Statistical Properties of Real-Valued Sequences Generated by Chebyshev maps

Hiroshi Fujisaki

Graduate School of Natural Science and Technology,

Kanazawa University

fujisaki@t.kanazawa-u.ac.jp

Abstract: Recently binary or real-valued sequences generated by Chebyshev maps are proposed as spreading sequences in DS/CDMA systems. In this note, we consider sequences of real-valued functions of bounded variation, which include binary functions, of iterates generated by Chebyshev maps, and evaluate the rate of mixing of such sequences using the Perron-Frobenius operator associated with the Chebyshev maps.

1   Introduction

During the past decade, binary or real-valued sequences generated by one-dimensional ergodic maps have been proposed for spreading sequences and its statistical properties in DS/CDMA system have been studied. In such ergodic maps, especially Chebyshev maps have been intensively studied in the following.

Binary pseudo-random sequences generated by Chebyshev maps, or briefly referred to as Chebyshev binary sequences, are firstly proposed in [1], and a sufficient condition is derived for a kind of ergordic maps to generate sequences of i.i.d. (independent and identically distributed) binary random variables in [2].

Based on the evaluations of correlation functions of so called Chebyshev binary sequences in [3], correlational properties of such sequences in DS/CDMA system are particularly examined, and the ones with exponentially decaying auto-correlation are recently proposed in [4].

On the other hand, some real-valued sequences generated by Chebyshev maps are recently proposed in [5] as optimum spreading sequences in terms of the AIP (average interference parameter) defined in [6]. Because these sequences are based on a linear combination of Chebyshev polynomials, correlational properties of such sequences can be evaluated by using the N-th order dependency moments $(N = 1, 2, 3, \cdots)$ of real-valued trajectory generated by Chebyshev maps, which are already derived in [7].

In this note, we consider sequences of real-valued functions of bounded variation of iterates generated by Chebyshev maps. First we define the modified Perron-Frobenius operator associated with the Chebyshev maps, and examine the spectrum of this operator. Then we evaluate the mixing property of the Chebyshev maps using this operator, and explicitly obtain the rate of mixing.

2 Perron-Frobenius Operator Associated with Chebyshev Map

Consider the Chebyshev map of degree k $(k = 2, 3, \cdots)$,

$$T_k(\omega) = \cos(k \cos^{-1} \omega), \quad \omega \in [-1, 1]. \tag{1}$$

Let $d\omega$ be the Lebesgue measure on $J = [-1, 1]$.

Definition 1. (Perron-Frobenius operator) The Perron-Frobenius operator associated with the Chebyshev map of degree k, denoted by P_{T_k} is defined by the formula

$$\int_J F(\omega) G(T_k(\omega)) d\omega = \int_J P_{T_k}\{F(\omega)\} G(\omega) d\omega \tag{2}$$

for $F \in L^1(J, d\omega)$ and G is in all bounded almost everywhere measurable functions $L^\infty(J, d\omega)$.

The inverse of the map T_k consists of the following k maps $g_i : [-1, 1] \rightarrow \left[\cos \dfrac{(i+1)\pi}{k}, \cos \dfrac{i\pi}{k}\right]$ $(i = 0, 1, 2, \cdots, k - 1)$ as expressed in [8]:

$$g_i(\omega) = \cos\left(\frac{i\pi + \cos^{-1}\{(-1)^i \omega\}}{k}\right),$$
$$i = 0, 1, \cdots, k - 1. \tag{3}$$

Thus P_{T_k} can be expressed in the form:

$$P_{T_k} H(\omega) = \sum_{i=0}^{k-1} |g_i'(\omega)| H(g_i(\omega)) \tag{4}$$

for $H \in L^1(J, d\omega)$.

Note that a nontrivial nonnegative solution of the equation $P_{T_k} H = H$ gives the density function of an absolutely continuous invariant measure for T_k. The density function for T_k, denoted by $f^*(\omega)$, is known to be

$$f^*(\omega) = \frac{1}{\pi\sqrt{1 - \omega^2}}. \tag{5}$$

Let p be a prime number. We have

Lemma 1. [7] For a Chebyshev polynomial of degree n $(n = 1, 2, \cdots)$, the Perron-Frobenius operator associated with the Chebyshev map of degree p satisfies

$$P_{T_p}\{T_n(\omega) f^*(\omega)\} = \begin{cases} T_{\frac{n}{p}}(\omega) f^*(\omega) & \text{for } p \mid n \\ 0 & \text{for } p \nmid n. \end{cases} \tag{6}$$

3 Modified Perron-Frobenius Operator Associated wi Chebyshev Map

To evaluate the mixing property of the Chebyshev maps with respect to the measure $f^*(\omega)d\omega$, we introduce

Definition 2. (Modified Perron-Frobenius operator) The modified Perron-Frobenius operator associated with the Chebyshev map of degree k $(k = 2, 3, \cdots)$, denoted by $\hat{P}_{T_k}$ is defined by the formula

$$\int_J F(\omega)G(T_k(\omega))f^*(\omega)d\omega$$
$$= \int_J \hat{P}_{T_k}\{F(\omega)\}\, G(\omega)f^*(\omega)d\omega \tag{7}$$

for $F \in L^1(J, d\omega)$ and $G \in L^\infty(J, d\omega)$.

We can express $\hat{P}_{T_k}$ in the form:

$$\hat{P}_{T_k}H(\omega) = \frac{1}{k}\sum_{i=0}^{k-1} H(g_i(\omega)). \tag{8}$$

Note that

$$P_{T_k}f^*(\omega) = f^*(\omega) \tag{9}$$

becomes

$$\hat{P}_{T_k}1 = 1. \tag{10}$$

Let p be a prime number. From lemma 1 and definition 2, we immediately have

Lemma 2. For a given Chebyshev polynomial of degree n $(n = 1, 2, \cdots)$, the modified P-F operator associated with the Chebyshev map of degree p satisfies

$$\hat{P}_{T_p}T_n(\omega) = \begin{cases} T_{\frac{n}{p}}(\omega) & \text{for } p \mid n \\ 0 & \text{for } p \nmid n. \end{cases} \tag{11}$$

The following theorem is already derived in [9].

Theorem 1. Assume that τ is a piecewise C^1-map of an interval onto itself which is not monotone. Let $\lambda \in C$ and $|\lambda| < 1$. Then λ is an eigenvalue of the Perron-Frobenius operator associated with τ with infinite multiplicity.

Now let us consider the space $L^2(J, f^*(\omega)d\omega)$. It follows from lemma 2 that the $L^2(J, f^*(\omega)d\omega)$ is an invariant subspace of $\widehat{P}_{T_p}$. We will restrict $\widehat{P}_{T_p}$ to the subspace $L^2(J, f^*(\omega)d\omega)$. and we denote it by $\widehat{P}_{T_p}|_{L^2}$. We have

$$\widehat{P}_{T_p}|_{L^2}1 = 1, \tag{12}$$

which implies 1 is an eigenvalue of $\widehat{P}_{T_p}|_{L^2}$. Let us consider the eigenvalues of $\widehat{P}_{T_p}|_{L^2}$ except 1. Then we have

Theorem 2. Let λ $(\in C)$ be an eigenvalue of $\widehat{P}_{T_p}|_{L^2}$. Assume $\lambda \neq 1$, then $|\lambda| < 1$, and it has infinite multiplicity.

Proof. See Appendix A.

Next, let us consider the space $BV = BV(J)$ of all functions with bounded variation on J. It follows from (3) and (8) that the BV is an invariant subspace of $\widehat{P}_{T_p}$. We will restrict $\widehat{P}_{T_p}$ to the subspace BV, and we denote it by $\widehat{P}_{T_p}|_{BV}$. We have

$$\widehat{P}_{T_p}|_{BV}1 = 1, \tag{13}$$

which implies 1 is an eigenvalue of $\widehat{P}_{T_p}|_{BV}$. Let us consider the eigenvalues of $\widehat{P}_{T_p}|_{BV}$ except 1. Then we have

Theorem 3. Let λ $(\in C)$ be an eigenvalue of $\widehat{P}_{T_p}|_{BV}$. Assume $\lambda \neq 1$, then $|\lambda| \leq \dfrac{1}{p}$.

Proof. See Appendix B.

4 Mixing Property of Chebyshev Maps

The following theorem is already derived in [10].

Theorem 4. The Chebyshev maps T_k are mixing with respect to the measure $f^*(\omega)d\omega$, that is

$$\lim_{n \to \infty} \int_J F(\omega)G(T_k^n(\omega))f^*(\omega)d\omega$$
$$= \int_J F(\omega)f^*(\omega)d\omega \cdot \int_J G(\omega)f^*(\omega)d\omega \tag{14}$$

for $F, G \in L^2(J, f^*(\omega)d\omega)$.

This theorem, however, is not in practical use because there exists some function F which results in (14) converging not uniformly. So we restrict the condition of the function

F, then we get

Theorem 5. Let p be a prime number. Let $F \in BV(J)$, then we have

$$\left| \int_J F(\omega)G(T_p^n(\omega))f^*(\omega)d\omega \right.$$

$$- \int_J F(\omega)f^*(\omega)d\omega \cdot \int_J G(\omega)f^*(\omega)d\omega \right|$$

$$\leq \frac{1}{p^n}\frac{1}{\sqrt{3}}V_F\|G\| \tag{15}$$

for $G \in L^2(J, f^*(\omega)d\omega)$.

Proof. See Appendix C.

5   Concluding Remarks

In this note, we consider sequences of real-valued functions of bounded variation of iterates generated by Chebyshev maps, and evaluate explicitly the rate of mixing of such sequences by defining the modified Perron-Frobenius operator associated with the Chebyshev maps.

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A   Proof of Theorem 2

Let G ($\in L^2(J, f^*(\omega)d\omega)$) be an eigenfunction of $\hat{P}_{T_p}$ corresponding to the eigenvalue λ. The Chebyshev expansion of G has the form

$$G(\omega) = g_0 + \sqrt{2} \sum_{n=1}^{\infty} g_n T_n(\omega) \quad \text{a.e.} \quad \omega \in J, \tag{16}$$

where

$$g_0 = \int_{-1}^{1} G(\omega) f^*(\omega) d\omega, \tag{17}$$

and

$$g_n = \sqrt{2} \int_{-1}^{1} T_n(\omega) G(\omega) f^*(\omega) d\omega,$$
$$n = 1, 2, \cdots. \tag{18}$$

We can express G as follows

$$G(\omega) = g_0 \; + \; \sqrt{2} \sum_{n=0}^{\infty} g_{p^n} T_{p^n}(\omega)$$
$$+ \; \sqrt{2} \sum_{n=0}^{\infty} g_{2p^n} T_{2p^n}(\omega) + \cdots$$
$$+ \; \sqrt{2} \sum_{n=0}^{\infty} g_{(p-1)p^n} T_{(p-1)p^n}(\omega)$$

$$+ \quad \sqrt{2} \sum_{n=0}^{\infty} g_{(p+1)p^n} T_{(p+1)p^n}(\omega) + \cdots$$

$$= g_0 \quad + \quad \sqrt{2} \sum_{\substack{m=1 \\ p \nmid m}}^{\infty} \sum_{n=0}^{\infty} g_{mp^n} T_{mp^n}(\omega)$$

$$\text{a.e.} \quad \omega \in J. \tag{19}$$

From the equation

$$\widehat{P}_{T_p} G(\omega) = \lambda G(\omega), \quad \lambda \neq 0 \tag{20}$$

and (19), we have

$$g_0 \quad = \quad 0, \tag{21}$$

$$g_{mp^{n+1}} \quad = \quad \lambda g_{mp^n} \quad \text{for} \quad m \neq 0, \ p \nmid m. \tag{22}$$

Hence we have

$$g_{mp^n} = \lambda^n g_m \quad \text{for} \quad m \neq 0, \ p \nmid m. \tag{23}$$

Thus the eigenfunction of $\widehat{P}_{T_p}$ corresponding to the eigenvalue λ can be written in the form of

$$G(\omega) = \sqrt{2} \sum_{\substack{m=1 \\ p \nmid m}}^{\infty} \sum_{n=0}^{\infty} \lambda^n g_m T_{mp^n}(\omega). \tag{24}$$

Since $G \in L^2$, then

$$\lim_{n \to \infty} g_{mp^n} = 0, \tag{25}$$

and hence $|\lambda| < 1$.

Since the following functions

$$\sum_{n=0}^{\infty} \lambda^n T_{mp^n}(\omega), \quad m = 1, 2, \cdots, \ p \nmid m \tag{26}$$

belongs to the kernel $\left\{ H; (\widehat{P}_{T_p} - \lambda I)H = 0 \right\}$, and thus λ has infinite multiplicity.

B Proof of Theorem 3

We have

Theorem 6. Let $F(\omega)$ be a function of bounded variation over $-1 \leq \omega \leq 1$, and let f_m be the Chebyshev expansion coefficient of F, then

$$f_m \leq \frac{\sqrt{2}}{m\pi} V_F \quad \text{for} \quad m \neq 0, \tag{27}$$

where V_F is the total variation of F over $[-1, 1]$.

Proof. See [11] for example.

Let λ be an eigenvalue of $\widehat{P}_{T_p}|_{BV}$, and suppose $|\lambda| > \dfrac{1}{p}$.

Let G $(\in BV(J))$ be the eigenfunction corresponding to λ $(\neq 0)$. Let g_m $(m \neq 0,$ $p \nmid m)$ be the Chebyshev expansion coefficient of G. Then, from (23), we have

$$p^n |g_{mp^n}| = p^n |\lambda|^n |g_m|. \tag{28}$$

Since $p|\lambda| > 1$, for the right hand side of (28), we have

$$(p|\lambda|)^n |g_m| \to \infty \tag{29}$$

as $n \to \infty$.

On the contrary, from theorem 6, for the left hand side of (28), we have

$$p^n |g_{mp^n}| \leq \frac{\sqrt{2}}{m\pi} V_G \tag{30}$$

for any n $(n = 1, 2, \cdots)$, which contradicts (29) and hence completes the proof.

C Proof of Theorem 5

From lemma 2, for $G \in L^2(J, f^*(\omega)d\omega)$, we have

$$\int_J F(\omega) G(T_p^n(\omega)) f^*(\omega) d\omega$$
$$= \int_J \widehat{P}_{T_p}^n \{F(\omega)\} G(\omega) f^*(\omega) d\omega. \tag{31}$$

Thus we get

$$\int_J F(\omega) G(T_p^n(\omega)) f^*(\omega) d\omega$$
$$= f_0 g_0 + \sum_{m=1}^{\infty} f_{p^n m} g_m, \tag{32}$$

where f_m and g_m $(m = 0, 1, 2, \cdots)$ are the Chebyshev expansion coefficients of F and G respectively.

The use of Schwartz's inequality and theorem 6 give

$$\left| \int_J F(\omega) G(T_p^n(\omega)) f^*(\omega) d\omega \right.$$
$$\left. - \int_J F(\omega) f^*(\omega) d\omega \cdot \int_J G(\omega) f^*(\omega) d\omega \right|$$
$$\leq \left(\sum_{m=1}^{\infty} |f_{p^n m}|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} |g_m|^2 \right)^{\frac{1}{2}}$$
$$\leq \frac{1}{p^n} \frac{\sqrt{2}}{\pi} V_F \|G\| \left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right)^{\frac{1}{2}} \tag{33}$$

and the conclusion of theorem 5 follows.