数理解析研究所講究録 1240 巻 2001 年 21-25

RECURRENCE SPEED OF TRANSFORMATIONS WITH CONTINUOUS INVARIANT MEASURES

GEON HO CHOE

1. INTRODUCTION

In the theory of dynamical systems the concept of recurrence is one of the important topics of research. Henri Poincaré observed that an orbit of a given transformation defined on a phase space returns to a neighborhood of a starting point infinitely many times with probability 1 with some suitable conditions imposed on T. This problem is related to stability of planetary systems in the universe, etc. In this paper we investigate the speed of recurrence when the error of recurrence is given. The origin of the problem was M. Boshernitzan's paper[3] and we obtain new results based on his fundamental theorems.

First, we briefly introduce terminology and notation. Let $\mathcal{A}$ be a σ-algebra on a set X. A mapping is said to be measure preserving if $\mu(T^{-1}A) = \mu(A)$ for $A \in \mathcal{A}$. By a measure preserving system $(X, \mathcal{A}, \mu, T)$ we mean a probability space $(X, \mathcal{A}, \mu)$ together with a measure preserving map $T : X \to X$. The map T is usually called a transformation and is not necessarily one-to-one. By a metric measure preserving system $(X, \mathcal{A}, \mu, d, T)$ we mean a measure preserving dynamical system $(X, \mathcal{A}, \mu, T)$ together with a metric d on X such that the σ-algebra $\mathcal{A}$ is the Borel σ-algebra generated by the metric d.

Let (X, d) be a metric space and $\mathcal{A}$ be the Borel σ-algebra. For $A \subset X$, $\varepsilon > 0$, let

$$H_{\alpha,\varepsilon}(A) = \inf \sum \operatorname{diam}(U_i)^\alpha,$$

where the infimum is taken over all countable coverings of A by subsets U_i with diameters $\operatorname{diam}(U_i) < \varepsilon$. The Hausdorff α-measure on X is defined by

$$H_\alpha(A) = \lim_{\varepsilon \downarrow 0} H_{\alpha,\varepsilon}(A) = \limsup_{\varepsilon \downarrow 0} H_{\alpha,\varepsilon}(A).$$

Then it is an outer measure. It is said to be σ-finite on A if A is a countable union of sets A_i with $H_\alpha(A_i) < \infty$. In this case, the metric space (X, d) has a countable base. There exists a unique value for α such that if $s < \alpha$ then $H_s(X) = \infty$ and if $s > \alpha$ then $H_s(X) = 0$. Such α is called the Hausdorff dimension of (X, d). For an introduction to Hausdorff measures, consult [2],[6].

Boshernitzan[3] proved the following fact.

1991 *Mathematics Subject Classification.* Primary 28D05.

Key words and phrases. Recurrence, invariant density, Hausdorff measure.

FIGURE 1. $y = R_k(x)\varepsilon_k(x)$ for $Tx = x + \theta \pmod 1$ with $\theta = \pi - 3$.

Fact 1.1. *Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure preserving system. Assume that for some $\alpha > 0$, the Hausdorff α-measure H_α agrees with the measure μ on the σ-algebra $\mathcal{A}$. Then for μ-almost all $x \in X$ we have*

$$\liminf_{n\to\infty} n^\beta \cdot d(T^n x, x) \le 1, \text{ with } \beta = \frac{1}{\alpha}.$$

For $X = [0,1]$ the Lebesgue measure μ coincides with the Hausdorff 1-measure H_1 on X. Hence Boshernitzan obtained the following corollary.

Fact 1.2. *Let $X = [0,1]$. If $T : X \to X$ is a Lebesgue measure preserving transformation, then, for a.e. x,*

$$\liminf_{n\to\infty} n \cdot |T^n x - x| \le 1.$$

The optimal value for the constant on the right hand side is not known. Probably the right hand side is bounded by a smaller constant depending on the transformation. See the simulations. The generalization of Fact 1.2 to absolutely continuous invariant measures is proved in Theorem 1.6. For the proofs and more simulations including computational techniques see [4],[5].

Definition 1.3. Let (X, d) be a metric space. The k-th first return time $R_k(x)$ is defined by

$$R_k(x) = \min\{s \ge 1 : d(T^s x, x) \le \frac{1}{2^k}\}.$$

The k-th recurrence error $\varepsilon_k(x)$ is defined by

$$\varepsilon_k(x) = d(T^{R_k(x)}x, x).$$

Example 1.4 (Translation by θ). Let $Tx = x + \theta \mod 1$ with $0 < \theta < 1$ irrational. The invariant measure is the Lebesgue measure and its entropy equals 0. Define a metric on $X = [0,1)$ by $||x - y||$ where $||t|| = \min\{|t - n| : n \in \mathbb{Z}\}$. For $0 < x < 1$ we have $\liminf_{n\to\infty} n \cdot ||T^n x - x|| = \liminf_{n\to\infty} n \cdot |T^n x - x|$. Note that R_k is constant since

$$\liminf_{n\to\infty} n \cdot |T^n x - x| = \liminf_{n\to\infty} n \cdot ||n\theta||.$$

Let p_k/q_k be the k-th convergent in the continued fraction expansion of θ. Then

$$\left|\theta - \frac{p_k}{q_k}\right| \le \frac{1}{{q_k}^2},$$

hence $q_k \cdot ||q_k\theta|| \le 1$. By choosing a subsequence of p_k/q_k, we may have a smaller upper bound. With $\theta = (\sqrt{5} - 1)/2$ we have $\liminf_{n\to\infty} n \cdot ||n\theta|| \le 1/\sqrt{5}$. For more details, consult [8]. Figure 1 displays the plot for $(k, R_k\varepsilon_k)$ for $\theta = \pi - 3$. The range of k is from

RECURRENCE SPEED

Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. The transformation T is said to be *ergodic* if $T^{-1}(A) = A$ modulo measure zero sets only if $\mu(A) = 0$ or 1. Take $A \in \mathcal{A}$ with positive measure. Define the first return time on A by

$$R_A(x) = \min\{j \geq 1 : T^j x \in A\}$$

for $x \in A$. It is not defined at x if the orbit of x does not return to A. The Poincaré Recurrence Theorem implies that it is finite for a.e. $x \in X$. The following fundamental fact was proved by M. Kac[7].

Fact 1.5 (Kac's Lemma). *Let T be an ergodic transformation on a probability space $(X, \mathcal{A}, \mu)$. If $\mu(A) > 0$, then*

$$\int_A R_A(x)\, d\mu_A = \frac{1}{\mu(A)},$$

where μ_A is the conditional measure on A defined by $\mu_A(E) = \mu(E)/\mu(A)$ for $E \subset A$.

Let $(X, \mathcal{A}, \mu, d, T)$ be a metric measure preserving system. Consider a ball A of radius $1/2^k$ centered at x_0. Then $R_A = R_k$ and in view of Kac's Lemma we expect that $R_k(x_0)$ is approximately equal to $1/\mu(A)$ in some sense. For a general introduction to ergodic theory, consult [9],[10].

For $X = [0, 1]$ a measure μ is said to *absolutely continuous* if $d\mu = \rho(x)dx$ for an integrable function $\rho(x) \geq 0$. We call $\rho(x)$ the invariant density. We generalize Fact 1.2 for transformations on $X = [0, 1]$ with absolutely continuous invariant measures. First we observe the following: Let $\rho(x) > 0$ be an integrable function on $X = [0, 1]$. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \left| \int_x^y \rho(t) dt \right|$$

for $x, y \in X$. Then (i) d is a metric on X, and (ii) μ coincides with the Hausdorff 1-measure H_1 on X.

Theorem 1.6. *Let $X = [0, 1]$ and let $\rho(x)dx$, $\rho(x) > 0$, be an absolutely continuous probability measure on X. Let $T : X \to X$ be an ergodic transformation that preserves $\rho(x)dx$. Then*

$$\liminf_{n \to \infty} n \cdot |T^n x - x| \leq \frac{1}{\rho(x)}$$

and

$$\liminf_{k \to \infty} R_k(x) \varepsilon_k(x) \leq \frac{2}{\rho(x)}.$$

FIGURE 2. $(x, R_k(x)\varepsilon_k(x))$ and $(x, \varepsilon_k(x) \cdot 2^k)$ for $k = 8$ and $Tx = 3x \pmod 1$

2. Simulations for Theorem 1.6

In investigating the behavior of $R_k(x)\varepsilon_k(x)$ as $k \to \infty$ we use the subsequence $n_1 = R_1(x)$, $n_2 = R_2(x)$, $n_3 = R_3(x)$, $\cdots$ since it is practically impossible to consider the whole sequence $1, 2, 3, \cdots$ in finding the limit infimum. Computer experiments are done using the mathematical software Maple V, which allows us to use up to 10,000 significant decimal digits. In computing the first return time it is crucial to take a sufficient number of significant digits.

Take a starting point $x_0 = \pi - 3$. It is a good choice as a starting point for an orbit of a transformation on the unit interval in general for most of simulation. Many statistical investigations have shown no regularity in the digits of π and it seems that they have prefect randomness as far as practical applications are concerned. See [1] for more on the randomness of digits of π.

Let $Tx = 3x \mod 1$. The invariant measure is the Lebesgue measure. The plot of $(x, R_k(x)\varepsilon_k(x))$ at $x = T^j x_0$, $1 \le j \le 1000$, is given on the left side in Figure 2. We take $k = 8$ and employ 300 significant decimal digits in Maple computations. Note that most of the points lie below $y = 2$. On the right $(x, \varepsilon_k(x) \cdot 2^k)$ are plotted. The factor 2^k is used for normalization. The points seem to be random.

3. Conjectures on the averages of R_k, ε_k and $R_k\varepsilon_k$

If $X = [0, 1]$ and $d\mu = \rho(x)dx$ for a piecewise continuous T-invariant density function $\rho(x) > 0$, then from extensive simulations it seems reasonable to assume that for sufficiently large k the points $T^{R_k(x)}(x)$, $0 < x < 1$, are almost uniformly distributed in the interval $[x - 1/2^k, x + 1/2^k]$ and that the average of the distance $|T^{R_k(x)}(x) - x|$ is close to $1/2^{k+1}$. In simulations the distribution of k-th recurrence error seems to be uniform with respect to the Lebesgue measure in the interval $[x - 1/2^k, x + 1/2^k]$, hence we conjecture that $\mathrm{Ave}[\varepsilon_k] \approx 1/2^{k+1}$. Since

$$\mu\left(\left[x - \frac{1}{2^k}, x + \frac{1}{2^k}\right]\right) = \int_{x-1/2^k}^{x+1/2^k} \rho(t)dt \approx \rho(x) \cdot \frac{1}{2^{k-1}},$$

we have the following conjecture for sufficiently large k:

$$R_k(x) \approx \frac{1}{\mu([x - 1/2^k, x + 1/2^k])} \approx \frac{2^{k-1}}{\rho(x)},$$

$$\mathrm{Ave}[R_k] = \int_0^1 R_k(x)\rho(x)dx \approx \int_0^1 \frac{2^{k-1}}{\rho(x)}\rho(x)dx = 2^{k-1}$$

and

$$\mathrm{Ave}[R_k \cdot \varepsilon_k] = \int_0^1 R_k(x)\varepsilon_k(x)\rho(x)dx \approx \int_0^1 \frac{2^{k-1}}{\rho(x)} \frac{1}{2^{k+1}} \rho(x)dx = \frac{1}{4},$$

where ' $\approx$' denotes 'being approximately equal to in a suitable sense' and 'Ave' denotes 'the average with respect to the invariant density $\rho(x)$'.

Note that for irrational translations modulo 1 the above conjecture is false, and we need an additional condition other than ergodicity. See Figure 1. Probably we need a condition of positive entropy or 'mixing' or both. The preceding discussion suggests the possibility of using $\{R_k\}_{k=1}^{\infty}$ to obtain information on the limit infimum in Theorem 1.6.

A probability measure μ on the unit interval is said to be singular if there exists a Lebesgue measure zero subset X_0 such that $\mu(X_0) = 1$. For example, a $(p, 1-p)$-Bernoulli measure is singular continuous if $0 < p < 1/2$. For transformation with singular continuous invariant measures, the limits on the right hand side are all zero for a.e. x with respect to the Lebesgue measure in corresponding statements for Theorem 1.6. This is due to the observation that at the points where a singular continuous measure μ is concentrated in a certain sense the density function becomes infinite.

References

[1] J. Arndt and C. Haenel, *Pi-Unleashed, 2nd ed.*, Springer-Verlag, Berlin, 2001.

[2] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York, 1965.

[3] M.D. Boshernitzan, *Quantitative recurrence results*, Invent. Math. 113 (1993), no. 3, 617–631.

[4] G.H. Choe, *Recurrence of transformations with absolutely continuous invariant measures*, to appear in Appl. Math. Comp.

[5] G.H. Choe, *Computational Ergodic Theory*, in preparation.

[6] G.A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.

[7] M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc., 53 (1947) 1002–1010.

[8] A.Ya. Khinchin, *Continued Fractions*, Univ. of Chicago Press, Chicago, 1964.

[9] K. Petersen, *Ergodic Theory* , Cambridge Univ. Press, New York, 1983.

[10] P. Walters, *An Introduction to Ergodic Theory*, 2nd ed., Springer-Verlag, New York, 1982.

Department of Mathematics, Korea Advanced Institute of Science and Technology, Daejeon.

E-mail address: `choe@euclid.kaist.ac.kr`