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Quasi-random walk methods

Christian Lécot*
Laboratoire de Mathématiques, Université de Savoie
Campus scientifique, 73376 Le Bourget-du-Lac cedex, France

Abstract

We present particle methods for solving one-dimensional nonlinear reaction-diffusion or convection-diffusion equations. This is done by a fractional step iteration in which diffusion is simulated by random walk. We investigate the effect of replacing pseudo-random numbers by quasi-random numbers in the random walk step. The application of quasi-random sequences is not straightforward, because of correlations, and a reordering technique is used in every time step. For simple problems, we show that a significant improvement in magnitude of error and convergence rate is achieved over standard random walk methods.

1 Introduction

We are interested in mathematical models that involve a combination of reaction and diffusion or convection and diffusion. The solutions may have sharp gradients or traveling fronts. Because of this, standard computational algorithms often require very fine grids to resolve the sharp gradients. With a traveling front solution the method would incorporate moving refined grid. An alternative is to consider a particle-based method that is automatically adapting to sharp gradients [3]. The methods considered are fractional step methods: the equation to be solved is split into two evolution equations, each of which is solved separately. The reaction equation is solved with a numerical ordinary differential equation solver: this is equivalent to altering the particle masses [1]. The nonlinear advection equation is approximated by advecting the particles in a velocity field induced by the particles [10]. In both cases one of the fractional steps is the heat equation. The numerical solution is obtained by random walking the particles.

The major drawback with a probabilistic method using pseudo-random numbers is that convergence can be extremely slow. For example, Monte Carlo integration converges at a rate $\mathcal{O}(N^{-1/2})$ where N is the number of nodes. Much of the effort in the development of Monte Carlo has been in construction of variance reduction methods which speed up the computation. An alternative approach to acceleration is to change the choice of sequence. Quasi-Monte Carlo methods use quasi-random sequences instead of pseudo-random. The quasi-Monte Carlo integration with N nodes yields a deterministic error bound of the form $\mathcal{O}(N^{-1}(\log N)^{s-1})$ in dimension s. There are quasi-Monte Carlo methods not only

*E-mail: Christian.Lecot@univ-savoie.fr

for numerical integration, but also for various other computational problems and it was found that in certain types of such problems they significantly outperform Monte Carlo methods. The refinements of quasi-Monte Carlo methods and the expanding scope of their applications are presented in [7].

The efficiency of a quasi-Monte Carlo method depends on the quality of the sample points that are used. These points should form a low-discrepancy point set, i.e., a point set with small star discrepancy. We recall from [6] some basic notations and concepts. If $s \geq 1$ is a fixed dimension, then $I^s := [0,1)^s$ is the s-dimensional half-open unit cube and λ_s denotes the s-dimensional Lebesgue measure. For a point set P consisting of $\mathbf{p}_0, \ldots, \mathbf{p}_{N-1} \in I^s$ and for an arbitrary subset E of I^s we define the *local discrepancy* by

$$D_N(E,P) := \frac{1}{N} \sum_{0 \leq j < N} c_E(\mathbf{p}_j) - \lambda_s(E),$$

where c_E is the characteristic function of E. The *star discrepancy* of the point set P is defined by

$$D_N^*(P) := \sup_J |D_N(J,P)|,$$

where J runs through all subintervals of I^s with one vertex at the origin. The idea of a (t,m,s)-net is to consider point sets P for which $D_N(J,P) = 0$ for a large family of intervals J. Such point sets should have a small star discrepancy. For integers $b \geq 2$ and $0 \leq t \leq m$, a *(t,m,s)-net in base b* is a point set P consisting of b^m points in I^s such that $D_N(J,P) = 0$ for every subinterval J of I^s of the form

$$J = \prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i + 1}{b^{d_i}} \right),$$

with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ and of measure $\lambda_s(J) = b^{t-m}$. The sequence analog of this concept is as follows. If $b \geq 2$ and $t \geq 0$ are integers, a sequence $\mathbf{p}_0, \mathbf{p}_1, \ldots$ of points in I^s is a *(t,s)-sequence in base b* if, for all integers $n \geq 0$ and $m > t$, the points $\mathbf{p}_j$ with $nb^m \leq j < (n+1)b^m$ form a (t,m,s)-net in base b.

A quasi-random walk method for simulation of diffusion is used in this paper. The basic idea is to write the desired result of the simulation as an integral and to replace the pseudo-random points by low-discrepancy sequences. But the improved accuracy of quasi-Monte Carlo methods may be lost for problems in which the integrand is not smooth. This problem can be overcome by a special technique involving reordering the particles by position after each time step. This approach, along with a convergence proof, is developed in [4] for the diffusion equation and in [5] for linear convection-diffusion problems.

The paper is organized as follows. First a quasi-random particle method for solving a 1-D reaction-diffusion equation of the form $u_t = \nu u_{xx} + f(u)$ is presented in section 2. This is followed by the description of a quasi-random walk method for the Burgers equation $u_t + uu_x = \nu u_{xx}$ in section 3. Finally, in section 4 we conclude by summarizing the results and discussing possible directions for future work.

2 A reaction-diffusion equation

In this section, we study quasi-random particle methods for approximating solutions of nonlinear reaction-diffusion equations in one species and in one spatial dimension. These

equations can be studied as initial-value problems and has the form:

$$\frac{\partial u}{\partial t}(x,t) = \nu \frac{\partial^2 u}{\partial x^2}(x,t) + f(u)(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

The reaction-diffusion equation appears in problems governed by the simultaneous action of gradient diffusion and local multiplication of concentration, which may be heat, chemical species, population density, or strength of nerve signal. The forcing term f is assumed to satisfy the conditions

$$f(u) > 0 \quad \text{for } 0 < u < 1, \quad f(0) = f(1) = 0, \tag{3}$$

$$f'(u) \leq 1 \quad \text{for } 0 < u \leq 1, \quad f'(0) = 1. \tag{4}$$

The initial data is subject to the constraints

$$\lim_{x \to -\infty} u_0(x) = 1, \quad \lim_{x \to +\infty} u_0(x) = 0 \quad \text{and} \quad 0 \leq u_0(x) \leq 1. \tag{5}$$

We refer to [8, 9] for analysis of the random particle method presented below.

We choose integers $b \geq 2$, $m \geq 0$ and we put $N := b^m$. For the quasi-random walk, we need a low-discrepancy sequence $P = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\} \subset I^2$ satisfying

$$\forall n \geq 0 \quad \{p_{j,1} \ :\ nN \leq j < (n+1)N\} \text{ is a } (0, m, 1)\text{-net in base } b. \tag{6}$$

$$\forall j \geq 0 \quad p_{j,2} > 0. \tag{7}$$

The strategy is to represent the gradient u_x by weighted particles. Let H denote the Heaviside function

$$H(x) := \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{otherwise.} \end{cases}$$

We begin the method by determining a step function approximation $u^{(0)}$ to the exact initial data

$$u^{(0)}(x) = \sum_{0 \leq j < N} w_j^{(0)} H(x_j^{(0)} - x), \tag{8}$$

where $x_j^{(0)}$ represents the location and $w_j^{(0)}$ is the *mass* of the jth particle. We assume

$$\forall j \quad 0 < w_j < 1 \quad \text{and} \quad \sum_{0 \leq j < N} w_j^{(0)} = 1. \tag{9}$$

Let Δt be the time step. We put $t_n := n\Delta t$ and $u_n(x) := u(x, t_n)$. Given the computed solution

$$u^{(n)}(x) = \sum_{0 \leq j < N} w_j^{(n)} H(x_j^{(n)} - x) \tag{10}$$

at time t_n, the solution at time t_{n+1} is obtained in two distinct steps.
First we assume that the particles have been labeled so that

$$x_0^{(n)} \leq \ldots \leq x_{N-1}^{(n)}. \tag{11}$$

Step 1. The first step is the numerical solution of the reaction equation

$$\frac{\partial u}{\partial t}(x,t) = f(u)(x,t), \quad x \in \mathbb{R},\ t > t_n, \tag{12}$$

$$u(x,t_n) = u^{(n)}(x), \quad x \in \mathbb{R}. \tag{13}$$

The solution of this equation can be obtained using an explicit ordinary differential equation (ODE) solver. When Euler's method is used, the intermediate solution is

$$\tilde{u}^{(n+1)} = u^{(n)} + \Delta t f(u^{(n)}). \tag{14}$$

This is equivalent to altering the weights so that the new weights satisfy

$$\tilde{u}^{(n+1)} = \sum_{0 \le j < N} w_j^{(n+1)} H(x_j^{(n)} - x). \tag{15}$$

This gives

$$w_j^{(n+1)} = w_j^{(n)} + \Delta t \Big(f\Big(\sum_{j \le k < N} w_k^{(n)} \Big) - f\Big(\sum_{j < k < N} w_k^{(n)} \Big) \Big). \tag{16}$$

Step 2. It remains to solve the diffusion equation for the gradient

$$\frac{\partial u_x}{\partial t}(x,t) = \nu \frac{\partial^2 u_x}{\partial x^2}(x,t), \quad x \in \mathbb{R},\ t > t_n, \tag{17}$$

$$u_x(x,t_n) = \tilde{u}_x^{(n+1)}(x), \quad x \in \mathbb{R}. \tag{18}$$

Let $\lfloor r \rfloor$ denote the greatest integer $\le r$ and put

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2) dy, \quad x \in \mathbb{R}.$$

Each particle takes a step drawn quasi-randomly from a Gaussian distribution with zero mean and variance $2\nu\Delta t$.

$$x_{\lfloor Np_{j,1} \rfloor}^{(n+1)} = x_{\lfloor Np_{j,1} \rfloor}^{(n)} + \sqrt{2\nu\Delta t}\,\Phi^{-1}(p_{j,2}), \quad nN \le j < (n+1)N. \tag{19}$$

We refer to [4] for a convergence analysis of the quasi-Monte Carlo simulation of the diffusion equation.

There are several ways to obtain a method which is higher-order in time. We may consider a second-order ODE solver in place of Euler's method. Define the intermediate solution as

$$\tilde{u}^{(n+1)} = u^{(n)} + \frac{\Delta t}{2} \left(f(u^{(n)}) + f\left(u^{(n)} + \Delta t f(u^{(n)}) \right) \right). \tag{20}$$

This is Heun's method for solving (12)-(13). Then the new weights are as follows:

$$\begin{aligned} w_j^{(n+1)} \;=\; & w_j^{(n)} + \frac{\Delta t}{2} \Big(f\Big(\sum_{j \le k < N} w_k^{(n)} \Big) - f\Big(\sum_{j < k < N} w_k^{(n)} \Big) \Big) \\ & + \frac{\Delta t}{2} \Big(f\Big(\sum_{j \le k < N} \hat{w}_k^{(n+1)} \Big) - f\Big(\sum_{j < k < N} \hat{w}_k^{(n+1)} \Big) \Big), \end{aligned} \tag{21}$$

where

$$\widehat{w}_k^{(n+1)} := w_k^{(n)} + \Delta t\Big(f\Big(\sum_{k\le \ell<N} w_\ell^{(n)}\Big) - f\Big(\sum_{k<\ell<N} w_\ell^{(n)}\Big)\Big).$$

As noticed in [9], the error due to the operator splitting remains $\mathcal{O}(\Delta t)$. To increase the accuracy, we may employ the following splitting algorithm known as Strang splitting. Let $t_{n+1/2} := t_n + \Delta t/2$.

Step 1. Solve (12)-(13) using Heun's scheme and consider the intermediate solution at time $t_{n+1/2}$:

$$\begin{aligned}\widetilde{u}^{(n+1/2)} &= u^{(n)} + \frac{\Delta t}{4}\Big(f(u^{(n)}) + f(u^{(n)} + \frac{\Delta t}{2} f(u^{(n)}))\Big)\\ &= \sum_{0\le j<N} w_j^{(n+1/2)} H(x_j^{(n)} - x). \end{aligned} \tag{22}$$

Step 2. Perform a quasi-random walk

$$x_{\lfloor Np_{j,1}\rfloor}^{(n+1)} = x_{\lfloor Np_{j,1}\rfloor}^{(n)} + \sqrt{2\nu\Delta t}\,\Phi^{-1}(p_{j,2}), \quad nN \le j < (n+1)N. \tag{23}$$

Assume that the $x_j^{(n+1)}$ have been sorted by position and put

$$u^{(n+1/2)}(x) := \sum_{0\le j<N} w_j^{(n+1/2)} H(x_j^{(n+1)} - x).$$

Step 3. Use Heun's scheme to solve

$$\frac{\partial u}{\partial t}(x,t) = f(u)(x,t), \quad x \in \mathbb{R},\ t > t_{n+1/2}, \tag{24}$$

$$u(x, t_{n+1/2}) = u^{(n+1/2)}(x), \quad x \in \mathbb{R}. \tag{25}$$

This gives

$$\begin{aligned} u^{(n+1)} &= u^{(n+1/2)} + \frac{\Delta t}{4}\Big(f(u^{(n+1/2)}) + f(u^{(n+1/2)} + \frac{\Delta t}{2} f(u^{(n+1/2)}))\Big)\\ &= \sum_{0\le j<N} w_j^{(n+1)} H(x_j^{(n+1)} - x). \end{aligned} \tag{26}$$

We examine one model example to study the effectiveness of quasi-random walk (QRW) method, as discussed above, when compared with standard random walk (SRW) method using pseudo-random numbers [9]. These experiments allow us to estimate the rate of convergence of each method, at least for the model problem for which an exact answer is available. For $\nu = 1$ and a forcing term $f(u) = u(1-u)$, equation (1)-(2) becomes the Kolmogorov equation and has a moving wave solution of the form $u(x,t) = g(x - \alpha t)$ with speed $\alpha = 5/\sqrt{6}$ and wave form

$$g(x) = \frac{1}{\big(1 + (\sqrt{2}-1)\exp(x/\sqrt{6})\big)^2}. \tag{27}$$

If the simulation is advanced up to T with a timestep Δt, we define the *averaged error*

$$E_N := \frac{\Delta t}{T}\sum_{n=1}^{T/\Delta t} \|u^{(n)} - u_n\|_\infty, \tag{28}$$

where $\|v\|_\infty$ denotes the essential supremum of the function v. The QRW method uses a $(0,2)$-sequence of Faure in base 2 for random walk [2]. We compute the solution up to $T = 1$ using three different schemes and time steps for solving the reaction equation:

1. Euler's method with $\Delta t = 2^{-10}$,
2. Heun's method with $\Delta t = 2^{-9}$,
3. Heun's method with Strang splitting and $\Delta t = 2^{-7}$.

The QRW method is compared with the SRW method in Fig. 1. For all the calculations, the number of simulated particles ranges from $N = 32$ to $N = 1,048,576$, with N being chosen as powers of two. In all cases, the averaged error is computed at each N, and a line is fitted to the log-log data to estimate the convergence rate. This assumes that over this range of N, the error may be modeled as cN^{-d}. One finds

1. For Euler's method

$$D_N(\text{SRW}) \approx \frac{0.64}{N^{0.50}}, \quad D_N(\text{QRW}) \approx \frac{0.77}{N^{0.70}}. \tag{29}$$

2. For Heun's method

$$D_N(\text{SRW}) \approx \frac{0.69}{N^{0.51}}, \quad D_N(\text{QRW}) \approx \frac{0.72}{N^{0.69}}. \tag{30}$$

3. For Heun's method with Strang splitting

$$D_N(\text{SRW}) \approx \frac{0.70}{N^{0.51}}, \quad D_N(\text{QRW}) \approx \frac{0.72}{N^{0.69}}. \tag{31}$$

For each method, the quasi-random strategy produces sizable gains over pseudo-random Monte Carlo. One easily sees the trend toward steady reduction of the error as more and more particles are used. Similar results have been observed for example in [4] for simple diffusion problems.

3 The Burgers equation

In this section we present a quasi-random walk method used to solve the quasilinear diffusion equation

$$\frac{\partial u}{\partial t}(x,t) + \left(u\frac{\partial u}{\partial x}\right)(x,t) = \nu\frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{32}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{33}$$

with viscosity $\nu > 0$. The equation is attributed to Burgers and was advanced as a one-dimensional model for the Navier-Stokes equations. We assume that u_0 is constant outside a compact set

$$u_0(x) = \begin{cases} u_-, & \text{if } x < -A \\ u_+, & \text{if } x > A. \end{cases} \tag{34}$$

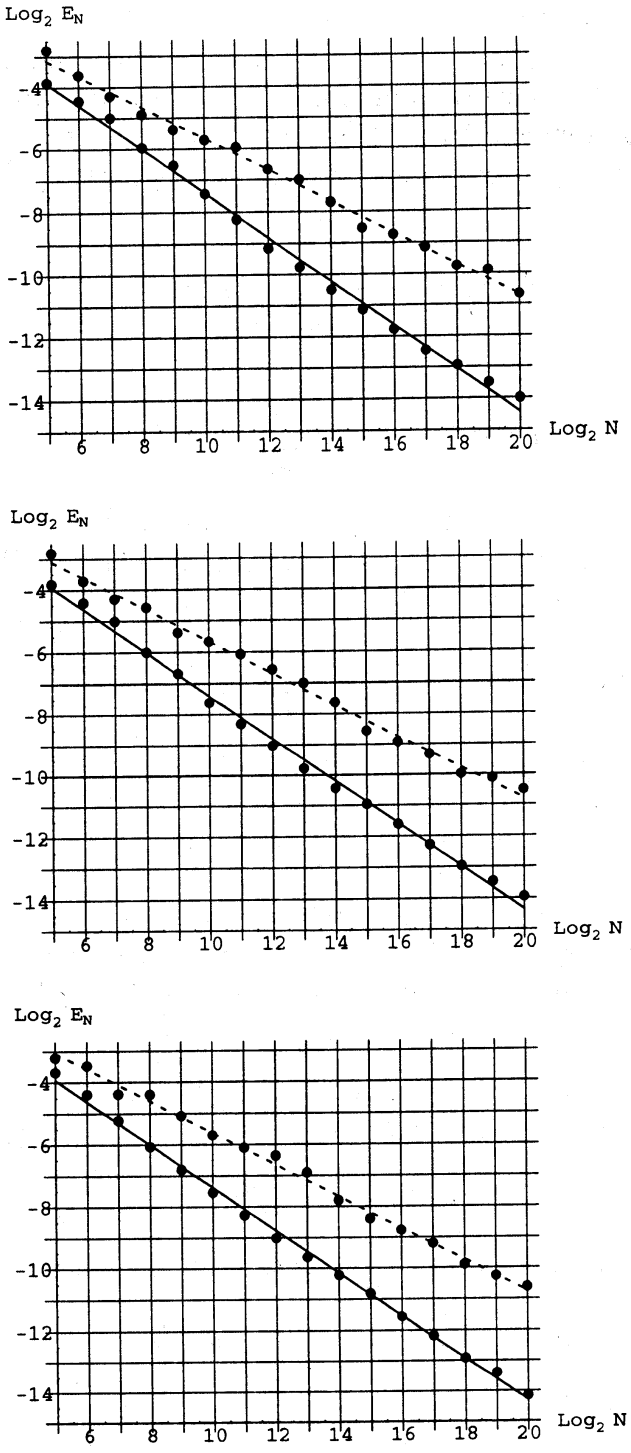

Figure 1: Kolmogorov equation: Euler's method (top), Heun's method (middle) and Heun's method with Strang splitting (bottom): SRW (*dotted line*) vs. QRW (*solid line*) averaged error.

We follow [10] for the description of the random particle method for this equation. The algorithm is based on a viscous splitting. We refer to [11] for information on the numerical solution of conservation laws.

We choose integers $b \geq 2$, $m \geq 0$, we put $N := b^m$ and we choose a spatial parameter $h > 0$. For the quasi-random walk, we need a low-discrepancy sequence $P = \{\mathbf{p}_0, \mathbf{p}_1, \ldots\} \subset I^2$ satisfying (6) and (7). We suppose that the gradient u_x of the solution is approximated by a collection of weighted particles. The initial step function approximation is given by

$$u^{(0)}(x) = u_- + h \sum_{0 \leq j < N} \varepsilon_j H(x - x_j^{(0)}), \tag{35}$$

where

$$\varepsilon_j = \pm 1 \quad \text{and} \quad \sum_{0 \leq j < N} \varepsilon_j = \frac{u_+ - u_-}{h}. \tag{36}$$

Here $x_j^{(0)}$ is the location and h is the absolute *strength* of the jth particle.

Let Δt be the time step and put $t_n := n\Delta t$, $u_n(x) := u(x, t_n)$. We denote the computed solution after n time steps as $u^{(n)}$,

$$u^{(n)}(x) = u_- + h \sum_{0 \leq j < N} \varepsilon_j H(x - x_j^{(n)}). \tag{37}$$

We assume that the particles have been labeled so that

$$j < k \;\Rightarrow\; x_j^{(n)} \leq x_k^{(n)} \quad \text{and} \quad \varepsilon_j \geq \varepsilon_k. \tag{38}$$

The approximate solution at time t_{n+1} is obtained in two steps.

Step 1. Given particles at position $x_j^{(n)}$, $0 \leq j < N$, we need to evolve the positions in such a way that the associated step function approximates the solution of the inviscid Burgers equation

$$\frac{\partial u}{\partial t}(x,t) + \left(u \frac{\partial u}{\partial x}\right)(x,t) = 0, \quad x \in \mathbb{R},\ t > t_n, \tag{39}$$

$$u(x, t_n) = u^{(n)}(x), \quad x \in \mathbb{R}. \tag{40}$$

We solve the Riemann problems associated with each jump separately. Let

$$[u^{(n)}]_j := u^{(n)}(x_j^{(n)} + 0) - u^{(n)}(x_j^{(n)} - 0)$$

be the strength of the jump at $x_j^{(n)}$.

- If $[u^{(n)}]_j \leq 0$, then we have a shock and the speed of the discontinuity is

$$v_j^{(n)} := \frac{1}{2}\left(u^{(n)}(x_j^{(n)} + 0) + u^{(n)}(x_j^{(n)} - 0)\right).$$

We define it as the velocity of the jth particle.

- If $[u^{(n)}]_j > 0$, then we have a rarefaction wave. Assume that $p+q$ particles are located at $x_j^{(n)}$:

$$\begin{aligned} x_j^{(n)} &= x_{j+1}^{(n)} = \ldots = x_{j+p-1}^{(n)}, \text{ with } \varepsilon_j = \ldots = \varepsilon_{j+p-1} = +1, \\ x_j^{(n)} &= x_{j+p}^{(n)} = \ldots = x_{j+p+q-1}^{(n)}, \text{ with } \varepsilon_{j+p} = \ldots = \varepsilon_{j+p+q-1} = -1. \end{aligned}$$

We define the velocity of the $(j+k)$th particle to be

$$v_{j+k}^{(n)} := \begin{cases} u^{(n)}(x_j^{(n)} - 0) + (k + \frac{1}{2})h, & \text{if } 0 \le k < p - q \\ u^{(n)}(x_j^{(n)} + 0), & \text{if } p - q \le k < p + q. \end{cases}$$

As time proceeds, the solutions start to interact: at this time we consider the approximating step function as new initial data. We set

$$\delta t_1^{(n)} := \min_{j,k} \left\{ \frac{x_k^{(n)} - x_j^{(n)}}{v_j^{(n)} - v_k^{(n)}} \ : \ x_j^{(n)} < x_k^{(n)} \text{ and } v_j^{(n)} > v_k^{(n)} \right\},$$

so that $t_n + \delta t_1^{(n)}$ is the first time of intersection of the trajectories of at least two particles that were at different positions at time t_n. For $t_n \le t \le t_n + \delta t_1^{(n)}$, the location of the jth particle at time t is

$$x_j^{(n)}(t) = x_j^{(n)} + (t - t_n)v_j^{(n)}. \tag{41}$$

If $\delta t_1^{(n)} \ge \Delta t$ we set

$$x_j^{(n+1/2)} := x_j^{(n)}(t_{n+1}),$$

and we go to step 2, otherwise we replace t_n by $t_n + \delta t_1^{(n)}$ and we repeat step 1 until we reach t_{n+1}. Then

$$u^{(n+1/2)}(x) := u_- + h \sum_{0 \le j < N} \varepsilon_j H(x - x_j^{(n+1/2)}).$$

Step 2. The next step involves solving the diffusion equation for u_x

$$\frac{\partial u_x}{\partial t}(x,t) = \nu \frac{\partial^2 u_x}{\partial x^2}(x,t), \quad x \in \mathbb{R}, \ t > t_n, \tag{42}$$

$$u_x(x, t_n) = u_x^{(n+1/2)}(x), \quad x \in \mathbb{R}. \tag{43}$$

This is accomplished by adding a quasi-random component to the position of the particles. We assume that the particles have been sorted in order of position:

$$x_0^{(n+1/2)} \le \ldots \le x_{N-1}^{(n+1/2)}. \tag{44}$$

Then

$$x_{[Np_{j,1}]}^{(n+1)} = x_{[Np_{j,1}]}^{(n+1/2)} + \sqrt{2\nu\Delta t}\Phi^{-1}(p_{j,2}), \quad nN \le j < (n+1)N. \tag{45}$$

In order to compare the performance of quasi-random walk (QRW) method described above and standard random walk (SRW) employing pseudo-random numbers [10], both

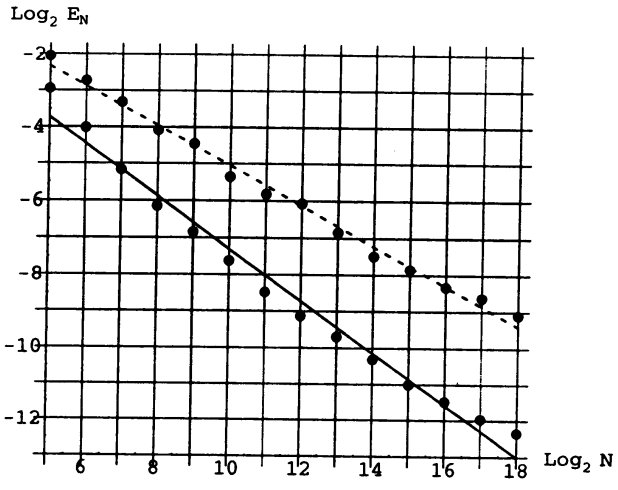

Figure 2: Burgers equation: SRW (*dotted line*) vs. QRW (*solid line*) averaged error

methods are used to solve a model problem which has an exact known solution. We can use the Cole-Hopf transformation to solve (32)-(33). If $u_0(x) = 1 - H(x)$, we obtain:

$$u(x,t) = \frac{\Phi\left((t-x)/\sqrt{2\nu t}\right)\exp((t-2x)/4\nu)}{\Phi\left((t-x)/\sqrt{2\nu t}\right)\exp((t-2x)/4\nu) + \Phi\left(x/\sqrt{2\nu t}\right)}. \tag{46}$$

We choose $\nu = 0.1$ and the solution is computed up to $T = 1$ with a time step $\Delta t = 2^{-11}$. The QRW method utilizes a $(0,2)$-sequence of Faure in base 2. For all the calculations, the number N of particles ranges from $N = 32$ to $N = 262,144$, with N being chosen as powers of two. Figure 2 shows a log-log plot of the averaged error (28). The least-squares fit convergence rates are as follows:

$$D_N(\text{SRW}) \approx \frac{1.31}{N^{0.54}}, \quad D_N(\text{QRW}) \approx \frac{0.89}{N^{0.71}}. \tag{47}$$

We see that the QRW method still clearly outperforms the SRW method for this example, although the problem being dealt with here is more complicated than simple diffusion problem.

4 Conclusion

In this paper we examine the use of quasi-random sequences of points in place of pseudo-random points in random walk simulation of diffusion. Two problems are considered: one-dimensional nonlinear reaction-diffusion equation and Burgers equation. The algorithm has been tailored to fit in a fractional step scheme. The results for the two problems reveal that quasi-Monte Carlo methods can be successfully applied to simple one-dimensional equations, producing errors which are smaller than standard random walk. A key element in successfully applying the quasi-random sequences is a technique involving renumbering the particles after each time step.

The algorithms are presently suited for solving scalar, one-dimensional equations. One desirable extension would be to systems of equations such as the Hodgkin-Huxley equations. A further possible extension is to problems in more than one space dimension, such as the Navier-Stokes equations. On the theoretical side it will be especially interesting to prove convergence of the schemes.

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