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DISCRETE VERSION OF LIAO'S CLOSING LEMMA AND THE C^1 STABILITY CONJECTURE : HAS THE C^1 STABILITY CONJECTURE BEEN SOLVED ?

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ABSTRACT. R. Mañé published a proof of the C^1 stability conjecture for diffeomorphisms[5]. In the proof R. Mañé used the discrete version of Liao's Closing Lemma without proof. However, the author cannot be convinced of this version of Liao's Closing Lemma. We consider length of γ-strings. We prove the discrete version of Liao's Closing Lemma in consideration of length of γ-strings. In this paper we claim need of reconstruction of a proof of the C^1 stability and Ω-stability conjecture for diffeomorphisms and flows.

1. INTRODUCTION

R. Mañé published a proof of the C^1 stability conjecture for diffeomorphisms[5]. In [5] R. Mañé used the discrete version of Liao's Closing Lemma *without proof.* Liao's Closing Lemma is a kind of Shadowing Lemma to show existence of a periodic orbit near a given periodic pseudo-orbit. Mañé cited this lemma from [3]. However, in [3] the original flow version of the Closing Lemma is only applied to a proof of a theorem. The original version of the Closing Lemma is stated in [2] in Chinese. Moreover, a proof of Lemma 3.6 in [2] is incorrect. Thus, there exists a counter example. But the original flow version maybe holds by minor corrections or at least in similar setting to Mañé's diffeomorphism version. The author however cannot be convinced of Mañé's discrete version of Liao's Closing Lemma, Lemma II.2[5]. Mañé's version has no bounds for length of γ-strings(that is, length of parts of a given pseudo-orbit). Mañé's discrete version is very powerful because there exist no bounds for length of γ-strings. However we need bound for length of γ-strings to guarantee shadowing property. We consider length of γ-strings to guarantee shadowing property. We prove the discrete version of Liao's Closing Lemma in consideration of length of γ-strings. In the framework of the argument of Mañé[5], we need not only the existence of a periodic orbit but also the periodic orbit to shadow a given periodic pseudo-orbit. If Lemma II.2[5] does not hold, then Theorem I.4 and Theorem II.1 in [5] collapse. If one would like to declare that the C^1 stability conjecture has been solved, one should show us clear and rigorous proof of Lemma II.2[5]. In this paper we claim need of reconstruction of a proof of the C^1 stability and Ω-stability conjecture for diffeomorphisms[5,6] and flows[1].

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In section 2 we give definitions and precise statements of results. After we investigate several information obtained from uniform γ-strings, we prove the discrete version of Liao's Closing Lemma in consideration of length of γ-strings. Also we prove Lemma II(Pliss's Lemma).

2. Discrete Version of Liao's Closing Lemma

Let M be a closed manifold with dimension $m \geq 2$ and let $\text{Diff}^r(M)$, $r \geq 1$, be the space of C^r diffeomorphisms of M endowed with the C^r topology. Given a compact f-invariant subset Λ of $f \in \text{Diff}^r(M)$ we say that a splitting $TM|\Lambda = E \oplus F$ is a *dominated splitting* if it is a continuous, Df-invariant and there exist a Riemannian norm $||\cdot||$ on TM, and $C > 0$, $0 < \lambda < 1$ such that

$$||(Df^n)|E(x)|| \cdot ||(Df^{-n})|F(f^n(x))|| \leq C\lambda^n$$

for all $x \in \Lambda$ and all $n \geq 0$. A splitting $TM|\Lambda = E \oplus F$ is *homogeneous* if the dimension of the subspace $E(x)$, $x \in \Lambda$, is constant. We say that a subbundle $E \subset TM|\Lambda$ is *contracting* if it is continuous, Df-invariant and there exist $G > 0$ and $0 < \mu < 1$ such that

$$||(Df)^n|E(x)|| \leq G\mu^n \quad \text{for all } x \in \Lambda \text{ and } n \geq 0.$$

We say that a pair of points $(x, f^n(x))$ contained in Λ, $n > 0$, is a *γ-string* if

$$\prod_{j=1}^{n} ||(Df^{-1})|F(f^j(x))|| \leq \gamma^n$$

and we say that it is a *uniform γ-string* if $(f^k(x), f^n(x))$ is a γ-string for all $0 \leq k < n$. For further information and details we refer the reader to Mañé[5], Shub[8].

At first, we state discrete version of Liao's Closing Lemma in consideration of length of γ-strings.

Theorem I. *Let Λ be a compact f-invariant subset of M. Let $TM|\Lambda = E \oplus F$ be a homogeneous dominated splitting such that E is contracting. Given $N \in Z^+$, $0 < \hat{\gamma} < 1$ and $\beta > 0$, there exists $\alpha = \alpha(N, \hat{\gamma}, \beta) > 0$ such that if $(x_i, f^{n_i}(x_i))$, $i = 1, \cdots, k$, are (uniform) $\hat{\gamma}$-strings satisfying*

(i) $d(f^{n_i}(x_i), x_{i+1}) < \alpha$ *for all* $1 \leq i < k$, *and* $d(f^{n_k}(x_k), x_1) < \alpha$;

(ii) $1 \leq n_i \leq N$ *for all* $1 \leq i \leq k$,

then there exists a periodic point y of f with period $\sum_{i=1}^{k} n_i$ such that

$$d(f^n(y), f^n(x_1)) < \beta \quad \text{for} \quad 0 \leq n \leq n_1$$

and setting $N_i = \sum_{j=1}^{i} n_j$,

$$d(f^{N_i+n}(y), f^n(x_{i+1})) < \beta \quad \text{for} \quad 0 \leq n \leq n_{i+1}, \quad 1 \leq i < k.$$

Remark. Mañé[5] claims that α depends only on $\hat{\gamma}$, β. That is, Mañé's discrete version has no bound for length of $\hat{\gamma}$-strings. However, in Liao's original flow version[2] correspondent to α depends on correspondents to $\hat{\gamma}$, β, and an upper bound of length of $\hat{\gamma}$-strings respectively. Moreover, Liao's original flow version[2] has lower bound for length of $\hat{\gamma}$-string.

From now on, we shall call above α *connecting range*, above β *shadowing range*, and above $\hat{\gamma}$ *contracting rate*. The essence of our problem is not the number of γ-strings but the length of γ-strings consisting of a periodic pseudo-orbit. More precisely, the main problem is whether a sufficiently long uniform γ-string can be decomposed into appropriate (uniform) γ'-strings with $\gamma < \gamma' < 1$. For simplicity, we consider the case of $k = 1$ in the setting of Theorem I. That is, $(x_1, f^{n_1}(x_1))$ is a uniform $\hat{\gamma}$-string satisfying $d(f^{n_1}(x_1), x_1) < \varepsilon$ for small $\varepsilon > 0$. If we treat $(x_1, f^{n_1}(x_1))$ as only a $\hat{\gamma}$-string, then we can show existence of a periodic point x with period n_1 but cannot guarantee whether x shadows x_1. However we can apply Lemma II(below) to $(x_1, f^{n_1}(x_1))$. Lemma II guarantees a decomposition of a (uniform) $\hat{\gamma}$-string into uniform γ_3-strings for some $1 > \gamma_3 > \hat{\gamma}$. Hence there exists a sequence $0 = m_0 < m_1 < \cdots < m_p = n_1$ such that $(f^{m_i}(x_1), f^{m_{i+1}}(x_1))$ is a uniform γ_3-string for all $0 \le i < p$. In original flow version, a quasi-hyperbolic arc [2] has similar properties to a uniform $\hat{\gamma}$-string $(x_1, f^{n_1}(x_1))$ in above situation. However, correspondent to $(f^{m_i}(x_1), f^{m_{i+1}}(x_1))$ has an upper bound of length of strings in original version. Lemma II does not inform us about length of a uniform γ_3-string $(f^{m_i}(x_1), f^{m_{i+1}}(x_1))$ at all. Certainly Lemma 3.6[2] is applicable to a uniform $\hat{\gamma}$-string $(x_1, f^{n_1}(x_1))$ with the decomposition into uniform γ_3-strings $(f^{m_i}(x_1), f^{m_{i+1}}(x_1))$. But the diffeomorphism case is different from the flow case. Continuing the similar argument to the flow case[2] is hard because discreteness and no upper bound for length of uniform γ_3-strings. If one would like to declear that the C^1 stability conjecture has been solved, one should show us the way of finding connecting range α from only shadowing range β and contracting rate $\hat{\gamma}$ without upper bound N for length of γ_3-strings.

Proof of Theorem I. Without loss of generality we can suppose that the given Riemannian metric is adapted to (f, Λ), uniformly on Λ, that is, there are constants $0 < \lambda < 1$, $C > 0$ such that

(1) $||Df|E(x)|| < \lambda$ for any x in Λ ;

(2) $||(Df)^n|E(x)|| \cdot ||(Df^{-1})^n|F(f^n(x))|| < C\lambda^n$ for any x in Λ and $n \ge 1$.

Let $\varepsilon' > 0$ be such that the exponential map $\exp_x : TM \to M$ is a diffeomorphism on the ball of radius ε' for every x in M. For small $0 < \varepsilon < \varepsilon'$, define $\mathcal{B}_p(\varepsilon) = E_p(\varepsilon) \times F_p(\varepsilon)$ and $B_p(\varepsilon) = \exp_p(\mathcal{B}_p(\varepsilon))$, where $E_p(\varepsilon)$ and $F_p(\varepsilon)$ are the closed balls in $E(p)$ and $F(p)$ about 0 of radius ε, respectively.

From now on we fix ε_0 such that $0 < \varepsilon_0 < \varepsilon'$. If z and x are two points in M with $d(f(x), z) \le \varepsilon_0$, define a map $\widetilde{F}_{z,x} : T_x M \to T_z M$ by $\widetilde{F}_{z,x} = D(\exp_z^{-1})_{f(x)} Df_x$.

If the points z and x belong to Λ, the splitting $E \oplus F$ allows us to write $\widetilde{F}_{z,x}$ as the block matrix

$$\begin{pmatrix} A_{z,x} & B_{z,x} \\ C_{z,x} & D_{z,x} \end{pmatrix}$$

where $A_{z,x} \in L(E(x), E(z))$, $B_{z,x} \in L(F(x), E(z))$, $C_{z,x} \in L(E(x), F(z))$, $D_{z,x} \in L(F(x), F(z))$. Here $L(E_1, E_2)$ is a space of continuous linear maps of E_1

Let $\widehat{F}_{z,x}$ be the map with the diagonal block matrix

$$\begin{pmatrix} A_{z,x} & O \\ O & D_{z,x} \end{pmatrix}.$$

In this setting we obtain two preliminary lemmas.

Lemma 1. *For all $\eta > 0$ we find a constant $0 < \delta \le \varepsilon_0$ such that if two points x, z in Λ satisfying $d(f(x), z) < \delta$, then*

$$||\widetilde{F}_{z,x} - \widehat{F}_{z,x}|| < \eta \ , \quad ||\widehat{F}_{z,x}|\widetilde{E}_x|| < \lambda \ .$$

Lemma 2. *For given $N \in Z^+$, $\beta > 0$ and $\eta' > 0$ with $\beta \le \varepsilon_0$, there are $0 < \delta$ $(< \beta)$ and $r = r(N, \beta, \eta', \delta) > 0$ such that if $d(z, f^n(y)) < \delta$ and $1 \le n \le N$, then we have*

(i) $f^n(B_y(r)) \subset \exp_z(\mathcal{B}_z(\beta))$,

(ii) $f^j(B_y(r)) \subset \exp_{f^j(y)}(\mathcal{B}_{f^j(y)}(\beta))$ *for* $0 \le j \le n$,

(iii) $Lip[(\widetilde{F}_{z,f^{n-1}(y)} \circ Df^{n-1} - \exp_z^{-1} \circ f^n \circ \exp_y)|\mathcal{B}_y(r)] < \eta'$.

Remark. r depends on δ.

Now return the proof of Theorem I. For $0 < \alpha < \min\{\varepsilon_0, \beta\}$, where β is given by Theorem I, let $(x_i, f^{n_i}(x_i))$, $i = 1, \cdots, k$, be a (uniform) $\widehat{\gamma}$-string satisfying

$$d(f^{n_i}(x_i), x_{i+1}) < \alpha \quad \text{for all } 1 \le i < k \quad \text{and} \quad d(f^{n_k}(x_k), x_1) < \alpha.$$

Let $X = \{x_1, \cdots, x_k\}$. We define the following maps:

(i) $i : X \to \Lambda \subset M$ is the inclusion map, i.e., $i(x_j) = x_j$ for all $1 \le j \le k$.

(ii) $h : X \to X$ is a shift with $h(x_j) = x_{j+1}$ for all $1 \le j < k$ and $h(x_k) = x_1$.

Let $\Gamma(X, i^*TM)$ be the space of continuous sections of X with sup norm $||\xi|| = \sup_{0 \le j \le k} ||\xi(x_j)||$. Continuity of section ξ on X means that there exists a continuous section ζ on M satisfying $\zeta \circ i = \xi$. We will construct a hyperbolic linear operater $\mathbf{F}$ on $\Gamma(X, i^*TM)$ which depends only on X. By $0 < \alpha \le \varepsilon_0$ we can define $\mathbf{F}$ by the formula

$$\mathbf{F}(\sigma)(x_1) = \hat{F}_{i(x_1), f^{n_k - 1}(ih^{-1}(x_1))} \circ (Df)^{n_k - 1}\sigma(h^{-1}(x_1))$$

$$\mathbf{F}(\sigma)(x_j) = \hat{F}_{i(x_j), f^{n_{j-1} - 1}(ih^{-1}(x_j))} \circ (Df)^{n_{j-1} - 1}\sigma(h^{-1}(x_j)) \quad \text{for} \quad 1 < j \le k,$$

where $\quad \sigma \in \Gamma(X, i^*TM)$.

We shall show that $\mathbf{F}$ is hyperbolic. Take $\hat{\lambda}$ such that $1 > \hat{\lambda} > \max\{\lambda, \hat{\gamma}\}$. Then there exists a constant $0 < \alpha_0 (< \varepsilon_0)$ such that if $\alpha_0 \ge \alpha > 0$ then

$$\left(\prod_{l=1}^{n_j} ||(Df^{-1})|F(f^l(x_j))|| \right) \cdot ||[D(\exp_{x_{j+1}}^{-1})_{f^{n_j}(x_j)}]^{-1}|| < \hat{\lambda} \ \text{ for } \ j = 1, \cdots, k-1,$$

$$\left(\prod_{l=1}^{n_k} ||(Df^{-1})|F(f^l(x_k))|| \right) \cdot ||[D(\exp_{x_1}^{-1})_{f^{n_k}(x_k)}]^{-1}|| < \hat{\lambda},$$

$$\left(\prod_{l=0}^{n_j - 1} ||(Df)|E(f^l(x_j))|| \right) \cdot ||D(\exp_{x_{j+1}}^{-1})_{f^{n_j}(x_j)}|| < \hat{\lambda} \ \text{ for } \ j = 1, \cdots, k-1,$$

$$\left(\prod^{n_k - 1} ||(Df)|E(f^l(x_k))|| \right) \cdot ||D(\exp_{x_1}^{-1})_{f^{n_k}(x_k)}|| < \hat{\lambda}.$$

(Because E is contracting and $(x_j, f^{n_j}(x_j))$ is $\hat{\gamma}$-string for $j = 1, \cdots, k$.) Hence for some $0 < \alpha < \alpha_0$, $\mathbf{F}$ is hyperbolic.

We define $\mathbf{G}: \Gamma_r(X, i^*TM) \to \Gamma(X, i^*TM)$ by

$\mathbf{G}(\sigma)(x_1) = \exp_{i(x_1)}^{-1} \circ f^{n_k} \circ \exp_{ih^{-1}(x_1)}(\sigma(h^{-1}(x_1)))$,

$\mathbf{G}(\sigma)(x_j) = \exp_{i(x_j)}^{-1} \circ f^{n_{j-1}} \circ \exp_{ih^{-1}(x_j)}(\sigma(h^{-1}(x_j)))$ for $1 < j \le k$,

where $\Gamma_r(X, i^*TM)$ is the closed ball in $\Gamma(X, i^*TM)$ about 0 of radius r. Let $K = \max_{1 \le k \le N} \|Df^k|\Lambda\|$. We shall show that $\mathbf{G}$ is Lipschitz close to $\mathbf{F}$. Using the norm on $\Gamma(X, i^*TM)$, we can calculate the Lipschitz distance from $\mathbf{G}$ to $\mathbf{F}$ on the ball $\Gamma_{r'}(X, i^*TM) = \Gamma(r')$:

$$Lip[(\mathbf{F} - \mathbf{G})|\Gamma(r')] < K \times \max_{1<j\le k} \{\|\hat{F}_{i(x_j), f^{n_{j-1}-1}(ih^{-1}(x_j))} - \tilde{F}_{i(x), ih^{-1}(x)}\|,$$
$$\|\hat{F}_{i(x_1), f^{n_k-1}(ih^{-1}(x_1))} - \tilde{F}_{i(x_1), f^{n_k-1}(ih^{-1}(x_1))}\|\} + \max_{1<j\le k}$$
$$\{Lip[(\tilde{F}_{i(x_j), f^{n_{j-1}-1}(ih^{-1}(x_j))} \circ Df^{n_{j-1}-1} - \exp_{i(x_j)}^{-1} \circ f^{n_{j-1}} \circ \exp_{ih^{-1}(x_j)})|\mathcal{B}_{ih^{-1}(x_j)}(r')],$$
$$Lip[(\tilde{F}_{i(x_1), f^{n_k-1}(ih^{-1}(x_1))} \circ Df^{n_k-1} - \exp_{i(x_1)}^{-1} \circ f^{n_k} \circ \exp_{ih^{-1}(x_1)})|\mathcal{B}_{ih^{-1}(x_1)}(r')]\}.$$

Now, we use $N \in Z^+$, $\beta > 0$ given in Theorem I. Moreover we take $\eta' > 0$ and $0 < \delta < \min\{\beta, \delta(\eta)\}$. (Note that $\delta(\eta)$ is given by Lemma 1 for η.) Then Lemma 2 allows us to find a constant $r(N, \beta, \delta, \eta') > 0$ such that for every $0 < r' < r(N, \beta, \delta, \eta')$

$$Lip[(\tilde{F}_{i(x_1), f^{n_k-1}(ih^{-1}(x_1))} \circ Df^{n_k-1} - \exp_{i(x_1)}^{-1} \circ f^{n_k} \circ \exp_{ih^{-1}(x_1)})|\mathcal{B}_{ih^{-1}(x_j)}(r')] < \eta'$$

and

$$Lip[(\tilde{F}_{i(x_j), f^{n_{j-1}-1}(ih^{-1}(x_j))} \circ Df^{n_{j-1}-1} - \exp_{i(x_j)}^{-1} \circ f^{n_{j-1}} \circ \exp_{ih^{-1}(x_j)})|\mathcal{B}_{ih^{-1}(x_j)}(r')]$$
$$< \eta' \quad \text{for} \quad 1 < j \le k.$$

Now, we take α, r' satisfying $0 < \alpha < \delta$, $0 < r' < r(N, \beta, \delta, \eta')$. Then we have

(a) $Lip[(\mathbf{F} - \mathbf{G})|\Gamma(r')] \le K\eta + \eta'$,

(b) $\|\mathbf{G}(0)\| < \alpha$,

(c) $\|\mathbf{F}|\Gamma(X, i^*E)\| < \hat{\lambda} < 1$,

(d) $\|\mathbf{F}^{-1}|\Gamma(X, i^*F)\| < \hat{\lambda} < 1$.

In order to apply Proposition 7.7 [8], we must use the box norm on $\Gamma_r(X, i^*TM) = \Gamma_r(X, i^*E) \oplus \Gamma_r(X, i^*F)$. It is easy to see the equivalence of the box norm $\|\cdot\|_{box}$ and the given Riemannian norm $\|\cdot\|$ on $E^s \oplus E^u$. Thus there is a constant $c > 0$ such that $c^{-1}\|\cdot\|_{box} \le \|\cdot\| \le c\|\cdot\|_{box}$ on $E \oplus F$.

Using the box norm, we can rewrite the estimate of (a) and (b):

(a′) $Lip_{box}[(\mathbf{F} - \mathbf{G})|\Gamma(r')] \le c^2(K\eta + \eta')$

(b′) $\|\mathbf{G}(0)\|_{box} < c\alpha$.

Let $\Gamma^s(r)$ be a closed ball in $\Gamma_r(X, i^*E)$ about 0 of radius r. Similarly for $\Gamma^u(r)$. If r'' is less than r'/c, the box $\Gamma^s(r'') \times \Gamma^u(r'')$ is contained in $\Gamma(r')$, and we have

$$Lip_{box}[(\mathbf{F} - \mathbf{G})|\Gamma^s(r'') \times \Gamma^u(r'')] \le Lip_{box}[(\mathbf{F} - \mathbf{G})|\Gamma(r')] \le c^2(K\eta + \eta').$$

In order to apply Proposition 7.7 [8], we must have

(e) $\hat{\lambda}+c^2(K\eta+\eta')<1$;

(f) $c\alpha<r''\{1-\hat{\lambda}-c^2(K\eta+\eta')\}$.

Therefore, we first choose η and η' satisfying (e). We take $\delta>0$ such that $\delta<\min\{\beta,\delta(\eta)\}$. So we get $r=r(N,\beta,\delta,\eta')$ by Lemma 2. Then, we find constants r' and $r''<r'/c$, as above. Finally we choose $\alpha<\min\{\delta,\alpha_0\}$ small enough so that (f) holds. Hence Proposition 7.7 [8] gives a fixed point $\sigma\in\Gamma(r')$ for $\mathbf{G}$. Then $y=\exp_{x_1}\sigma(x_1)$ is a periodic point of f with period $\sum_{j=1}^{k}n_j$ satisfying $d(f^l(y),f^l(x_1))<\beta$ for all $0\le l\le n_1$ and, setting $N_j=\sum_{m=1}^{j}n_m$, $d(f^{N_j+l}(y),f^l(x_{j+1}))<\beta$ for $0\le l\le n_{j+1}$, $1\le j<k$. Λ

Remark. (1) Setting $N_j=\sum_{m=1}^{j}n_m$, $d(f^{N_k}(y),x_1))<[c^2/(1-\hat{\lambda}-c^2(K\eta+\eta'))]\alpha$ and $d(f^{N_j}(y),x_{j+1}))<[c^2/(1-\hat{\lambda}-c^2(K\eta+\eta'))]\alpha$ for $1\le j<k$.

(2) By some minor modifications of the above arguments, we can give rigorous proofs of Step V and Lemma B in [4].

(3) In the framework of the arguments of Mañé[5], Theorem I is not effective. Hence, if Lemma II.2[5] does not hold, then it is hard to prove Theorem I.4 and Theorem II.1 in [5].

The following lemma is essentially due to Pliss[7].

Lemma II. *For all* $0<\gamma_0<\gamma_3<1$ *there exist* $N(\gamma_0,\gamma_3)>0$ *and* $K(\gamma_0,\gamma_3)>0$ *such that if* $(x,g^n(x))$ *is a* γ_0*-string and* $n\ge N(\gamma_0,\gamma_3)$, *then there exist a seqeunce of positive integers* $0<n_1\cdots<n_s\le n$, $s>1$, *such that* $(x,g^{n_i}(x))$ *is a uniform* γ_3*-string for all* $1\le i\le s$. *Moreover, if* $m<nK(\gamma_0,\gamma_3)$ *then* $m\le s$. *Let* $K(n)=\max\{m\in Z^+ \mid m<nK(\gamma_0,\gamma_3)\}$. *Then* $s\ge K(n)$.

Proof. Let $H=\sup\{|\log\|(Dg^{-1})|F(x)\|\,|;x\in\Lambda\}+\alpha$, where $\alpha>0$ is small enough. Let $N(\gamma_0,\gamma_3)=2H/\log(\gamma_3/\gamma_0)$. Let $(x,g^n(x))$ be a γ_0-string with $n\ge N(\gamma_0,\gamma_3)$. Define a sequence of positive numbers $\{p(k)\}$ by

$$p(0)=1,\quad p(k)=\|(Dg^{-1})|F(g^{n+1-k}(x))\|\quad\text{for all }1\le k\le n.$$

Then it is obvious that $|\log p(k)|<H$ for $1\le k\le n$. Moreover, $\sum_{k=0}^{n}\log p(k)=\sum_{k=1}^{n}\log p(k)\le n\log\gamma_0$. (Because $(x,g^n(x))$ is a γ_0-string.) Define a sequence of positive numbers $\{q(k)\}$ by

$$q(0)=p(0)=1,\qquad q(k)=p(k)\gamma_3^{-1}\qquad\text{for }1\le k\le n.$$

Define $f(\nu)=\sum_{k=0}^{\nu}\log q(k)$. Then

$$f(n)=\sum_{k=1}^{n}\log q(k)\le n\log\gamma_0+n\log\gamma_3^{-1}=n\log(\gamma_0/\gamma_3)<0\qquad\text{(a)}.$$

Let ν_1 be a minimal number such that $f(\nu_1)\ge f(\nu)$ for $0\le\nu\le n$. Obviously $0\le\nu_1<n$ because $f(0)=0$, $f(n)<0$. Let ν_2 be a minimal number satisfying:

(i) $\nu_1<\nu_2$;

(ii) $f(\nu_2)\ge f(\nu)$ for $\nu_2\le\nu\le n$;

(iii) $0\le f(\nu_1)-f(\nu_2)<H$.

Continuing in this fashion, we obtain a sequence of numbers $\{\nu_j | 1 \leq j \leq s\}$ satisfying

(I) $f(\nu_j) \geq f(\nu)$ for $\nu_j \leq \nu \leq n$;

(II) $0 \leq f(\nu_{j-1}) - f(\nu_j) < H$ for $2 \leq j \leq s$;

(III) $\nu_s = n$.

By (I) we have

$$\log \gamma_3^{(\nu-\nu_j)} \geq \log \prod_{k=\nu_j+1}^{\nu} ||(Dg^{-1})|F(g^{n+1-k}(x))|| \text{ for } \nu_j < \nu \leq n \text{ and } 1 \leq j \leq s.$$

Hence $\prod_{k=\nu_j+1}^{\nu} ||(Dg^{-1})|F(g^{n+1-k}(x))|| \leq \gamma_3^{\nu-\nu_j}$ for $\nu_j < \nu \leq n$ and $1 \leq j \leq s$.

Setting $n_j = n - \nu_{l+1-j}$ for $1 \leq j \leq s$, we obtain $0 < n_1 < \cdots < n_l \leq n$ such that

$$\prod_{k=1}^{i} ||(Dg^{-1})|F(g^{n_j+1-k}(x))|| \leq (\gamma_3)^i \quad \text{for} \quad 1 \leq i \leq j \text{ and } 1 \leq j \leq s.$$

Hence $(x, g^{n_j}(x))$ is a uniform γ_3-string for $1 \leq j \leq s$.

Summation of the inequalities (II) from $j = 2$ to $j = s$ yields $f(\nu_1) - f(\nu_s) < Hs$. Since $f(0) = 0 \leq f(\nu_1)$, $f(\nu_s) > -Hs$. Let $K(\gamma_0, \gamma_3) = H^{-1}\log(\gamma_3/\gamma_0)$. Then we claim that $k < nK(\gamma_0, \gamma_3)$ implies $k \leq s$. Suppose that $k > s$. Then $-Hk < -Hs < f(\nu_s)$. If $\nu_{s-1} = n-1$ then $-Hk < f(n)$. But $-Hk > -nK(\gamma_0, \gamma_3)H = -n\log(\gamma_3/\gamma_0) = n\log(\gamma_0/\gamma_3) \geq f(n)$ by (a) above. This is a contradiction. If $\nu_{s-1} = n-2$ then $-Hk < f(n-1) < f(n) \leq f(n-2)$ by the construction of $\{\nu_j\}$. This contradicts to $-Hk > f(n)$. By the similar argument of the case $\nu_{s-1} = n-2$, we can induce a contradiction for the case $\nu_{s-1} = n - m$, $n > m \geq 3$.

Since $n \geq N(\gamma_0, \gamma_3)$, $n > 2H/\log(\gamma_3/\gamma_0)$ so $nH^{-1}\log(\gamma_3/\gamma_0) > 2$ hence $nK(\gamma_0, \gamma_3) > 2$. Therefore $s \geq 2$. Λ

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