数理解析研究所講究録 1245 巻 2002 年 102-116

The Cowling–Price theorem for semisimple Lie groups

広大・理学研究科　江端　満彦 (Mitsuhiko Ebata)
Department of Mathematics Graduate School of Science,
Hiroshima University
広大・総科　江口　正晃 (Masaaki Eguchi)
Faculty of Integrated Arts and Sciences, Hiroshima University
放送大学　熊原　啓作 (Keisaku Kumahara)
The University of the Air
尾道大学　小泉　伸 (Shin Koizumi)
Faculty of Economy, Management and Information sciences

1. Introduction

The mathematical uncertainty principle, roughly speaking, states that a nonzero function and its Fourier transform cannot both be sharply localized. First of all, in the case of Euclidean space, G. H. Hardy showed that if a measurable function f on $\mathbf{R}$ satisfies $|f(x)| \leqq Ce^{-ax^2}$ and $|\hat{f}(y)| \leqq Ce^{-by^2}$ and $ab > 1/4$, then $f = 0$ (a.e.). Here we use the Fourier transform defined by $\hat{f}(y) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)e^{\sqrt{-1}xy}\,dx$. M. G. Cowling and J. F. Price [?] generalized Hardy's theorem as follows. Suppose that $1 \leqq p, q \leqq \infty$ and one of them is finite. If a measurable function f on $\mathbf{R}$ satisfies $\|\exp\{ax^2\}f(x)\|_{L^p(\mathbf{R})} < \infty$ and $\|\exp\{by^2\}\hat{f}(y)\|_{L^q(\mathbf{R})} < \infty$ and $ab \geqq 1/4$, then $f = 0$ (a.e.). The case where $p = q = \infty$ and $ab > 1/4$ is covered by Hardy's theorem. S. C. Bagchi and S. K. Ray [?] showed that if $ab > 1/4$, then Hardy's theorem on $\mathbf{R}$ is equivalent to the Cowling-Price theorem.

Some generalizations of Hardy's theorem and the Cowling–Price theorem to various homegeneous spaces were obtained (e.g. [?], [?], [?], [?] and [?]). In these papers, the theorems were proved by using the estimates of matrix elements of representations and the Phragmén–Lindelöf theorem.

The purpose of this paper is to prove an analogue of the Cowling–Price theorem for semisimple Lie groups. On the other hand, J. Sengupta [11] proved the Cowling–Price theorem on Riemannian symmetric spaces, by using the argument that the Fourier transform is decomposed into the composition of the Radon transform and the Euclidean Fourier transform. We consider the Helgason–Fourier transform as the Fourier transform on homogeneous vector bundles over Riemannian symmetric spaces. By using a similar argument to [11], we get the Cowling–Price theorem for the vector bundles. Form this result and the estimate of the Plancherel measures, we obtain the Cowling–Price theorem for semisimple Lie groups.

2. Notaion and preminaries

The standard symbols $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ shall be used for the sets of the integers, the real numbers and the complex numbers, respectively. For $z \in \mathbf{C}$, $\Re z$ and $\Im z$ denote its real and complex part, respectively. If U is a manifold, then we denote by $C(U)$ the set of continuous complex valued functions on U and by $C_0^\infty(U)$ the set of compactly supported smooth functions on U. If $S \subset U$ and f is a function on U, then $f|_S$ denotes the restriction of f to S. If V is a vector space over $\mathbf{R}$, $V_{\mathbf{C}}$, V^* and $V_{\mathbf{C}}^*$ denote its complexification, its real dual and its complex dual, respectively. For a Lie group L, $\hat{L}$ denotes the set of equivalence classes of irreducible unitary representations of L. As usual, we use lower case German letters to denote the corresponding Lie algebras.

If $\mathcal{H}$ is a complex separable Hilbert space, $\mathbf{B}(\mathcal{H})$ denotes the Banach space comprised of all bounded operators on $\mathcal{H}$ with operator norm $\|\cdot\|_\infty$. For $T \in \mathbf{B}(\mathcal{H})$ and $1 \leqq p < \infty$, we indicate the p-th norm by $\|T\|_p$, that is, $\|T\|_p = (\mathrm{tr}(T^*T)^{p/2})^{1/p}$, T^* being the adjoint operator of T. For a complex separable Hilbert space $\mathcal{H}$ and a σ-finite measure space (X, μ), we denote by $L^p(X, \mathbf{B}(\mathcal{H}))$ the noncommutative L^p-space relative to the gage $(L^2(X, \mathbf{B}(\mathcal{H})), L^\infty(X, \mathbf{B}(\mathcal{H})))$.

Let G be a connected semisimple Lie group with finite centre, K a maximal compact subgroup of G and G/K the associated Riemannian symmetric space of noncompact type. Let $G = KAN$ be an Iwasawa decomposition. Each $g \in G$ can be uniquely decomposed as $g =$

$\kappa(g)\exp(H(g))n(g)$. We denote by θ the Cartan involution fixing the elements in K. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ defined by θ. Denote by d the real rank of G. Let Δ be the set of restricted roots, Δ^+ the set of all positive restricted roots and ρ the half the sum of the elements in Δ^+. Denote by $\mathfrak{a}_+$ the positive Weyl chamber in $\mathfrak{a}$ and set $A_+ = \exp\mathfrak{a}_+$. Then $G = K\mathrm{Cl}(A_+)K$ is a Cartan decomposition, where $\mathrm{Cl}(A_+)$ denotes the closure of A_+ in A. Let dk be the Haar measure on K normalized as $\int_K dk = 1$. We normalize the Lebesgue measure dH on $\mathfrak{a}$ by multiplying $(2\pi)^{-d/2}$. We write for dg the Haar measure on G given by $dg = D(\exp H)dk_1dk_2dH$, where $D(\exp H) = \prod_{\alpha\in\Delta^+}|\sinh\alpha(H)|^{m(\alpha)}$ and $m(\alpha)$ denotes the multiplicity of α. Let M be the centralizer of A in K. Then $P = MAN$ is a minimal parabolic subgroup of G. The Killing form of $\mathfrak{g}$ induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}$ and $\mathfrak{a}^*$. We write $|H| = \langle H, H\rangle^{1/2}$. Let W be the restricted Weyl group. When $g = k\exp X$ for $k \in K$ and $X \in \mathfrak{p}$, we set $\sigma(g) = |X|$. For $\nu \in \mathfrak{a}^*$, there exists a unique element $H_\nu \in \mathfrak{a}$ such that $\nu(H) = \langle H, H_\nu\rangle$ for all $H \in \mathfrak{a}$. For $H \in \mathfrak{a}$ and $r \in \mathbf{R}_{>0}$, we set $B(H,r) = \{X \in \mathfrak{a} \mid |X - H| < r\}$.

For $\tau \in \hat{K}$, we denote by $\hat{M}(\tau)$ the subset of $\hat{M}$ contained in the restriction of τ to M. For $\delta, \tau \in \hat{K}$, we write $\hat{M}(\delta,\tau) = \hat{M}(\delta) \cap \hat{M}(\tau)$. We denote the degree of τ by $d(\tau)$ and the character of τ by χ_τ. We set $\xi_\tau = d(\tau)\bar{\chi_\tau}$. We set for $k, k_1, k_2 \in K,\ g \in G$ that

$$
\begin{aligned}
L^p_\tau(G) &= \left\{ f \in L^p(G) \,\middle|\, f *_K \xi_\tau = f \right\}, \\
L^p(G,\tau) &= \left\{ F \in L^p(G, \mathrm{End}(V_\tau)) \,\middle|\, F(gk) = \tau(k)^{-1}F(g) \right\}, \\
L^p(G,V_\tau) &= \left\{ f : G \to V_\tau \,\middle|\, \int_G \|f(g)\|^p_{V_\tau} dg < \infty,\ f(gk) = \tau(k)^{-1}f(g) \right\} \\
L^p(G,\tau,\tau) &= \left\{ F \in L^p(G, \mathrm{End}(V_\tau)) \,\middle|\, F(k_1gk_2) = \tau(k_2)^{-1}F(g)\tau(k_1)^{-1} \right\}.
\end{aligned}
$$

Let $\mathcal{D}_\tau(G)$ (resp. $\mathcal{D}(G,\tau)$, $\mathcal{D}(G,V_\tau)$, $\mathcal{D}(G,\tau,\tau)$) be the subset of $L^p_\tau(G)$ (resp. $L^p(G,\tau)$, $L^p(G,V_\tau)$, $L^p(G,\tau,\tau)$) comprised of all compactly supported C^∞-functions. For $f \in \mathcal{D}_\tau(G)$, we set $F_f(g) = \int_K f(gk)\tau(k)dk$. Then the mapping $f \mapsto F_f$ is a topological isomorphism of $\mathcal{D}_\tau(G)$ onto $\mathcal{D}(G,\tau)$ and its inverse is the mapping $F \mapsto d(\tau)\mathrm{Tr}F$, $(F \in \mathcal{D}(G,\tau))$ (cf.

[8], p. 397). For $f \in L^p(G, V_\tau)$ and $v \in V_\tau$, we define $f \otimes v$ by

$$\langle (f \otimes v)(g), w \rangle_{V_\tau} = \langle w, v \rangle_{V_\tau} f(g), \quad \text{for all } w \in V_\tau.$$

For $f \in L^p(G, V_\tau)$ and $v \in V_\tau$, we have

$$\|f \otimes v\|_{L^p(G,\tau)} = \|f\|_{L^p(G,V_\tau)} \|v\|_{V_\tau}, \tag{1}$$

and thus $f \otimes v \in L^p(G, \tau)$. For $F_1, F_2 \in \mathcal{D}(G, \tau)$, we define the convolution $F_1 * F_2$ by

$$(F_1 * F_2)(g) = \int_G F_1(x^{-1}g) F_2(x) dx. \tag{2}$$

This definition is arranged so that $F_1 * F_2 \in \mathcal{D}(G, \tau)$. And we also define the convolution for $\Psi \in \mathcal{D}(G, \tau, \tau)$ and $f \in \mathcal{D}(G, V_\tau)$ by

$$(\Psi * f)(g) = \int_G \Psi(x^{-1}g) f(x) dx. \tag{3}$$

It is easy to show that $\Psi * (f \otimes v) = (\Psi * f) \otimes v$.

3. The vector-valued Helgason–Fourier transform

Let (σ, H_σ) be a unitary representation of M and $\nu \in \sqrt{-1}\mathfrak{a}^*$. We denote by $\pi_{\sigma,\nu}$ the representation induced from $\sigma \otimes \nu \otimes 1$ of P to G. The representation space $\mathcal{H}^{\sigma,\nu}$ is

$$\mathcal{H}^{\sigma,\nu} = \{\varphi \in L^2(K, H_\sigma) \mid \varphi(km) = \sigma(m)^{-1}\varphi(k),\ m \in M,\ k \in K\},$$

with the norm

$$\|\varphi\|_{\mathcal{H}^{\sigma,\nu}} = \int_K \|\varphi(k)\|^2_{H_\sigma} dk.$$

The action of $\pi_{\sigma,\nu}$ on $\mathcal{H}^{\sigma,\nu}$ is given by

$$(\pi_{\sigma,\nu}(g)\varphi)(k) = e^{-(\nu+\rho)H(g^{-1}k)} \varphi(\kappa(g^{-1}k)).$$

It is known that $(\pi_{\sigma,\nu}, \mathcal{H}^{\sigma,\nu})$ is unitary. We set

$$\mathcal{H}^{\sigma,\nu}_\tau = \{\varphi \in \mathcal{H}^{\sigma,\nu} \mid \varphi *_K \xi_\tau = \varphi\}.$$

For $P \in \mathrm{Hom}_M(V_\tau, H_\sigma)$, $v \in V_\tau$, we write $\varphi_{P\otimes v}(k) = P(\tau(k)^{-1}v)$. Then the mapping $P \otimes v \mapsto \varphi_{P\otimes v}$ is a bijection of $\mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$ onto $\mathcal{H}_\tau^{\sigma,\nu}$. For $f \in L^1(G)$, its Fourier transform on G is defined by

$$\pi_{\sigma,\nu}(f) = \int_G f(g)\pi_{\sigma,\nu}(g)dg. \tag{4}$$

R. Camporesi defined the Helgason–Fourier transform of $f \in L^1(G, V_\tau)$ by

$$\tilde{f}(k,\nu) = \int_G e^{-(\nu+\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))^{-1}f(g)dg, \tag{5}$$

for $k \in K$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. The Plancherel formula for $f \in \mathcal{D}(G, V_\tau)$ is given by

$$\|f\|^2_{L^2(G,V_\tau)} = \sum_{P'} c_{P'} \sum_{\sigma'}$$
$$\frac{1}{d_{\tilde{\sigma}'}} \int_{\mathfrak{a}'^*} \int_K \langle T_{\tilde{\sigma}'}\tilde{f}(k, \nu' + \sqrt{-1}\mu_1), T_{\tilde{\sigma}'}\tilde{f}(k, \nu' - \sqrt{-1}\mu_1)\rangle_{V_\tau} p_{\sigma'}(\nu')d\nu' dk,$$

(see [2], p. 286). The relation between (4) and (5) is given by the following proposition.

Proposition 3.1. *If $f \in L^1_\tau(G)$ and $T \otimes v \in \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$, then we have*

$$(\pi_{\sigma,\nu}(f)\varphi_{T\otimes v})(k) = T(\tilde{f}_v(k,\nu)),$$

where $f_v(g) = F_f(g)v$.

Proof. We have

$$\begin{aligned} T(\tilde{f}_v(k,\nu)) &= T\Big(\int_G e^{-(\nu+\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))^{-1}\int_K f(gk_1)\tau(k_1)dk_1 v dg\Big) \\ &= \int_G f(g)e^{-(\nu+\rho)H(g^{-1}k)}T(\tau(\kappa(g^{-1}k))^{-1}v)dg \\ &= \int_G f(g)\pi_{\sigma,\nu}(g)\varphi_{T\otimes v}(k)dg \\ &= (\pi_{\sigma,\nu}(f)\varphi_{T\otimes v})(k). \quad \square \end{aligned}$$

Let $f \in L^1(G, V_\tau)$. We have

$$\begin{aligned}
\tilde{f}(k,\nu) &= \int_G e^{-(\nu+\rho)H(g^{-1})}\tau(\kappa(g^{-1}))^{-1}f(kg)dg \\
&= \int_A\int_N\int_K e^{-(\nu+\rho)H(k_1^{-1}n^{-1}a^{-1})}\tau(\kappa(k_1^{-1}n^{-1}a^{-1}))^{-1}f(kank_1)dadndk_1 \\
&= \int_A\int_N e^{-(\nu+\rho)H(a^{-1})}f(kan)dadn = \int_A e^{\nu H(a)}\int_N e^{\rho H(a)}f(kan)dnda.
\end{aligned}$$

For $k \in K$ and $a \in A$, we set

$$Rf(k,a) = \int_N e^{\rho H(a)}f(kan)dn. \tag{6}$$

We call Rf the (vector-valued) Radon transform of f. And also, define the Fourier transform of $f \in L^1(K \times A)$ on A by

$$\mathcal{F}_A f(k,\nu) = \int_A e^{\nu H(a)}f(k,a)da$$

for $k \in K$.

Let $\sigma \in \hat{M}(\delta,\tau)$. In the following, we write $m_\sigma = [\tau|_M : \sigma]$ and $n_\sigma = [\delta|_M : \sigma]$. Let $\{P_{\sigma,j}\}_{j=1,2,\cdots,m_\sigma}$ and $\{Q_{\sigma,j}\}_{j=1,2,\cdots,n_\sigma}$ be bases of $\mathrm{Hom}_M(V_\tau, H_\sigma)$ and $\mathrm{Hom}_M(V_\delta, H_\sigma)$, respectively, such that

$$\mathrm{Tr}(P_{\sigma,i}^*P_{\sigma,j}) = d(\sigma)\delta_{ij}, \quad \mathrm{Tr}(Q_{\sigma,i}^*Q_{\sigma,j}) = d(\sigma)\delta_{ij}.$$

For $\sigma \in \hat{M}(\delta,\tau)$, we set $T_{\sigma,ij} = Q_{\sigma,j}^*P_{\sigma,i} \in \mathrm{Hom}_M(V_\tau, V_\delta)$. Let $\{v_\ell\}_{\ell=1,2,\cdots,d(\tau)}$ and $\{w_\ell\}_{\ell=1,2,\cdots,d(\delta)}$ be orthonormal bases of V_τ and V_δ, respectively.

Lemma 3.2. *The set*

$$\left\{ \frac{1}{\sqrt{d(\sigma)}}T_{\sigma,ij} \;\middle|\; \sigma \in \hat{M}(\delta,\tau),\ i = 1,2,\cdots,m_\sigma,\ j = 1,2,\cdots,n_\sigma \right\}$$

is an orthonormal basis of $\mathrm{Hom}_M(V_\tau, V_\delta)$.

Proof. For $k = 1,2,\cdots,m_\sigma$ and $\ell = 1,2,\cdots,n_\sigma$, we have

$$\langle T_{\sigma,ij}, T_{\sigma,k\ell}\rangle_{\mathrm{Hom}_M(V_\tau,V_\delta)} = \mathrm{Tr}(T_{\sigma,k\ell}^*T_{\sigma,ij}) = d(\sigma)\delta_{ik}\delta_{j\ell}.$$

In a similar fashion,

$$\langle T_{\sigma,ij}, T_{\mu,k\ell}\rangle_{\mathrm{Hom}_M(V_\tau,V_\delta)} = 0,$$

for $\mu, \sigma \in \hat{M}(\delta,\tau)$ such that $\mu \not\cong \sigma$. For $T \in \mathrm{Hom}_M(V_\tau, V_\delta)$, we obtain

$$T = 1_{V_\delta} T 1_{V_\tau} = \sum_{\mu\in\hat{M}(\delta)} \sum_{j=1}^{n_\mu} \sum_{\sigma\in\hat{M}(\tau)} \sum_{i=1}^{m_\sigma} Q^*_{\mu,j} Q_{\mu,j} T P^*_{\sigma,i} P_{\sigma,i}.$$

Let

$$_{(\sigma,i)}\mathrm{Hom}_M(V_\tau, V_\delta)_{(\mu,j)} = \left\{ Q^*_{\mu,j} Q_{\mu,j} T P^*_{\sigma,i} P_{\sigma,i} \;\middle|\; T \in \mathrm{Hom}_M(V_\tau, V_\delta) \right\}.$$

Then we have

$$\text{(7)} \quad \mathrm{Hom}_M(V_\tau, V_\delta) = \sum_{\sigma\in\hat{M}(\tau)} \sum_{\mu\in\hat{M}(\delta)} \sum_{i=1}^{m_\sigma} \sum_{j=1}^{n_\mu} {}_{(\sigma,i)}\mathrm{Hom}_M(V_\tau, V_\delta)_{(\mu,j)}.$$

From $Q_{\mu,j} T P^*_{\sigma,i} \in \mathrm{Hom}_M(H_\sigma, H_\mu)$, we have $_{(\sigma,i)}\mathrm{Hom}_M(V_\tau, V_\delta)_{(\mu,j)} = 0$ for $\mu \not\cong \sigma$. Since $Q_{\sigma,j} T P^*_{\sigma,i} \in \mathrm{End}_M(H_\sigma)$, there exists $c(T) \in \mathbf{C}$ such that $Q_{\sigma,j} T P^*_{\sigma,i} = c(T) 1_{H_\sigma}$. Therefore,

$$Q^*_{\sigma,j} Q_{\sigma,j} T P^*_{\sigma,i} P_{\sigma,i} = c(T) T_{\sigma,ij}.$$

Especially, we have

$$Q^*_{\sigma,j} Q_{\sigma,j} T_{\sigma,ij} P^*_{\sigma,i} P_{\sigma,i} = T_{\sigma,ij}.$$

Hence $c(T_{\sigma,ij}) = 1$ and $_{(\sigma,i)}\mathrm{Hom}_M(V_\tau, V_\delta)_{(\sigma,j)} = \mathbf{C} T_{\sigma,ij}$. From (7), we obtain

$$\begin{aligned} \mathrm{Hom}_M(V_\tau, V_\delta) &= \sum_{\sigma\in\hat{M}(\tau)} \sum_{\sigma\in\hat{M}(\delta)} \sum_{i=1}^{m_\sigma} \sum_{j=1}^{n_\sigma} {}_{(\sigma,i)}\mathrm{Hom}_M(V_\tau, V_\delta)_{(\sigma,j)} \\ &= \sum_{\sigma\in\hat{M}(\delta,\tau)} \sum_{i=1}^{m_\sigma} \sum_{j=1}^{n_\sigma} \mathbf{C} T_{\sigma,ij}. \quad \square \end{aligned}$$

Let $\delta, \tau \in \hat{K}$. For $T \in \mathrm{Hom}_M(V_\tau, V_\delta)$, we set

$$E(T,\nu,g) = \int_K \delta(k) T \tau(\kappa(g^{-1}k))^{-1} e^{-(\nu+\rho)H(g^{-1}k)} dk.$$

The function $E(T,\nu,g)$ is so called the Eisenstein integral. In case $\xi_\delta *_K f = f$, R. Camporesi gave the expression of the Helgason–Fourier transform $\tilde{f}(k,\nu)$ in terms of the Eisenstein integrals.

Proposition 3.3. ([2]) *If $\xi_\delta *_K f = f$, for $f \in \mathcal{D}(G, V_\tau)$, then*

$$\tilde{f}(k,\nu) = \sum_{\sigma\in\hat{M}(\delta,\tau)} \frac{d(\delta)}{d(\tau)} \sum_{i=1}^{m_\sigma}\sum_{j=1}^{n_\sigma} T^*_{\sigma,ij}\delta(k)^{-1}\int_G E(T_{\sigma,ij},\nu,g)f(g)dg.$$

We define the Helgason–Fourier transform $\hat{F}(k,\nu) \in \mathrm{End}(V_\tau)$ of $F \in \mathcal{D}(G,\tau)$ by

$$\hat{F}(k,\nu) = \int_G e^{-(\nu+\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))^{-1}F(g)dg.$$

From the definition of $\hat{F}$ for $F \in \mathcal{D}(G,\tau)$, we have $\hat{F}(k,\nu)v = \widetilde{(Fv)}(k,\nu)$ for $v \in V_\tau$. For $\Psi \in \mathcal{D}(G,\tau,\tau)$, we have

$$\begin{aligned}
\hat{\Psi}(k,\nu) &= \int_G e^{-(\nu+\rho)H(g^{-1}k)}\tau(\kappa(g^{-1}k))^{-1}\Psi(g)dg \\
&= \int_{AN} e^{(\nu+\rho)(\log a)}\Psi(kan)dadn \\
&= \int_{AN} e^{(\nu+\rho)(\log a)}\Psi(an)dadn\tau(k)^{-1}.
\end{aligned}$$

Therefore, we define the Fourier transform of $\Psi \in \mathcal{D}(G,\tau,\tau)$ by

$$\hat{\Psi}(\nu) = \int_{AN} e^{(\nu+\rho)(\log a)}\Psi(an)dadn.$$

Remark. R. Camporesi (cf. [2]) defined the Fourier transform of $\Psi \in \mathcal{D}(G,\tau,\tau)$ by

$$\hat{\Psi}_\sigma(\nu)(P) = \sum_{j=1}^{m_\sigma}\frac{1}{d(\sigma)}\int_G \mathrm{Tr}(\Psi(g)E(P^*_{\sigma,j}P,\nu,g))dgP_{\sigma,j},$$

for $P \in \mathrm{Hom}_M(V_\tau, H_\sigma)$. Each $v \in V_\tau$ can be decomposed into

$$v = \sum_{\sigma\in\hat{M}(\tau)}\sum_{i=1}^{m_\sigma} P^*_{\sigma,i}P_{\sigma,i}v \in \sum_{\sigma\in\hat{M}(\tau)} \mathrm{Hom}_M(V_\tau,H_\sigma)\otimes H_\sigma.$$

Accordingly, for $v \in V_\tau$, the relation between $\hat{\Psi} \in \mathrm{End}(V_\tau)$ and $\hat{\Psi}_\sigma$ is

$$\begin{aligned}\hat{\Psi}(\nu)v &= \sum_{\sigma\in\hat{M}(\tau)}\sum_{i=1}^{m_\sigma} P^*_{\sigma,i}\hat{\Psi}_\sigma(\nu)(P_{\sigma,i})v \\ &= \sum_{\sigma\in\hat{M}(\tau)}\sum_{i=1}^{m_\sigma}\sum_{j=1}^{m_\sigma} P^*_{\sigma,i}\frac{1}{d(\sigma)}\int_G \mathrm{Tr}(\Psi(g)E(P^*_{\sigma,j}P_{\sigma,i},\nu,g))dgP_{\sigma,j}v.\end{aligned}$$

We have the following proposition.

Proposition 3.4. *If $f \in \mathcal{D}(G, V_\tau)$ and $v \in V_\tau$, then the Helgason–Fourier transform of $f(g) \otimes v \in \mathcal{D}(G,\tau)$ is given by*

$$(\widehat{f(g)\otimes v})(k,\nu) = \tilde{f}(k,\nu)\otimes v.$$

From the definition of $\hat{\Psi}$ for $\Psi \in \mathcal{D}(G,\tau,\tau)$, we have

Proposition 3.5. *If $\Psi \in \mathcal{D}(G,\tau,\tau)$, then we have $\hat{\Psi}(k,\nu) = \hat{\Psi}(\nu)\tau(k)^{-1}$.*

The next proposition can be proved by using a similar argument to the K-biinvariant case.

Proposition 3.6. *Let $\Psi \in \mathcal{D}(G,\tau,\tau)$ and $F \in \mathcal{D}(G,\tau)$. Then we have*

$$(\widehat{\Psi * F})(k,\nu) = \hat{\Psi}(\nu)\hat{F}(k,\nu).$$

From Proposition 3.6, we have

Corollary 3.7. *If $\Psi \in \mathcal{D}(G,\tau,\tau)$ and $f \in \mathcal{D}(G,V_\tau)$, then*

$$(\widetilde{\Psi * f})(k,\nu) = \hat{\Psi}(\nu)\tilde{f}(k,\nu).$$

4. The Cowling–Price theorem for vector-valued Helgason–Fourier transform

In this section, we shall prove the Cowling–Price theorem for a vector-valued function over G/K. The following is the Cowling–Price theorem for a vector-valued function on $\mathbf{R}^n$.

Lemma 4.1. *Let $a, b > 0$, $1 \leqq p, q \leqq \infty$, $\min(p, q) < \infty$ and V a finite-dimensional vector space. Let f be a measurable V-valued function on $\mathbf{R}^n$ such that*

$$\|e^{ax^2} f(x)\|_{L^p(\mathbf{R}^n,V)} < \infty, \quad \|e^{by^2} \hat{f}(y)\|_{L^q(\mathbf{R}^n,V)} < \infty.$$

If $ab \geqq 1/4$, then $f = 0$ (a.e.).

Proof. For $v \in V$, we set $h(x) = \langle f(x), v\rangle_V$. Then

$$\langle \hat{f}(y), v\rangle_V = \langle \int_{\mathbf{R}^n} e^{\sqrt{-1}xy} f(x)dx, v\rangle_V = \hat{h}(y). \tag{8}$$

We have

$$\begin{aligned}\int_{\mathbf{R}^n} |e^{ax^2} h(x)|^p dx &= \int_{\mathbf{R}^n} e^{pax^2} |\langle f(x), v\rangle_V|^p dx \\ &\leqq \int_{\mathbf{R}^n} e^{pax^2} \|f(x)\|_V^p \|v\|_V^p dx < \infty.\end{aligned}$$

Similarly, from (8), we have $\|e^{by^2}\hat{h}(y)\|_{L^q(\mathbf{R}^n)} < \infty$. Applying the Cowling–Price theorem to h, we obtain $h = 0$ (a.e.). Then $f = 0$ (a.e.). $\square$

Let $\psi \in C_0^\infty(A)$ be a non-negative W-invariant function with $\mathrm{supp}(\psi \circ \exp) \subsetneqq B(0, 1)$ and $\int_{\mathfrak{a}_+} \psi(\exp H)D(\exp H)dH = 1$. For $H \in \mathfrak{a}_+$, $\varepsilon > 0$ and $k_1, k_2 \in K$, we set

$$\Psi_\varepsilon(k_1 \exp H k_2) = \varepsilon^{-d} D(\exp H)^{-1} D(\exp \varepsilon^{-1} H)\psi(\exp \varepsilon^{-1} H)\tau(k_1 k_2)^{-1},$$

and

$$\psi_\varepsilon(\exp H) = \varepsilon^{-d} D(\exp H)^{-1} D(\exp \varepsilon^{-1} H)\psi(\exp \varepsilon^{-1} H).$$

From the smoothness of Ψ_ε, Ψ_ε is well-defined on G. And also, we set

$$\int_{\mathfrak{a}_+} \psi_\varepsilon(\exp H)D(\exp H)dH = 1.$$

The next lemma is given by the same way to [13].

Lemma 4.2. (cf. [13]) *If $f \in L^p(G, V_\tau)$ and $1 \leqq p \leqq \infty$, then*

$$\lim_{\varepsilon \to 0} \|\Psi_\varepsilon * f - f\|_{L^p(G,V_\tau)} = 0.$$

We have the following Cowling–Price theorem for V_τ-valued functions.

Theorem 4.3. *Let $1 \leqq p, q \leqq \infty$ and $a, b, C > 0$. Let f be a measurable V_τ-valued function such that*

$$\|e^{a\sigma(g)^2} f(g)\|_{L^p(G,V_\tau)} < C, \quad \|e^{b|\nu|^2} \tilde{f}(k,\nu)\|_{L^q(K\times\sqrt{-1}\mathfrak{a}^*,V_\tau,\mu(\nu)d\nu dk)} < C,$$

where $\mu(\nu)$ is a positive function on $\sqrt{-1}\mathfrak{a}^*$ of polynomial order. If $ab > 1/4$, then $f = 0$ (a.e.).

Proof. At first, by using a similar argument of J. Sengupta [11], we have $\tilde{f} = 0$.

Secondly, we shall show $f = 0$ (a.e.). If $1 \leqq p', q' \leqq \infty$ and $r^{-1} = p'^{-1} + q'^{-1} - 1 \geqq 1$, then the Young inequality implies that

$$\begin{aligned} \frac{1}{d(\tau)}\|u\|_{V_\tau}\|\Psi * f\|_{L^r(G,V_\tau)} &= \|\Psi * F_{\langle f,u\rangle}\|_{L^r(G,\tau,\tau)} \\ &\leqq \|\Psi\|_{L^{p'}(G,\tau,\tau)}\|F_{\langle f,u\rangle}\|_{L^{q'}(G,\tau,\tau)} \\ &= \frac{1}{d(\tau)}\|u\|_{V_\tau}\|\Psi\|_{L^{p'}(G,\tau,\tau)}\|f\|_{L^{q'}(G,V_\tau)}, \end{aligned}$$

for $u \in V_\tau$ and $\Psi \in \mathcal{D}(G, \tau, \tau)$. From the assumption, we have $\Psi * f \in L^1(G, V_\tau) \cap L^2(G, V_\tau)$. Corollary 3.7 implies $\widetilde{\Psi * f} = \hat{\Psi}\tilde{f} = 0$. Then we obtain $\Psi * f = 0$. As shown in Lemma 4.2, we can compose $\{\Psi_\varepsilon\}_{\varepsilon>0}$ such that

$$\lim_{\varepsilon \to 0} \|\Psi_\varepsilon * f - f\|_{L^1(G,V_\tau)} = 0.$$

Therefore, this proves $f = 0$ (a.e.). □

5. The Cowling–Price theorem for semisimple Lie groups

We need the following lemma, which can be proved by a slight modification of [3].

Lemma 5.1. *Let* $1 \leqq p \leqq \infty$, $s > 0$ *and* $A > 0$. *Let* g *be an entire function on* $\mathbf{C}$ *such that*

$$\begin{aligned} |g(x+\sqrt{-1}y)| &\leqq Ae^{\pi x^2}, \quad (x, y \in \mathbf{R}), \\ \left(\int_{\mathbf{R}} |g(x)|^p |x|^s dx\right)^{1/p} &\leqq A. \end{aligned}$$

Then g *is a constant function on* $\mathbf{C}$. *Moreover, if* $p < \infty$ *then* $g = 0$.

Let $\mu(\sigma, \nu)$ be the Harish-Chandra μ-function. We need an analogous one for general μ-functions.

Lemma 5.2. *Let* $\sigma \in \hat{M}$ *and* $\nu \in \sqrt{-1}\mathfrak{a}^*$. *Then there exist* $B_1, B_2 > 0$, $t \geqq 0$ *and* $s \in \mathbf{R}$ *such that*

$$B_1\mu(\sigma,\nu) \leqq \prod_{\alpha\in\Delta^+} \left|\frac{\langle\nu,\alpha\rangle}{4\langle\rho,\alpha\rangle}\right|^t \left(1+\left|\frac{\langle\nu,\alpha\rangle}{4\langle\rho,\alpha\rangle}\right|\right)^s \leqq B_2\mu(\sigma,\nu).$$

Proof. In [15, p. 47], there exist $a_i, b_i, (i = 1, \cdots, m), c_j, d_j, (j = 1, \cdots, n)$ such that

$$\mu(\sigma,\nu) = \prod_{\alpha\in\Delta^+} \prod_{\substack{1\leqq i\leqq m \\ 1\leqq j\leqq n}} \frac{\Gamma(\langle\nu,\alpha\rangle/4\langle\rho,\alpha\rangle - a_i)\Gamma(-\langle\nu,\alpha\rangle/4\langle\rho,\alpha\rangle - c_j)}{\Gamma(\langle\nu,\alpha\rangle/4\langle\rho,\alpha\rangle - b_i)\Gamma(-\langle\nu,\alpha\rangle/4\langle\rho,\alpha\rangle - d_j)}.$$

In [14, p. 96], Trombi proved that a_i and c_j must be real numbers. By considering zeros of the Plancherel measure (cf. [10], p. 536) and the property of $\Gamma(z)$, b_i and d_j must be real numbers. Let m_0 and n_0 be numbers of the case $b_i \neq 0$ and $d_j \neq 0$, respectively. In a similar fashion to [7], we can find $B_1, B_2 > 0$ such that

$$B_1\mu(\sigma,\nu) \leqq \prod_{\alpha\in\Delta^+} \left|\frac{\langle\nu,\alpha\rangle}{4\langle\rho,\alpha\rangle}\right|^t \left(1+\left|\frac{\langle\nu,\alpha\rangle}{4\langle\rho,\alpha\rangle}\right|\right)^s \leqq B_2\mu(\sigma,\nu),$$

where

$$\begin{aligned} t &= m+n-m_0-n_0 \\ s &= -ma_i + m_0b_i - nc_j + n_0d_j - m - n + m_0 + n_0. \quad \square \end{aligned}$$

Finally, we shall prove the Cowling–Price theorem for G.

Theorem 5.3. *Let $1 \leqq p,q \leqq \infty$ and $a,b,C,C_\sigma > 0$. Let f be a measurable function on G such that*

$$\|e^{a\sigma(g)^2} f(g)\|_{L^p(G)} < C, \quad \|e^{b|\nu|^2}\pi_{\sigma,\nu}(f)\|_{L^q(\sqrt{-1}\mathfrak{a}^*,\mathbf{B}(\mathcal{H}^\sigma))} < C_\sigma.$$

If $ab > 1/4$, then $f = 0$ (a.e.).

Proof. It is sufficient to prove the case when $f = \xi_\delta *_K f *_K \xi_\tau$. First assumption and $f_v \in L^1(G, V_\tau)$ imply that

$$\|e^{a\sigma(g)^2} f_v(g)\|_{L^p(G,V_\tau)} < C. \tag{9}$$

From $f \in L^1_\tau(G)$ and Proposition 3.1, we have

$$(\pi_{\sigma,\nu}(f)\varphi_{P\otimes v})(k) = P(\tilde{f}_v(k,\nu))$$

for $P \otimes v \in \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$. We also have $\pi_{\sigma,\nu}(f)\varphi_{P\otimes v} \in \mathcal{H}_\delta^{\sigma,\nu} \cong \mathrm{Hom}_M(V_\delta, H_\sigma) \otimes V_\delta$. Therefore, we obtain

$$\tilde{f}_v(k,\nu) = 1_{V_\tau}\tilde{f}_v(k,\nu) = \sum_{\sigma\in\hat{M}(\delta,\tau)}\sum_{i=1}^{m_\sigma} P_{\sigma,i}^* P_{\sigma,i}\tilde{f}_v(k,\nu)$$

$$= \sum_{\sigma\in\hat{M}(\delta,\tau)}\sum_{i=1}^{m_\sigma}\sum_{j=1}^{n_\sigma}\sum_{\ell=1}^{d(\delta)}\langle\pi_{\sigma,\nu}(f)\varphi_{P_{\sigma,i}\otimes v}, \varphi_{Q_{\sigma,j}\otimes w_\ell}\rangle_{\mathcal{H}^{\sigma,\nu}} P_{\sigma,i}^* Q_{\sigma,j}(\delta(k)^{-1}w_\ell),$$

where $P_{\sigma,i}^* Q_{\sigma,j} \in \mathrm{Hom}_M(V_\delta, V_\tau)$. And we see that

$$\|\varphi_{P_{\sigma,j}\otimes v}\|^2 = \langle\varphi_{P_{\sigma,j}\otimes v}, \varphi_{P_{\sigma,j}\otimes v}\rangle_{\mathcal{H}^{\sigma,\nu}} = \frac{1}{d(\tau)}\langle v,v\rangle_{V_\tau}\mathrm{Tr}(P_{\sigma,j}^* P_{\sigma,j}) = \frac{d(\sigma)}{d(\tau)}\|v\|_{V_\tau}^2$$

(cf. [2], p. 281). Hence we obtain

$$\|\tilde{f}_v(k,\nu)\|_{L^q(K\times\sqrt{-1}\mathfrak{a}^*,V_\tau,\mu(\nu)dkd\nu)}$$

$$\leqq \sum_{\sigma\in\hat{M}(\delta,\tau)}\sum_{i=1}^{m_\sigma}\sum_{j=1}^{n_\sigma}\sum_{\ell=1}^{d(\delta)}\|\langle\pi_{\sigma,\nu}(f)\varphi_{P_{\sigma,i\otimes v}}, \varphi_{Q_{\sigma,j}\otimes w_\ell}\rangle_{\mathcal{H}^{\sigma,\nu}}\|_{L^q(\sqrt{-1}\mathfrak{a}^*,\mu(\nu)d\nu)}$$
$$\times\|P_{\sigma,i}^* Q_{\sigma,j}(\delta(k)^{-1}w_\ell)\|_{L^q(K,V_\tau)}$$

$$\leqq \sum_{\sigma\in\hat{M}(\delta,\tau)}\sum_{i=1}^{m_\sigma}\sum_{j=1}^{n_\sigma}\sum_{\ell=1}^{d(\delta)}\|\pi_{\sigma,\nu}(f)\|_{L^q(\sqrt{-1}\mathfrak{a}^*,\mathbf{B}(\mathcal{H}^\sigma),\mu(\nu)d\nu)}\frac{d(\sigma)}{(d(\tau)d(\delta))^{1/2}}\|v\|_{V_\tau}$$

$$\leqq \sum_{\sigma\in\hat{M}(\delta,\tau)}\frac{d(\sigma)d(\delta)}{d(\tau)}m_\sigma n_\sigma\|v\|_{V_\tau}\|\pi_{\sigma,\nu}(f)\|_{L^q(\sqrt{-1}\mathfrak{a}^*,\mathbf{B}(\mathcal{H}^\sigma),\mu(\nu)d\nu)}.$$

$$z(\nu,\rho) = \prod_{\alpha\in\Delta^+} \left|\frac{\langle\nu,\alpha\rangle}{4\langle\rho,\alpha\rangle}\right|^t.$$

Lemma 5.2, the second assumption and the Hölder inequality imply that

$$\|e^{b_1|\nu|^2}\tilde{f}_v(k,\nu)\|_{L^1(K\times\sqrt{-1}\mathfrak{a}^*,V_\tau,z(\nu,\rho)dkd\nu)} < \infty, \tag{10}$$

where $C_1 > 0$ and $0 \leqq b_1 < b$ such that $ab_1 > 1/4$. Applying (9) and (10) to Theorem 4.3 and Lemma 5.1, we conclude $f = 0$ (a.e.). □

参考文献

[1] S. C. Bagchi and S. K. Ray, *Uncertainty principle like Hardy's theorem on some Lie groups*, J. Austral. Math. Soc. **65** Ser. A (1998), 289-302.

[2] R. Camporesi, *The Helgason Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces*, Pacific J. Math. **179** (1997), 263-300.

[3] M. G. Cowling and J. F. Price, *Generalizations of Heisenberg's inequality*, Lecture Notes in Math 992, Springer-Verlag, Berlin, 1983, 443-449.

[4] M. Cowling, A. Sitaram and M. Sundari, *Hardy's uncertainty principle on semisimple groups*, Pacific J. Math. **192**, (2000), 293-296.

[5] M. Ebata, M. Eguchi, S. Koizumi and K. Kumahara, *A generalization of the Hardy theorem to semisimple Lie groups*, Proc. Japan Acad. **75**, Ser. A (1999), 113-114.

[6] M. Eguchi, S. Koizumi and K. Kumahara, *An L^p version of the Hardy theorem for motion groups,* J. Austral. Math. Soc. **68**, Ser. A (2000), 55-67.

[7] M. Eguchi and K. Kumahara, *An L^p Fourier Analysis on Symmetric Spaces,* J. Funct. Anal. **47**, (1987), 230-246.

[8] S. Helgason, *Groups and Geometric Analysis*, Academic Press, New York, 1984.

[9] A. Knapp, *Representation Theory of Semisimple Groups, an Overview based on Examples*, Princeton University Press, 1986.

[10] A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple Lie groups*, Ann. of Math. **93** (1971), 489-578.

[11] J. Sengupta, *The uncertainty principle on Riemannian symmetric spaces of the noncompact type*, (preprint).

[12] A. Sitaram and M. Sundari, *An analogue of Hardy's theorem for very rapidly decreasing functions on semi-simple Lie groups*, Pacific J. Math. **177** (1997), 187-200.

[13] R. J. Stanton and P. A. Tomas, *Pointwise inversion of the spherical transform on $L^p(G/K)$, $1 \leq p < 2$*, Proc. Amer. Math. Soc. **73** (1979), 398-404.

[14] P. C. Trombi, *Harmonic analysis of $C^p(G: F)(1 \leq p < 2)$*, J. Funct. Anal. **40** (1981), 84-125.

[15] N. R. Wallach, *Real Reductive Groups* II, Academic Press, Inc., Boston, MA, 1992.

[16] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups*, Vol. I, II, Springer-Verlag, New York, 1972.