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Optimizing the largest eigenvalue of positive matrices having their rows belonging to polytopes

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1 Introduction

Problem of minimizing the maximum real part of the eigenvalue of a matrix arises in a variety of real world, for example, stability analysis of control systems[11], economic structure models[6] and so on[10, 12].

Previous studies discuss an eigenvalue optimization problem whose optimal value is the largest real eigenvalue, that is , the maximum real part of the eigenvalue is attained by a real eigenvalue not a complex one. Hence, the eigenvalue optimization problems impose proper restrictions on a matrix, for example, symmetry and irreducibility. In other words, eigenvalue optimization problem is to find a matrix among a class of symmetric matrices or a class of positive matrices under an additional real constraints.

This study considers a class of positive matrices whose row vectors are defined by several linear inequalities. The main aim of this study is to develop an algorithm for minimizing/maximizing the largest eigenvalue of a positive matrix whose row vector belongs to a each polytope.

We omit the almost all proofs of lemmas and theorems in the sequel. The details can be found in [5]

2 Frobenius's Theorems and multifractional problems

For every positive matrix X, the largest eigenvalue of X is denoted by $\Lambda(X)$. Let $I = \{1, 2, \ldots, n\}$. The i^{th} row vector of X of order n is denoted by x_i for all $i \in I$. We consider a perturbation of the i^{th} row vector x_i of X as $x_iC^i \leq b^i$ for all $i \in I$, where C^i is some matrix and b^i is some row vector. The polyhedron $\{x|xC^i \leq b^i\}$ is denoted by S^i for all $i \in I$. We put the following assumption in order to keep the matrix X consisting of rows $\{x_1, \ldots, x_n\}$ positive.

Assumption 1 *Assume that the polyhedron S^i is a nonempty bounded set in the positive orthant of R^n for all $i \in I$.*

Since S^i is a polytope for all $i \in I$, the product set $\Pi_{i\in I}S^i$ is a polytope. For every positive matrix X, the largest eigenvalue of X is denoted by $\Lambda(X)$. From Frobenius' theorem [6], $\Lambda(X)$ is a real function on $\Pi_{i\in I}S^i$. By $X \in \Pi S^i$, $x_i \in S^i$ for all $i \in I$. We consider a pair of the following two problems:

$$\min_{X\in\Pi S^i} \Lambda(X) \tag{1}$$

$$\text{and} \quad \max_{X\in\Pi S^i} \Lambda(X) \tag{2}$$

as the problem of finding the bounds for the largest eigenvalue of X with perturbed rows. Then we have the existence of optimal solutions of Problem (1) and Problem (2) as follows:

Theorem 2.1 *Each of Problem (1) and Problem (2) has a real optimal solution.*

From Frobenius' theorem [6], we have

$$\Lambda(X) = \max_{w>0} \min\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\} = \min_{w>0} \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\}$$

for all $X \in \Pi_{i \in I} S^i$. Therefore, we can transform Problem (1) and Problem (2) into the following two multifractional problems:

$$\begin{array}{ll} \min & \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{x_n}\right\} \\ \text{s.t.} & w > 0 \text{ and } X \in \Pi S^i \end{array} \tag{3}$$

and

$$\begin{array}{ll} \max & \min\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{x_n}\right\} \\ \text{s.t.} & w > 0 \text{ and } X \in \Pi S^i \end{array} \tag{4}$$

respectively. Both Problem (3) and Problem (4) have n homogeneous ratios of a single variable w_i to a bilinear term $x_i w$.

Let V^i be the vertex set of S^i for all $i \in I$, then the polytope S^i is the convex hull of V^i. We consider two following problem for any positive vector w:

$$\min_{X \in \Pi S^i} \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\} \tag{5}$$

$$\text{and} \quad \max\left\{\min_{x_1 \in V^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in V^n} \frac{x_n w}{w_n}\right\} \tag{6}$$

Lemma 2.1 *For any positive vector w, the optimal value of Problem* (5) *is equal to that of the following problem:*

$$\max\left\{\min_{x_1 \in S^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in S^n} \frac{x_n w}{w_n}\right\}$$

Proof.

Choose $w > 0$ arbitrarily. Assume that

$$\min_{X \in \Pi S^i} \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\} \neq \max\left\{\min_{x_1 \in S^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in S^n} \frac{x_n w}{w_n}\right\}.$$

Let $\tilde{x}_i$ be an optimal solution of $\min_{X \in \Pi S^i}(x_i w)/w_i$ for all $i \in I$, then it follows from $\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{bmatrix} \in \Pi S^i$ that

$$\begin{aligned} \min_{X \in \Pi S^i} \max\left\{\frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n}\right\} &\leq \max\left\{\frac{\tilde{x}_1 w}{w_1}, \ldots, \frac{\tilde{x}_n w}{w_n}\right\} \\ &= \max\left\{\min_{x_1 \in S^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in S^n} \frac{x_n w}{w_n}\right\} \end{aligned}$$

and hence,

$$\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\}<\max\left\{\min_{x_1\in S^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in S^n}\frac{x_nw}{w_n}\right\} \tag{7}$$

Let $\hat{X}$ be an optimal solution of $\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\}$, then,

$$\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\}\geq\frac{\hat{x}_iw}{w_i} \tag{8}$$

for all $i\in I$. Since $\hat{x}_i\in S^i$ for all $i\in I$, it follows from (8) that

$$\begin{aligned}\max\left\{\min_{x_1\in S^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in S^n}\frac{x_nw}{w_n}\right\} &\leq \max\left\{\frac{\hat{x}_1w}{w_1},\ldots,\frac{\hat{x}_nw}{w_n}\right\}\\ &\leq \min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\},\end{aligned}$$

and hence

$$\max\left\{\min_{x_1\in S^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in S^n}\frac{x_nw}{w_n}\right\}<\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\}. \tag{9}$$

From (7) and (9), it is a contradiction. □

Then we have the following theorem:

Theorem 2.2 *The optimal value of Problem (1) is equal to that of the following problem:*

$$\min_{w>0}\max\left\{\min_{x_1\in V^1}\frac{x_1w}{w_1},\ldots,\min_{x_1\in V^n}\frac{x_nw}{w_n}\right\}. \tag{10}$$

Proof.

It follows from Lemma 2.1 and $V^i\subseteq S^i$ that for problem (6)and (5)

$$\begin{aligned}\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\} &= \max\left\{\min_{x_1\in S^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in S^n}\frac{x_nw}{w_n}\right\}\\ &\leq \max\left\{\min_{x_1\in V^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in V^n}\frac{x_nw}{w_n}\right\}.\end{aligned}$$

Since the i^{th} term $\min_{x_i\in V^i}(x_iw)/w_i$ of Problem (6) is a linear programming problem whose feasible region is the polytope S^i, it has an optimal solution which belongs to the vertex set V^i. Therefore, we have

$$\min_{x_i\in V^i}\frac{x_iw}{w_i}=\min_{x_i\in S^i}\frac{x_iw}{w_i} \tag{11}$$

for all $i\in\{1,\ldots,n\}$. This implies that

$$\min_{X\in\Pi S^i}\Lambda(X)=\min_{w>0}\min_{X\in\Pi S^i}\max\left\{\frac{x_1w}{w_1},\ldots,\frac{x_nw}{w_n}\right\}=\min_{w>0}\max\left\{\min_{x_1\in V^1}\frac{x_1w}{w_1},\ldots,\min_{x_n\in V^n}\frac{x_nw}{w_n}\right\}.$$

□

Theorem 2.3 *The optimal value of Problem (2) is equal to that of the following problem:*

$$\max_{w>0}\min\left\{\min_{x_1\in V^1}\frac{x_1w}{w_1},\ldots,\min_{x_1\in V^n}\frac{x_nw}{w_n}\right\}. \tag{12}$$

3 Coloring matrix and algorithms

Firstly, we will define "coloring matrix" for matrix of order n which is similar to a basis matrix of a linear programming as follows: A matrix A of order n is called a coloring matrix if the i^{th} row vector a_i of A is a vertex of S^i for all $i \in I$, i.e., $a_i \in V^i$ for all $i \in I$. By solving Problem (10) and Problem (12), we will find an optimal solution of Problem (1) and Problem (2), respectively. The following lemma states that a coloring matrix attaining the optimal value of Problem (10) is also an optimal solution of Problem (1).

Lemma 3.1 *There is a coloring matrix whose largest eigenvalue is the optimal solution of Problem* (1). *Let $(\bar{w}, \bar{X})$ be an optimal solution of Problem* (10). *Then $\bar{w}$ is a positive largest eigenvector of $\bar{X}$.*

If a coloring matrix has the largest eigenvalue that is equal to the optimal value of Problem (1), it is called the optimal coloring matrix for Problem (1). We have the following property for an optimal solution of Problem (10).

Theorem 3.1 *Let $\bar{X}$ be an optimal coloring matrix for Problem* (1). *Then, $(\bar{w}, \bar{X})$ is an optimal solution of Problem* (10) *if and only if $\bar{w}$ is a positive largest eigenvector of $\bar{X}$.*

From Lemma (3.1) and Theorem (3.1) we have only to find the least largest eigenvalue of a coloring matrix among those of all the coloring matrices. We develop the following algorithm:

Algorithm for Problem (1)

Step0 Choose a positive vector w^0 and set $k = 1$.

Step1 Find an optimal solution $x_i^k \in V^i$ and the optimal value γ_i^k of $\min_{x_i \in S^i}(x_i w^{k-1})/w_i^{k-1}$ for every $i \in I$. Let $X^k = [x_1^{kT}, \ldots, x_n^{kT}]^T$.

Step2 Find the largest eigenvalue λ and a largest eigenvector $\hat{w}$ of X^k

Step3 If $\lambda \geq \max_{i \in I} \gamma_i^k$, then X^k and λ are an optimal coloring matrix and the optimal value of Problem (1), respectively and stop. Otherwise let $\lambda_k = \lambda, w^k = \hat{w}$ and $k = k + 1$ and go to Step 1.

The algorithm has the following properties:

Lemma 3.2 $\lambda_{k+1} < \lambda_k$ *for* $k = 1, 2, \ldots$.

Lemma 3.3 $\min_{i \in I} \gamma_i^k \leq \lambda \leq \max_{i \in I} \gamma_i^k$ *for* $k = 1, 2, \ldots$.

Lemma 3.4 *Suppose that $\lambda \geq \max_{i \in I} \gamma_i^k$ in Step3, then $\min_{i \in I} \gamma_i^k = \lambda = \max_{i \in I} \gamma_i^k$ and λ is the least largest eigenvalue among those of all the coloring matrices.*

Theorem 3.2 *The coloring algorithm for Problem (1) provides its optimal solution after a finite number of iterations.*

Replacing min / max with max / min in the definition of γ_i^k of Step 1 and the stopping criteria of Step 3, we obtain the same algorithm for Problem (2) as the above one and we can show the similar properties to the above lemmas and theorem.

Algorithm for Problem (2)

Step0 Choose a positive vector w^0 and set $k = 1$.

Step1 Find an optimal solution $x_i^k \in V^i$ and the optimal value γ_i^k of $\max_{x_i \in S^i}(x_i w^{k-1})/w_i^{k-1}$ for every $i \in I$. Let $X^k = [x_1^{kT}, \ldots, x_n^{kT}]^T$.

Step2 Find the largest eigenvalue λ and a largest eigenvector $\hat{w}$ of X^k

Step3 If $\lambda \leq \min_{i \in I} \gamma_i^k$, then X^k and λ are an optimal coloring matrix and the optimal value of Problem (2), respectively and stop. Otherwise let $\lambda_k = \lambda, w^k = \hat{w}$ and $k = k + 1$ and go to Step1.

The iterates $\{\lambda_k\}$ of the proposed algorithm for Problem (2) have the similar properties to those in Lemma 3.2 and Lemma 3.3.

The algorithm has the following properties:

Lemma 3.5 $\lambda_{k+1} > \lambda_k$ *for* $k = 1, 2, \ldots$.

Lemma 3.6 $\min_{i \in I} \gamma_i^k \leq \lambda \leq \max_{i \in I} \gamma_i^k$ *for* $k = 1, 2, \ldots$.

Lemma 3.7 *Suppose that* $\lambda \leq \max_{i \in I} \gamma_i^k$, *then* $\min_{i \in I} \gamma_i^k = \lambda = \max_{i \in I} \gamma_i^k$. *That is,* λ *is the largest largest eigenvalue among those of all the coloring matrices.*

Theorem 3.3 *The proposed algorithm for Problem (2) provides its optimal solution after a finite number of iterations.*

4 Duality of the largest eigenvalue estimation problems

We define

$$\Phi(w) = \min_{X \in \Pi S^i} \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n} \right\} \text{ and } \Psi(w) = \max_{X \in \Pi S^i} \left\{ \frac{x_1 w}{w_1}, \ldots, \frac{x_n w}{w_n} \right\}$$

and consider the following two multifractional problem:

$$\max_{w > 0} \quad \Phi(w) \text{ and} \tag{13}$$

$$\min_{w > 0} \quad \Psi(w) \tag{14}$$

The following theorem states the weak duality between Problem. (1) and Problem (13)

Theorem 4.1 *For every positive vector* w *and for every matrix* $X \in \Pi S^i$, *we have* $\Phi(w) \leq \Lambda(X)$. *Furthermore, there are a matrix* $\bar{X} \in \Pi S^i$ *and a positive vector* $\bar{w}$ *such that* $\Phi(\bar{w}) = \Lambda(\bar{X})$.

We call Problem (13) a dual problem of Problem (1)

Corollary 1 *An optimal solution of Problem* (13) *is unique up to scalar multiplication.*

The following theorem provides the same results between Problem (2) and Problem (14) as that between (1) and Problem (13).

Theorem 4.2 *For every positive vector* w *and for every matrix* $X \in \Pi S^i$, *we have* $\Psi(w) \geq \Lambda(X)$. *Furthermore, there are a matrix* $\bar{X} \in \Pi S^i$ *and a positive vector* $\bar{w}$ *such that* $\Psi(\bar{w}) = \Lambda(\bar{X})$

Corollary 2 *An optional solution of Problem* (14) *is unique up to scalar multiplication.*

5 Numerical experiments

We now give a summary of our computational experiments with the proposed algorithm.

Let P^i be a given set of finite n-dimensional row vectors for all $i \in I$. In this section we consider that S^i is the convex hull of the given set of P^i for all $i \in I$. Then we can simplify the proposed algorithm for Problem (1) as follows: Replacing $\min_{x_i \in S^i}(x_i w^{k-1})/w_i^{k-1}$ in Step 1 of the proposed algorithm for Problem (1) with $\min_{x_i \in P^i}(x_i w^{k-1})/w_i^{k-1}$, we have the proposed algorithm for Problem (1). All steps of the proposed algorithm and we have set tolerance of a stopping criteria in the power method as 1.0×10^{-7}.

Since S^i is the convex hull of P^i for all $i \in I$, an equivalent formulation of $\max_{w>0} \Phi(w)$ is as follows:

$$\max_{w>0} \min \left\{ \min_{x_1 \in P^1} \frac{x_1 w}{w_1}, \ldots, \min_{x_n \in P^n} \frac{x_n w}{w_n} \right\}. \tag{15}$$

Problem (15) is a linear fractional problem with $\sum_{i \in I} |P^i|$ ratios that is discussed von Neumann [7] and Crouzeix, Ferland and Schaible [8]. We can apply some algorithms of the linear fractional programming to Problem (15) which is the dual problem of Problem (1).

Let e be a vector of all ones. From Corollary 2 we can add a constraint $e^T w = 1$ to Problem (15) without loss of generality. Notice that Problem (15) with the additional constraint $e^T w = 1$ has a unique optimal solution. Hence, we can solve the dual problem (15) of Problem (1) by a Dinkelbach-type algorithm due to Borde and Crouzeix [4] that converges super-linearly for Problem (15) with the additional constraint $e^T w = 1$. The Dinkelbach-type algorithm is as follows:

Step0 Choose a positive vector w^0 and set

$$\eta_1 = \min \left\{ \min_{x_1 \in P^1} \frac{x_1 w^0}{w_1^0}, \ldots, \min_{x_n \in P^n} \frac{x_n w^0}{w_n^0} \right\}$$

and set $k = 1$.

Step1 Solve

$$\max_{\substack{w > 0 \\ e^T w = 1}} \quad \min \left\{ \min_{x_1 \in P^1} \frac{x_1 w - \eta_k w_1}{w_1^{k-1}}, \ldots, \min_{x_n \in P^n} \frac{x_n w - \eta_k w_n}{w_n^{k-1}} \right\}. \tag{16}$$

and let w^k and ξ_k be an optimal solution and an optimal value, respectively.

Step2 If $\xi_k = 0$, then stop: η_k and w^k is an optimal value and solution of Problem (15) , respectively. Otherwise let

$$\eta_{k+1} = \min \left\{ \min_{x_1 \in P^1} \frac{x_1 w^k}{w_1^k}, \ldots, \min_{x_n \in P^n} \frac{x_n w^k}{w_n^k} \right\}$$

$k = k + 1$ and go to Step 1.

In Step 1 of the Dinkelbach-type algorithm we solve the linear programming problem (16) with the coefficient matrix multiplied by the iterates η_k. This means that we must pay sufficient attention to the numerical error in pivoting process and evaluating the reduced cost. All steps of the algorithm except for Step 1 has been implemented in C and the linear programming of Step 1 is solved by the commercial source code Xpress-MP in order to avoid the numerical errors. Tolerance of a stopping criterion of both Xpress-MP and Step 2 are set as 5.0×10^{-7}.

Two programs were run on Sun Ultra-1 with double precision arithmetic. The study conducts a $3 \times 3 \times 3$ factorial experiment in which each treatment has 10 replications. Factors used in this numerical experiment are summarized in the following manner:

Type of $\Pi_{i\in I}P^i$

Type1 The first type of $\Pi_{i\in I}P^i$ is generated as follows: For all $i \in I$, choose m points u_i^j (j=1, ... ,m) randomly from $\{x|x \in R^n, 0 < x \leq e\}$ and set $P^i = \{u_i^{jT}/||u_i^j||_\infty | j = 1,\ldots,m\}$. Note that all row vectors of P^i are positive and that at least one element of each row vector is 1. The coloring matrix of this product set $\Pi_{i\in I}P^i$ is modeled after the cross efficiency matrix of DEA [9].

Type2 The second type of $\Pi_{i\in I}P^i$ is generated as follows: First, we have generated m reciprocal matrices $X^k(k = 1,\ldots,m)$ such that for all $i < j$ the element x_{ij}^k of X^k are chosen uniformly at random from $\{1/7, 1/5, 1/3, 1, 3, 5, 7\}$, and $x_{ij}^k = 1/x_{ij}^k$ and $x_{ii}^k = 1$ for all $i \in I$. Next, P^i is given as a set of the i^{th} row vector of X^k for $k = 1,\ldots,m$. The coloring matrix of this product set $\Pi_{i\in I}P^i$ is similar to the pairwise comparison matrix of AHP[10].

Type3 the third type of $\Pi_{i\in I}P^i$ is generated as follows: First, we have generated m stochastic matrices $X^k(k = 1,\ldots,m)$ such that each column is randomly chosen from $\{x|e^Tx = 1, x_i > 0 \text{ for all } i \neq k, \text{and } x_k = 0\}$. Next, P^i is given as the set of the i^{th} row vector of X^k for $k = 1,\ldots,m$. For all coloring matrices,all the diagonal elements are zero.

Matrix size (n) 5,10, and 40

Number of points in $P^i(m)$: 4,8 and 16.

Table 1: Number of iterations in our algorithm (#C) and Number of iterations in the Power method (#P)

m		4		8		16	
		#C	#P	#C	#P	#C	#P
n=5	Type 1	2.2	6.9	2.3	6.7	2.7	6.5
	Type 2	2.6	17.7	2.3	16.7	2.3	16.3
	Type 3	2.4	16.5	2.5	15.8	2.8	19.0
n=10	Type 1	2.4	6.1	2.1	6.3	2.2	6.2
	Type 2	2.9	19.0	2.9	18.6	2.7	18.3
	Type 3	2.6	10.5	3.1	10.2	2.7	10.9
n=40	Type 1	2.5	5.3	2.6	5.2	3.0	5.0
	Type 2	2.9	18.8	3.0	19.0	3.4	17.8
	Type 3	2.8	6.0	3.0	6.0	3.0	6.0

Table 2: Number of iterations in the proposed algorithm (#C) and Number of iterations in the Dinkelbach-type algorithm (#D)

m		4		8		16	
		#C	#P	#C	#P	#C	#P
n=5	Type 1	2.2	4.2	2.3	4.0	2.7	4.0
	Type 2	2.6	9.0	2.3	9.2	2.3	8.6
	Type 3	2.4	4.2	2.5	4.0	2.8	4.0
n=10	Type 1	2.4	4.0	2.1	4.0	2.2	4.0
	Type 2	2.9	9.0	2.9	12.2	2.7	12.2
	Type 3	2.6	4.8	3.1	5.0	2.7	4.6
n=40	Type 1	2.5	4.0	2.6	4.0	3.0	4.0
	Type 2	2.9	18.8	3.0	17.0	3.4	16.7
	Type 3	2.8	4.4	3.0	4.1	3.0	4.0

The same initial point e/n is given to the both algorithms. Table 1 reports the average number of iterations in the proposed algorithm and the average number of iterations in the power method.

We see from Table 1 that the proposed algorithm converges within 4 iterations in all cases. From Table 1 we see that the number of iterations in the proposed algorithm is independent of the matrix-size and the number of the points in P^i for all types.

There are large difference of the average number of iterations in power method between Type 1 and the other types in the small matrix-size. The power method converges within several iterations for all cases in Type 1 but it converges within about 18 iterations in Type 2 and 3 for the small matrix-size. The number of iterations in the power method is much effected by the distribution of $x_i \in P^i$. For Type 3, the average number of iterations in the power method is in inverse proportion to the matrix-size. For all types the number of iterations in the power method does not increase for the matrix-size and the number of points.

The proposed algorithm and the Dinkelbach-type algorithm generate an eigenvalue problem and a linear programming problem as subproblem in each iteration, respectively, and the size of the coloring matrix is almost equal to that of the basis matrix in the linear programming problem of the Dinkelbach-type algorithm.

Table 2 reports the average number of iterations in the Dinkelbach-type algorithm and the proposed algorithm for 3^3 experiments. Since the numbers of iterations is coincide with the number of subproblems, the number of subproblems in the proposed algorithm is always less than that in the Dinkelbach-type algorithm. Especially, in Type 1, the proposed algorithm has less number of subproblems than the Dinkelbach-type algorithm. If the computational complexity of solving the subproblem in the Dinkelbach-type algorithm, the computational complexity of the proposed algorithm is comparable with that of the Dinkelbach-type algorithm. In fact, we may say from Table 1 that solving the subproblem in the proposed algorithm is as efficient as that in the Dinkelbach-type algorithm.

6 Extension and further research

In this study, we propose the algorithm for optimizing the largest eigenvalue of a matrix with linearly perturbed row vectors. For a given n-squared symmetric matrix X^i and $i = 0, \dots, m$, Jarre consider the affine space generated by $X^0, \dots, X^m$:

$$S = \left\{ X^0 + \sum_{i=1}^{m} \mu_i X^i \;\middle|\; \mu_i \in R \,, i = 1, \dots, m \right\}.$$

Let $\boldsymbol{\mu} = [\mu_1, \dots, \mu_m]$ and $Y(\boldsymbol{\mu}) = X^0 + \sum_{i=1}^{m} \mu_i X^i$, then largest eigenvalue optimization problem is as follows:

$$\inf \{ \Lambda(Y(\boldsymbol{\mu})) \mid \boldsymbol{\mu} \in R^m \} .$$

Jarre develop an interior-point algorithm [3] to finding the lower bound of the largest eigenvalue.

In further research, we will consider the largest eigenvalue optimization problem with the matrix set that is similar to that of Jarre's study as follows:

$$S = \left\{ \sum_{i=1}^{m} \mu_i X^i \,\middle|\, \sum_{i=1}^{m} \mu_i = 1, \begin{array}{l} \mu_i \geq 0 \\ i = 1, \ldots, m \end{array} \right\}.$$

This matrix set S is convex set of a matrices $\{X^1, \ldots, X^m\}$. We only assume that X^i is a positive matrix for all $i = 1, \ldots, m$. In order to solve the largest eigenvalue optimization problem with the matrix set S, we will develop the similar algorithm to the coloring algorithm. On computational experiments, we will compare the algorithm with the Jarre's algorithm.

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