P'_{\mathrm{III}}(\lambda\alpha,\mu\lambda^{-1}\beta,\lambda^2\gamma,\mu^2\lambda^{-2}\delta)$.

By Theorem 2.1 (3), the third Painleve equation of type D_8 reduced to type D_6. By Theorem 2.1 (4), the third Painleve equations P'_{III} of type D_7 can be normalized to (5).

2.2 Hamiltonian system

The Hamiltonian associated with (5) is

$$H = \frac{1}{t}\left(f^2g^2 - a_1fg + tg + \frac{1}{2}f\right), \tag{10}$$

where $f = q$ and $a = 1 + a_1$ ([24]). The Hamiltonian system $\mathcal{H}(a_1)$ is

$$\begin{cases} \dfrac{df}{dt} = \dfrac{1}{t}\left(2f^2g - a_1 f + t\right), \\ \dfrac{dg}{dt} = -\dfrac{1}{t}\left(2fg^2 - a_1 g + \dfrac{1}{2}\right). \end{cases} \tag{11}$$

We take an autiliary Hamiltonian

$$h(t) = tH + a_1^2/4. \tag{12}$$

Then we have

$$\begin{cases} f(t) = -\dfrac{1 - 2a_1 h'(t) + 2t h''(t)}{4h'(t)^2}, \\ g(t) = h'(t). \end{cases} \tag{13}$$

Therefore we have

Proposition 2.2. *h satisfies the differential equation*

$$(th''(t))^2 + 4h'(t)^2(th'(t) - h(t)) + a_1 h'(t) - \frac{1}{4} = 0. \tag{14}$$

Inversely, for a solution of $h(t)$ of (14), we have a solution (f, g) of (11) by (13) if

$$\frac{d^2h}{dt^2} \neq 0.$$

Proposition 2.3. *There exists the one-to-one correspondence from a solution h of* (14) *and a solution* (f, g) *of* (11).

The equation (14) admits a singular solution of the form

$$h = \lambda t + \mu,$$

$$-\frac{1}{4} + a_1\lambda - 4\lambda^2\mu = 0.$$

2.3 Transformation group

The transformation group of the third Painleve equation of type D_7 is isomorphic to the affine Wyle group $W(A_1)$ ([24]). The geneators of $W(A_1)$ are given by

$$\pi : a_1 \longrightarrow -a_1, \tag{15}$$

$$s : a_1 \longrightarrow -1 - a_1. \tag{16}$$

We will show the explicit expression of the action of $W(A_1)$.

Theorem 2.4. *1) If $(f(t), g(t))$ satisfies $H(a_1)$,*

$$(F, G) = \left(-f(-t) + \frac{a_1}{g(-t)} - \frac{1}{2g(-t)^2}, -g(-t)\right)$$

satisfies $H(-a_1)$.

2)If $(f(t), g(t))$ *satisfies* $H(a_1)$,

$$(F, G) = \left(-2tg(-t), \frac{f(-t)}{2t}\right)$$

satisfies $H(-1-a_1)$.

By Theorem 2.4, we have a Bäcklund transformation which gives

$$\pi \circ s : a_1 \longrightarrow a_1 + 1,$$

as follows:

$$(f, g) \longrightarrow \left(\frac{-2\,t^2}{f(t)^2} + \frac{2\,(1+a_1)\,t}{f(t)} - 2\,t\,g(t), \frac{f(t)}{2\,t}\right). \tag{17}$$

This transformation is found by Gromak ([2]).

2.4 τ-function

We define the τ-function of P'_{III} . For any solution (f, g), the τ-function $\tau(t)$ is defined by

$$\frac{d}{dt}\log\tau(t) = H(f, g, t), \tag{18}$$

up to constant multiplication.

Theorem 2.5. *The* τ*-function* $\tau(t)$ *is holomorphic in* $\mathbb{C} - \{0\}$ *and has simple zeros.*

In most papers, the third Painleé equations of type D_6 are represented as monodromy preserving deformation. Recently Kawamuko and Sakai showed that the equations of type D_7 and D_8 are represented as monodromy preserving deformations. Therefore the holomorphicity of the τ-function is followed from Miwa's result ([16], [13]).

Here we will give direct proof of the holomorphicity of the τ-function. From (14), if the auxiliary Hamiltonian h has a pole at $t = t_0$ $(t_0 \neq 0)$,

$$h \sim \frac{t_0}{t-t_0} + O((t-t_0)^0),$$

where $O((t-t_0)^0)$ is the Landau's O. Therefore we have

$$H = \frac{1}{t}\left(h - \frac{a_1^2}{4}\right) \sim \frac{1}{t-t_0} + O((t-t_0)^0).$$

By the definiion of the τ-function, $\tau(t)$ has a simple zero at $t = t_0$.

2.5 Toda equation

Let $h = h(t, f, g, a_1)$ be an auxiliary Hamiltonian (12). We difine a new auxiliary function h_1 by

$$h_1 = h - fg + \frac{2a_1+1}{4}. \tag{19}$$

We set (F, G) is the Bäcklund transformation of (f, g) by $\pi \circ s$:

$$(F,G)=\left(\frac{-2t^2}{f^2}+\frac{2\,(1+a_1)\,t}{f}-2\,t\,g,\frac{f}{2\,t}\right).$$

As the same as (13), we have

$$\begin{cases} f=2th_1', \\ g=\dfrac{-1+2\,h_1'+2\,a_1\,h_1'+2\,x\,h_1''}{8\,t\,{h_1'}^2}. \end{cases} \tag{20}$$

Lemma 2.6. *$h_1(t, f, g, a_1)$ equals the auxiliary Hamiltonian $h(t, F, G, a_1 + 1)$.*

Proof. We have

$$H(t,F,G,a_1+1)=H(t,f,g,a_1)-\frac{fg}{t} \tag{21}$$

by direct calculation. Therefore

$$\begin{aligned} h(t,F,G,a_1+1) &= tH(t,F,G,a_1+1)+\frac{(a_1+1)^2}{4} \\ &= tH(t,f,g,a_1)-fg+\frac{(a_1+1)^2}{4} \\ &= h(t,f,g,a_1)-fg+\frac{2a_1+1}{4}=h_1(t,f,g,a_1). \end{aligned}$$

□

We set $X = fg$. From (13) we have

$$X=\frac{-2\,t\,h''+2\,a_1\,h'-1}{4\,h'}. \tag{22}$$

From (20) we have

$$X=\frac{2\,t\,h_1''+2\,(a_1+1)h_1'-1}{4\,h_1'}. \tag{23}$$

We will consider the sequece of solutions transformed by

$$\ell=\pi\circ s,\quad \ell^2,\quad \ell^3,\quad \ell^4,\ldots$$

For an fixed solution $(f_0, g_0) = (f, g)$, we set

$$(f_n,g_n)=(\ell^n(f),\ell^n(g)),$$

which is a solution for $a_1 + n$. Let τ_n be a function defeined by

$$\frac{d}{dt}\tau_n=H(t,f_n,g_n,a_1+n).$$

Theorem 2.7. *τ_n satisfy the Toda equation*

$$\frac{d}{dt}t\frac{d}{dt}\log\tau_n=c(n)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \tag{24}$$

for some constants $c(n)$.

Proof. We set $X_n = f_n g_n$. From (21) we have

$$H(t, f_{n+1}, g_{n+1}, a_1 + n + 1) = H(t, f_n, g_n, a_1 + n) - \frac{X_n}{t}.$$

Therefore

$$X_n = t\frac{d}{dt}\log\frac{\tau_n}{\tau_{n+1}}. \tag{25}$$

Let h_n be the auxiliary Hamiltonian for (f_n, g_n). From (22), we have

$$X_n = \frac{-2\,t\,h_n'' + 2\,(a_1 + n)\,h_n' - 1}{4\,h_n'}.$$

Changing a_1 to $a_1 - 1$ in (23), we have

$$X_{n-1} = \frac{2\,t\,h_n'' + 2\,(a_1 + n)\,h_n' - 1}{4\,h_n'}.$$

Therefore we have

$$X_{n-1} - X_n = \frac{t\,h_n''}{h_n'} = t\frac{d}{dt}\log h_n'. \tag{26}$$

From (25) and (26), we obtain

$$h_n' = c(n)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}.$$

□

3 Polynomial generated by special solution

3.1 Algebraic solution

Theorem 3.1. *The third Painleve equation of type D_8 does not have trancendental classical solutions. The third Painleve equation of type D_8 has two rational solutions. There are no more algebraic solutions.*

The first part comes from [28] and the second part comes from ([18]) by the transformation in Theorem 2.1 (3). Actually, $P_{\mathrm{III}}(\alpha, \beta, 0, 0)$ has constant solutions $y = \pm\sqrt{-\beta/\alpha}$.

Lukashevich found special algebraic solutions and Gromak classified all algebraic solutions for type D_7:

Theorem 3.2. *([11], [12]) In case $a_1 = -1$, $H(a_1)$ has an algebraic solution*

$$f(t) = -t^{2/3}, \qquad g(t) = \frac{1}{6t^{2/3}} - \frac{1}{2t^{1/3}}.$$

$H(a_1)$ has one and only one algebraic solution if a_1 is an integer. These algebraic solutions are transformed by the the Bäcklund transformation ℓ^n.

3.2 Sequence of algebraic solution

First few algebraic solutions in Thorem 3.2 by the Bäcklund transformation ℓ is as followes.

If $a_1 = 0$

$$(f,g) = \left(\frac{-t^{\frac{1}{3}}}{3} - t^{\frac{2}{3}}, \frac{-1}{2t^{\frac{1}{3}}} \right).$$

If $a_1 = 1$

$$(f,g) = \left(\frac{-5t^{\frac{2}{3}} - 12t - 9t^{\frac{4}{3}}}{\left(1+3t^{\frac{1}{3}}\right)^2}, \frac{-1-3t^{\frac{1}{3}}}{6t^{\frac{2}{3}}} \right),$$

If $a_1 = 2$,

$$(f,g) = \left(\frac{-35x^{\frac{1}{3}} - 315x^{\frac{2}{3}} - 990x - 1350x^{\frac{4}{3}} - 891x^{\frac{5}{3}} - 243x^2}{3\left(5+12x^{\frac{1}{3}}+9x^{\frac{2}{3}}\right)^2}, \frac{-5-12t^{\frac{1}{3}}-9t^{\frac{2}{3}}}{2\left(1+3t^{\frac{1}{3}}\right)^2 t^{\frac{1}{3}}} \right)$$

We can calculate τ-functions of algebraic solutions by the Toda equation (24) more easily. From now on we set $\tau_n(t)$ as the τ-function of the algebraic solution for $a_1 = n$.

Theorem 3.3. *Let $s = 3t^{1/3}$. Then we have*

$$\tau_n(t) = \exp\left(-\frac{1}{2}ns - \frac{s^2}{8}\right) s^{-d_n/12} S_n(s),$$

up to constant multiplication. Here d_n is

$$d_n = \begin{cases} 9n^2 - 1 & n \text{ is even}, \\ 9n^2 - 4 & n \text{ is odd}. \end{cases} \tag{27}$$

$S_n(s)$ are monic polynomials of s with integral coefficients. $S_n(0) \neq 0$ and

$$\begin{aligned} nS_n(s)^2 + sS_n(s)^2 - 2S_n(s)S_n'(s) + 2sS_n'(s)^2 - 2sS_n(s)S_n''(s) \\ = \begin{cases} sS_{n+1}(s)S_{n-1}(s) & n \text{ is even}, \\ S_{n+1}(s)S_{n-1}(s) & n \text{ is odd}. \end{cases} \end{aligned} \tag{28}$$

Proof. We assume that

$$\tau_n(t) = \exp\left(-\frac{3}{2}t^{1/3} - \frac{9}{8}t^{2/3}\right) t^{-c_n/36} T_n(s).$$

By the Toda equation (24) we have a recurrent relation

$$\begin{aligned} nT_n(s)^2 + sT_n(s)^2 - 2T_n(s)T_n'(s) + 2sT_n'(s)^2 - 2sT_n(s)T_n''(s) \\ = T_{n+1}(s)T_{n-1}(s), \end{aligned} \tag{29}$$

and $2c_n + 12 = c_{n-1} + c_{n+1}$.

We set

$$T_n(s) = a_0^{(n)} + a_1^{(n)}s + a_2^{(n)}s^2 + O(s^3).$$

Then the left hand side of (29) is

$$(na_0^{(n)} - 2\,a_1^{(n)})a_0^{(n)} + \left((a_0^{(n)})^2 + 2na_0^{(n)}a_1^{(n)} - 8a_0^{(n)}a_2^{(n)}\right)s + O(s^2).$$

We will see that if n is even, $na_0^{(n)} - 2\,a_1^{(n)} = 0$ and $(a_0^{(n)})^2 + 2na_0^{(n)}a_1^{(n)} - 8a_0^{(n)}a_2^{(n)}$ is an odd integer and that if n is odd, $na_0^{(n)} - 2\,a_1^{(n)} = 0$ is an odd integer by induction.

If n is odd and $T_n(0) = a_0^{(n)}$ is an odd integer, $na_0^{(n)} - 2\,a_1^{(n)} = 0$ is an odd integer. From (29) we have

$$na_0^{(n)} - 2\,a_1^{(n)} = a_0^{(n-1)}a_0^{(n+1)}.$$

Therefore $a_0^{(n+1)}$ is also an odd integer.

Assume that n is even and $T_n(0) = a_0^{(n)}$ is an odd integer. We will show $na_0^{(n)} - 2\,a_1^{(n)} = 0$ later. Then $(1+n^2)(a_0^{(n)})^2 - 8a_0^{(n)}a_2^{(n)}$ is an odd integer. Therefore if we set

$$\begin{cases} S_n(s) = T_n(s) & \text{n is even,} \\ S_n(s) = \dfrac{T_n(s)}{s} & \text{n is odd,} \end{cases}$$

we have $S_n(0)$ is an odd integer.

We set

$$\tau_n(t) = \exp\left(-\frac{3}{2}t^{1/3} - \frac{9}{8}t^{2/3}\right)t^{-d_n/36}S_n(s).$$

Then S_n satisfy (28) and d_n satisfy (27).

Now we will show that $na_0^{(n)} - 2\,a_1^{(n)} = 0$ when n is even.

$$\frac{d}{dt}\log\tau_n \sim -\frac{d_n}{36}\frac{1}{t} + k_1t^{-2/3} + k_2t^{-1/3} + O(t^0),$$

where

$$\begin{aligned} k_1 &= \frac{a_1^{(n)}}{a_0^{(n)}} - \frac{n}{2}, \\ k_2 &= -\frac{3}{4} - \frac{3(a_1^{(n)})^2}{(a_0^{(n)})^2} + \frac{6a_2^{(n)}}{a_0^{(n)}}. \end{aligned}$$

Therefore the auxiliary hamiltonian h is the form

$$h \sim \frac{9n^2 - d_n}{36} + k_1t^{1/3} + k_2t^{2/3} + O(t).$$

By (14)

$$(th''(t))^2 + 4h'(t)^2(th'(t) - h(t)) + nh'(t) - \frac{1}{4} = \frac{(4 - 9n^2 + d_n)\,k_1^2}{81}t^{-4/3} + O(t^{-1}).$$

Thus when n is even, we have

$$k_1 = \frac{a_1^{(n)}}{a_0^{(n)}} - \frac{n}{2} = 0.$$

By Theorem 2.5, $S_n(s)$ have simple zeros. The polynomials $S_n(s)$ are analogue of Yablonskii-Vorob'ev polynomials for the second Painlevé equations. It is a conjecture that $S_n(s)$ can be represented by Shur polynomials like Yablonskii-Vorob'ev polynomials ([8]).

We will list $S_n(s)$ for $n = 0, 1, 2, 3, 4, 5$.

$$S_0(s) = 1,$$

$$S_1(s) = 1,$$

$$S_2(s) = s + 1,$$

$$S_3(s) = s^2 + 4s + 5,$$

$$S_4(s) = s^4 + 10s^3 + 40s^2 + 70s + 35,$$

$$S_5(s) = s^6 + 20s^5 + 175s^4 + 840s^3 + 2275s^2 + 3220s + 1925.$$

4 Transcendental classical solutions

4.1 Main Theorem

The third Painleve equations have classical solutions written by Bessel functions. The third Painlevé equations of type D_6 is in [24].

The Hamiltonian form of the third Painlevé equations of type D_6 as follows ([28]):

$$\begin{cases} t\dfrac{dq}{dt} = 2q^2p - q^2 + v_1 q + t, \\ t\dfrac{dp}{dt} = -2qp^2 + 2pq - v_1 p + \dfrac{1}{2}(v_1 + v_2). \end{cases} \tag{30}$$

Umemura and Watanabe show that (30) has transcendental classical solutions if and only if $v_1 + v_2$ or $v_1 - v_2$ is an even integer. For example, if $v_1 + v_2 = 0$, we have classical solutions $p = 0$ and

$$t\frac{dq}{dt} = -q^2 + v_1 q + t. \tag{31}$$

We will introduce a new variable u by

$$q = \frac{v_1}{2} + t\frac{d}{dt}(\log u).$$

Then (31) turns to be the linear equation

$$\frac{d^2u}{dt^2} + \frac{1}{t}\frac{du}{dt} - \frac{1}{t^2}\left(t + \frac{v_1^2}{4}\right)u = 0,$$

which is equivalent to Bessel's equation.

If $v_1 + v_2$ or $v_1 - v_2$ is an even integer, there exist a Bäcklund transformation on (30), we can reduce to the case $v_1 + v_2 = 0$. Therefore any transcendental classical solutions can be represented by Bessel functions.

The Bäcklund transformation group of the third Painlevé equations of type D_6 is the affine Wyle group $W(A_1) \oplus W(A_1)$ and walls of this action is the set

$$\{(v_1, v_2) \in \mathbb{C} \mid v_1 + v_2 \text{ or } v_1 - v_2 \text{ is an even integer}\}.$$

For other Painlevé equations (second, fourth, fifth and sixth), they have transcendental classical solutions on the walls of action of the affine Wyle group.

The third Painlevé equations of type D_7 has one parameter a and the Bäcklund transformation group is the affine Wyle group $W(A_1)$. Hence we may expect that the equation (5) has transcendental classical solutions on the walls. But the this expectation is incorrect. We will show that

Theorem 4.1. *The third Painlevé equations of type D_7 do not have transcendental classical solutions.*

In [2] algebraic solutions of the third Painlevé equations of type D_7 are classified. Combined with Theorem 4.1, we have

Theorem 4.2. *The third Painlevé equations of type D_7 have classical solutions if and only if a is an integer. If a is an integer,* (5) *has only one algebraic solutions with three-sheeted covering.*

4.2 Invariant divisor

In this section we will show a key lemma for Theorem 4.1.

Let K be an ordinary differential field which is an extention of $\mathbb{C}(t)$, the field of rational functions of t. Let $K[f, g]$ be the polynimial ring over K in two independent variables f and g. We consider the following derivation $X(a)$ on $K[f, g]$:

$$X(a) = t\frac{\partial}{\partial t} + \left(2f^2 g - af + t\right)\frac{\partial}{\partial f} - \left(2fg^2 - ag + \frac{1}{2}\right)\frac{\partial}{\partial g}. \tag{32}$$

The differential ring $(K[f, g], X(a))$ represents (11).

In [27], Umemura and Watanabe introduced the condition (J) for $X(a)$ as follows:

(J) For any ordinary differential field extension $K/\mathbb{C}(t)$, there exists no principal ideal I of $K[f, g]$ such that $0 \subsetneq I \subsetneq K[f, g]$ and $X(a)I \subset I$.

The following Proposition is the key in this paper.

Proposition 4.3. *The derivation $X(a)$ does not satisfy the condition (J).*

Proof. Assume that there exists a principal ideal I of $K[f, g]$ which is invariant under the action of $X(a)$. Let F in $K[f, g]$ be a generator if I. Then we have

$$X(a)F = GF, \tag{33}$$

for some $G \in K[f, g]$. We will show there is no such polynomial F in $K[f, g]$ in six steps.

Step 1. Two gradings in $K[f, g]$:

We will introduce two gradings in $K[f,g]$. At first, we define the weights of f and g to be -1 and 2 respectively. The weight of a monomial af^ig^j in $K[f,g]$ is $2j-i$ for any $a \in K$ $(a \neq 0)$. Let R_d be the K-linear subspace of $K[f,g]$ generated over K by all the monomials of weight d. $R_{-d} = K[f^2g]f^d$, $R_{2d} = K[f^2g]g^d$, $R_{2d-1} = K[f^2g]fg^d$, for any non-negative integer d and

$$K[f,g] = \bigoplus_{d \in \mathbf{Z}} R_d, \quad R_d \cdot R_{d'} \doteq R_{d+d'}.$$

We define three homogeneous derivations X_{-2}, X_0, X_1 by

$$\begin{aligned} X_1 &= (2f^2g+t)\frac{\partial}{\partial f} - 2fg^2\frac{\partial}{\partial g}, \\ X_0 &= t\frac{\partial}{\partial t} - af\frac{\partial}{\partial f} + ag\frac{\partial}{\partial g}, \\ X_{-2} &= -\frac{1}{2}\frac{\partial}{\partial g}. \end{aligned}$$

We have $X(a) = X_{-2} + X_0 + X_1$ and each X_i maps R_d to R_{d+i}.

In the second grading, we set the weights of f and g to be 2 and -1 respectively. The weight of a monomial af^ig^j in $K[f,g]$ is $2i-j$ for any $a \in K$ $(a \neq 0)$. Let R'_d be the K-linear subspace of $K[f,g]$ generated over K by all the monomials of weight d. $R'_{-d} = K[fg^2]g^d$, $R'_{2d} = K[fg^2]f^d$, $R'_{2d-1} = K[fg^2]f^dg$, for any non-negative integer d and

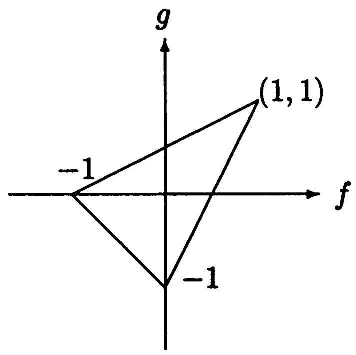
$$K[f,g] = \bigoplus_{d \in \mathbf{Z}} R'_d, \quad R'_d \cdot R'_{d'} = R'_{d+d'}.$$

We define three homogeneous derivations X'_{-2}, X'_0, X'_1 by

$$\begin{aligned} X'_1 &= 2f^2g\frac{\partial}{\partial f} - \left(\frac{1}{2} + 2fg^2\right)\frac{\partial}{\partial g}, \\ X'_0 &= t\frac{\partial}{\partial t} - af\frac{\partial}{\partial f} + ag\frac{\partial}{\partial g}, \\ X'_{-2} &= t\frac{\partial}{\partial f}. \end{aligned}$$

We have $X(a) = X'_{-2} + X'_0 + X'_1$ and each X'_i maps R'_d to R'_{d+i}.

The both gradings come from the Newton polygon of $X(a)$. The Newton polygon of $X(a)$ is as follows:



Here an integral point (i, j) represents the derivation $bf^{i+1}g^j\partial/\partial p + cf^i g^{j+1}\partial/\partial g$ $(b, c \in K)$. Since the Newton polygon of type D_7 is different from the polygon of type D_6 in [28], we choose different gradings.

We will determine the polynimial G. Since the highest part X_1 and X_1' are of weight one, G is at most of weight one in the both degree. Therefore

$$G = \lambda fg + \mu$$

for some $\lambda, \mu \in K$.

Step 2. the highest part of F with respect to the first grading

We will consider the highest part of F. Let F be a sum of homobeneous polynomials with respect to the first grading

$$F = F'_{m'} + F_{m'+1} + \cdots + F_1 + \cdots + F_m. \quad (m' \leq m)$$

$F_j \in R_j$, $F_{m'}, F_m \neq 0$ and if $m = m' = 0$, $F_0 \notin K$. The homogeneous part of (33) is

$$X_1 F_{k-1} + X_0 F_k + X_{-2} F_{k+2} = \lambda fg F_{k-1} + \mu F_k. \tag{34}$$

Firstly we claim that F_m is not divisible by f. If F_m is divisible by f, there exists an integer $k \geq 1$

$$F_m = Qf^k, \quad f \nmid Q, \quad Q \in K[f, g].$$

Since

$$X_1(F_m) = \lambda fg F_m,$$

we have

$$2f^{k+2}g\frac{\partial Q}{\partial f} + tf^k\frac{\partial Q}{\partial f} - 2f^{k+1}g^2\frac{\partial Q}{\partial g} + \left(2kf^{k+1}g + ktf^{k-1}\right)Q = \lambda g f^{k+1} Q.$$

This means $f | Q$, which is contradiction.

Therefore m is a non-negative and even integer. We set $m = 2p$ and

$$F_{2p} = g^p \sum_{j=0}^{k} b_j (f^2 g + t)^j,$$

for $b_0, b_1, ..., b_k \in K$. Then we have

$$X_1(F_{2p}) = fg^{p+1}\sum_{j=0}^{k} 2(j-p)b_j(f^2g+t)^j.$$

Hence we have $\lambda = 2(j-p)$ for a non-negative integer j and

$$F_{2p} = b(f^2g+t)^{p+\lambda/2}g^p \tag{35}$$

for $b \in K$.

When F is an invariant polynomial, $b^{-1}F$ is also an invariant polynomial for any $b \in K$. Hence we may assume $b = 1$. From now on, we assume $b = 1$ in (35).

Step 3. F_{2p-1} and F_{2p-2}

We will determine F_{2p-1} and F_{2p-2}. By (34), F_{2p-1} F_{2p-2} and F_{2p-3} satisfy the equations

$$\begin{aligned} X_1(F_{2p-1}) + X_0(F_{2p}) &= \lambda fgF_{2p-1} + \mu F_{2p}, & (36)\\ X_1(F_{2p-2}) + X_0(F_{2p-1}) &= \lambda fgF_{2p-2} + \mu F_{2p-1}. & (37) \end{aligned}$$

We can set

$$\begin{aligned} F_{2p-1} &= g^p f\sum_{j=0}^{k_1} d_jL^j,\\ F_{2p-2} &= g^{p-1}\sum_{j=0}^{k_2} e_jL^j, \end{aligned}$$

for $d_j, e_j \in K$ and $L = f^2g+t$. By (35) we have

$$\mu F_{2p} - X_0(F_{2p}) = \left(\mu + \frac{a\lambda}{2}\right)L^{p+\frac{\lambda}{2}}g^p - (a+1)t(p+\frac{\lambda}{2})L^{p+\frac{\lambda}{2}-1}g^p. \tag{38}$$

Moreover we have

$$\begin{aligned} X_1(F_{2p-1}) &- \lambda fgF_{2p-1}\\ &= g^p\sum_{j=0}^{k} d_j\left[(2-\lambda-2p+2j)L^{j+1} + (\lambda-1+2p-2j)tL^j\right]. \end{aligned} \tag{39}$$

Comparing (38) and (39), we have $k = p+\lambda/2-1$. But in this case the coefficient of $L^{p+\lambda/2-1}g^p$ of $X_1(F_{2p-1}) - \lambda fgF_{2p-1}$ becomes zero. Hence we have

$$\mu + \frac{a\lambda}{2} = 0. \tag{40}$$

Moreover we have

$$d_0 = 0,\ d_1 = 0, ..., d_{k-1} = 0, d_k = -(a+1)\left(p + \frac{\lambda}{2}\right).$$

Finally we have

$$F_{2p-1} = -\frac{1}{2}(a+1)\,(2p+\lambda)\,g^p f(f^2g+t)^{p+\lambda/2-1}.$$

In the same way we obtain

$$F_{2p-2} = -\frac{1}{8}(a+1)^2\,(2p+\lambda-2)\,(2p+\lambda)\,tg^{p-1}(f^2g+t)^{p+\lambda/2-2},$$

from (37).

Step 4. The highest part of F with respect to the second grading

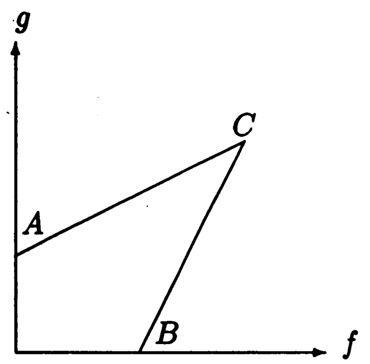
Let us decompose F with respect to the second grading

$$F = F'_{n'} + F'_{n'+1} + \cdots + F'_1 + \cdots + F'_n. \quad (n' \leq n)$$

$F'_j \in R'_j$, $F'_{n'}, F'_n \neq 0$ and if $n = n' = 0$, $F'_0 \notin K$.
In the same way F'_n is not divisible by g and n is an even integer $2q$. There exists a non-negative integer k such that $\lambda = 2(q-k)$ and

$$F'_{2q} = c\left(\frac{1}{2} + fg^2\right)^{q-\lambda/2} f^q \tag{41}$$

for $c \in K$.
The Newton polygon of F has the following figure:



The side AC represents F_{2p} and the side BC represents F'_{2q}. Therefore the coordinates of A, B are $(0,p), (q,0)$ and the coordinates of C are

$$\left(2p+\lambda, 2p+\frac{1}{2}\lambda\right) = \left(2q-\frac{1}{2}\lambda, 2q-\lambda\right).$$

Namely

$$p + \frac{3}{4}\lambda = q. \tag{42}$$

Step 5. F'_{2q-1} and F'_{2q-2}

We can calculate F'_{2q-1} and F'_{2q-2} in the same way as the Step 3:

$$F'_{2q-1} = \frac{1}{2}a\,(\lambda-2q)\,f^q g\left(fg^2+\frac{1}{2}\right)^{q-\lambda/2-1},$$

$$F'_{2q-2} = -\frac{1}{16}a^2(\lambda - 2q)(\lambda - 2q + 2)f^{q-1}\left(fg^2 + \frac{1}{2}\right)^{q-\lambda/2-2}.$$

We will compare F_{2p-1} and F'_{2q-1}. From (42) we have

$$f^{2p+\lambda-1}g^{2p+\lambda/2-1} = f^{2q-\lambda/2-1}g^{2q-\lambda-1}.$$

The coeffiecient of this monomial in F is

$$-\frac{1}{2}(a+1)\,(2p+\lambda) = \frac{1}{2}a\,(\lambda - 2q)\,.$$

From (42) we have

$$\lambda = -\frac{4p}{a+2}. \tag{43}$$

Step 6. F'_{2p-3}

F'_{2p-3} is determined by the equation

$$X_1(F'_{2p-3}) + X_0(F'_{2p-2}) + X_{-2}(F'_{2p}) = \lambda fgF'_{2p-3} - \frac{a\lambda}{2}F'_{2p-2}$$

We may assume that F'_{2p-3} has the form

$$F'_{2p-3} = f^{p-1}g\sum_{j=0}^{k_3} h_j M^j,$$

where $M = fg^2 + 1/2$. Then we have

$$\mu F'_{2p-2} - X_0(F'_{2p-2}) - X_{-2}(F'_{2p}) = \sum_{j=0}^{3} s_j f^{q-1}M^{q-\lambda/2-j},$$

where

$$s_0 = \left(-\frac{1}{2}\lambda + 2q\right)t,\ s_1 = \left(\frac{1}{4}\lambda - \frac{1}{2}q\right)t,\ s_2 = -al,\ s_3 = al\left(1 + \frac{\lambda}{4} - \frac{q}{2}\right),$$

and

$$l = -\frac{1}{16}a^2(\lambda - 2q)(\lambda - 2q + 2).$$

For any positive integer s we have

$$\begin{aligned} X_1(g^{p-1}M^{q-\lambda/2-s}) &- \lambda fg(g^{p-1}M^{q-\lambda/2-s}) \\ &= (2s-4)f^{q-1}M^{q-\lambda/2-s+1} + (\frac{3}{2} - s)f^{q-1}M^{q-\lambda/2-s}. \end{aligned} \tag{44}$$

Therefore $k_3 = q - \lambda/2 - 1$. Setting $s = 1$ in (44), we have

$$\begin{aligned} X_1(g^{p-1}M^{q-\lambda/2-1}) &- \lambda fg(g^{p-1}M^{q-\lambda/2-1}) \\ &= -2f^{q-1}M^{q-\lambda/2} + \frac{1}{2}f^{q-1}M^{q-\lambda/2-1}. \end{aligned}$$

Therefore

$$h_{q-\lambda/2-1} = -\frac{s_0}{2} \tag{45}$$

Setting $s = 2$ in (45), we have

$$X_1(g^{p-1}M^{q-\lambda/2-2}) - \lambda f g(g^{p-1}M^{q-\lambda/2-2}) = -\frac{1}{2}f^{q-1}M^{q-\lambda/2-2}$$

and the term $f^{q-1}M^{q-\lambda/2-1}$ does not appear. Therefore

$$\frac{1}{2}h_{q-\lambda/2-1} = s_1. \tag{46}$$

Comparing (45) and (46), we have

$$-\frac{s_0}{4} = s_1.$$

Namely $\lambda = 0$. Hence we have $\mu = 0$ and $p = q = 0$. This means $F \in K$. Since $I \subsetneq K$, this is contradiction. □

4.3 Proof of Theorem 4.1

The derivation $X(a)$ satisfies the condition (J) for any a. By Theorem 1.1 in [27] we see that every transcendental solution of the equation of type D_7 is non-classical.

By a quadratic transformation, the third Painlevé equation of type D_8 reduces to a third Painlevé equation of type D_6. The third Painlevé equation of type D_8

$$y'' = \frac{1}{y}y'^2 - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x}$$

has two algebraic solutions $y = \sqrt{-\beta/\alpha}$ and no transcendental classical solutions.

Thus we classified classical solutions of the third Painlevé equation of all type.

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