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TWISTED RADON TRANSFORMS ON GRASSMANN MANIFOLDS

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ABSTRACT. グラスマン多様体上の線型束の切片の満たす一般的微分作用素の具体的構成を、線型代数の概念の量子化というアイデアで構成し、グラスマン多様体上の線型束の切片やコホモロジー空間からの積分変換（ペンローズ変換、ポアソン変換、ラドン変換など）への応用について述べる。特にラドン変換の場合についてはGelfand の超幾何関数との関係や、通常のラドン変換を一般化した twisted ラドン変換について考察する。

1. Differential Equations Satisfied by Functions on Grassmannians

The Grassmann manifold is defined as follows.

$\mathbb{X}_k^n := \{k\text{-subspaces in } \mathbb{F}^n\}, \quad \mathbb{F} = \mathbb{R},\ \mathbb{C},\ \mathbb{H}$

$$M^o(n,k;\mathbb{F}) := \left\{ X = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nk} \end{pmatrix} \in M(n,k;\mathbb{F});\ \operatorname{rank} X = k \right\}$$

$$= M^o(n,k;\mathbb{F})/GL(k,\mathbb{F})$$

Here the column vectors $\mathbf{x}_\nu$ $(\nu = 1,\ldots,k)$ of an element X in $M(n,k;\mathbb{F})$ are linearly independent vectors in a k-subspace of $\mathbb{F}^n$.

Then we note that

$$\dim_{\mathbb{R}} \mathbb{X}_k^n = \bigl(nk - k^2 = (n-k)k\bigr) \times \begin{cases} 1 & \mathbb{F} = \mathbb{R}, \\ 2 & \quad\ \ \mathbb{C}, \\ 4 & \quad\ \ \mathbb{H}. \end{cases}$$

$$\mathbb{P}^{n-1}(\mathbb{F}) := \mathbb{X}_1^n = \mathbb{F}^n/\mathbb{F}^\times : \quad \text{A projective space}$$

The group $G = GL(n,\mathbb{F})$ acts on $\mathbb{X}_k^n$ (by left) and then

$$\mathbb{X}_k^n = GL(n,\mathbb{F})/P_{k,n} \qquad (= O(n)/O(k)\times O(n-k) \quad \text{if } \mathbb{F} = \mathbb{R})$$

$$P_{k,n} := \left\{ p = \begin{pmatrix} g_1 & 0 \\ y & g_2 \end{pmatrix};\ g_1 \in GL(k,\mathbb{F}),\ g_2 \in GL(n-k,\mathbb{F}),\ y \in M(n-k,k;\mathbb{F}) \right\}.$$

Consider the space of hyperfunction sections of a line bundle over the Grassmann manifold. Fix $\lambda = (\lambda_1,\lambda_2) \in \mathbb{C}^2$.

$$\begin{aligned} \mathcal{B}(G/P_{k,n};L_\lambda) &:= \{f \in \mathcal{B}(G);\ f(xp) = f(x)|\det g_1|^{\lambda_1}|\det g_2|^{\lambda_2}, \quad \forall p \in P_{k,n}\} \\ &\quad (= \mathcal{B}(O(n)/O(k)\times O(n-k)) \quad \text{if } \mathbb{F} = \mathbb{R}) \\ &= \{f \in \mathcal{B}(M^o(n,k;\mathbb{F}));\ f(Xg_1) = f(X)|\det g_1|^{-\lambda_1},\ \forall g_1 \in GL(k,\mathbb{F})\} \\ &\quad (\text{under the correspondence } G \ni x \mapsto {}^t x^{-1} \mapsto X \in M^o(n,k;\mathbb{F})) \end{aligned}$$

To study the differenatial equations satisfied by $\mathcal{B}(G/P_{k,n};L_\lambda)$ we give a notation related to the Lie algebra of $GL(n,\mathbb{C})$ as follows:

$$\mathfrak{g} := \mathbb{C} \otimes \mathrm{Lie}(G) \qquad (\simeq M(n,\mathbb{C}) \text{ if } G = GL(n;\mathbb{R}))$$

$$(Xf)(x) := \frac{d}{dt} f(xe^{tX})\big|_{t=0} \quad \left(E_{ij} = \sum_{\nu=1}^{n} x_{\nu i} \frac{\partial}{\partial x_{\nu j}},\ \ (x_{ij}) \in G\right)$$

$$(L_X f)(x) := \frac{d}{dt} f(e^{-tX} x)\big|_{t=0}$$

$$U^{\epsilon}(\mathfrak{g}) := \left(\bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g}\right) / \langle X \otimes Y - Y \otimes X - \epsilon[X,Y] \rangle$$

: Homogenized universal enveloping algebra

$$U^{\epsilon}(\mathfrak{g}) \simeq U(\mathfrak{g}) := U^1(\mathfrak{g}) \quad (X \leftrightarrow \epsilon X) \text{ if } \epsilon \neq 0. \quad S(\mathfrak{g}) := U^0(\mathfrak{g})$$

$$\lambda : \mathfrak{p}\ (:= \mathbb{C} \otimes \mathrm{Lie}(P)) \to \mathbb{C}$$

$$J^{\epsilon}_{\mathfrak{p}}(\lambda) := \sum_{X \in \mathfrak{p}} U^{\epsilon}(\mathfrak{g})(X - \lambda(X))$$

Note that $J_{\mathfrak{p}}(\lambda) := J^1_{\mathfrak{p}}(\lambda)$ is the left ideal of $U(\mathfrak{g})$ which kills $\mathcal{B}(G/P_{k,n}; L_\lambda)$ by identifying $U(\mathfrak{g})$ with the ring of left invariant differential operators on G.

If we identify $U(\mathfrak{g})$ with the ring of right invariant differential operators, the ideal killing $\mathcal{B}(G/P_{k,n}; L_\lambda)$ is given by

$$I^{\epsilon}_{\mathfrak{p}}(\lambda) := \bigcap_{g \in G} \mathrm{Ad}(g) J^{\epsilon}_{\mathfrak{p}}(\lambda) \quad \text{(two sided ideal of } U^{\epsilon}(\mathfrak{g}))$$

with $\epsilon = 1$ through the anti-automorphism $(X \mapsto -X,\ XY \mapsto (-Y)(-X))$.

Namely

$$M^{\epsilon}_{\mathfrak{p}}(\lambda) := U^{\epsilon}(\mathfrak{g})/J^{\epsilon}_{\mathfrak{p}}(\lambda) \quad \text{(a generalized Verma module of the scalar type)}$$

$$I^{\epsilon}_{\mathfrak{p}}(\lambda) = \mathrm{Ann}_{U^{\epsilon}(\mathfrak{g})}(M^{\epsilon}_{\mathfrak{p}}(\lambda)) \quad \text{if } \epsilon \neq 0.$$

Problem: Give an explicit construction of the generator system of $I^{\epsilon}_{\mathfrak{p}}(\lambda)$!

Idea: Consider the case when $\epsilon = 0$ and quantize it!.

We will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle X, Y \rangle = \mathrm{Trace}\, XY$.
Then $J^0_{\mathfrak{p}}(\lambda)$ is the defining ideal of the set

$$A_{\mathfrak{p},\lambda} := \left\{ X = \begin{pmatrix} \lambda_1 I_k & 0 \\ Y & \lambda_2 I_{n-k} \end{pmatrix};\ Y \in M(n-k, k; \mathbb{C}) \right\}$$

and our problem is to find the generator system of the defining ideal of the closure of the conjugacy class of a matrix in $M(n;\mathbb{C})$.

$$f \in \mathrm{Ad}(g) J^0_{\mathfrak{p}}(\lambda),\ \forall g \in G \ \Leftrightarrow\ f(\mathrm{Ad}(g) A_{\mathfrak{p},\lambda}) = 0,\ \forall g \in G.$$

Method I (minors, elementary devisors):

$$\mathrm{rank}(X - \lambda_1) \leq n - k \Rightarrow \det(X - \lambda_1)_{IJ} \in I^0_{\mathfrak{p}}(\lambda)$$

$$\text{with } I = \{i_1, \ldots, i_{n-k+1}\},\ J = \{j_1, \ldots, j_{n-k+1}\}.$$

$$I^0_{\mathfrak{p}}(\lambda) = \langle \det(X - \lambda_1)_{IJ}, \det(X - \lambda_2)_{I'J'};\ \#I = \#J = n - k + 1,$$
$$\#I' = \#J' = k + 1 \rangle \quad \text{if } \lambda_1 \neq \lambda_2.$$

$\Rightarrow$ elementary divisors + quantization (which means $\epsilon = 0 \mapsto 1$)!

Theorem 1. *Suppose $2k \le n$. If $\lambda_1 - \lambda_2 \notin \{k\epsilon, \cdots, (n-k)\epsilon\}$, then*

$$I^\epsilon_{\mathfrak{p}}(\lambda) = \Big\langle \det\Big(E_{i_\mu j_\nu} - (\lambda_1 + (\nu - n + k - 1)\epsilon)\delta_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le n-k+1 \\ 1\le\nu\le n-k+1}}, \quad \det\Big(E_{i'_\mu j'_\nu} - (\lambda_2 + (\nu - 1)\epsilon)\delta_{i'_\mu j'_\nu}\Big)_{\substack{1\le\mu\le k+1 \\ 1\le\nu\le k+1}}\Big\rangle.$$

If $\lambda_1 - \lambda_2 \in \{k\epsilon, \cdots, (n-k)\epsilon\}$, then

$$I^\epsilon_{\mathfrak{p}}(\lambda) = \Big\langle \frac{d}{dt}\det\Big(E_{i_\mu j_\nu} - (t + \lambda_1 + (\nu - n + k - 1)\epsilon)\delta_{i_\mu j_\nu}\Big)_{\substack{1\le\mu\le n-k+1 \\ 1\le\nu\le n-k+1}}\Big|_{t=0}, \quad \det\Big(E_{i'_\mu j'_\nu} - (\lambda_2 + (\nu - 1)\epsilon)\delta_{i'_\mu j'_\nu}\Big)_{\substack{1\le\mu\le k+1 \\ 1\le\nu\le k+1}}\Big\rangle.$$

Here $\det(A_{ij}) = \sum_\sigma \mathrm{sign}(\sigma) A_{\sigma(1)1}A_{\sigma(2)2}\cdots$ *and $I = \{i_1, \ldots, i_{n-k-1}\}$ etc.*
A similar result also holds in the case when $2k \ge n$.

Remark 1. i) The action of G on the space of the $\mathbb{C}$-span of the above (Capelli type) generators does not depend on ϵ. Hence the $\mathbb{C}$-span is a G-module.
ii) This theorem is generalized for any parabolic $\mathfrak{p}$ of $\mathfrak{gl}_n$ and any λ by [O3].
iii) $\epsilon = \lambda = 0$ ($\Rightarrow$ nilpotent variety) conjectured by Tanisaki, proved by Weyman [We] (by Kostant with moreover $\mathfrak{p}$ is a Borel subalgebra).
iv) In the case when $\mathfrak{g} = \mathfrak{o}_n$, a similar constuction is done by [Od] (together with quantized Pfaffians).

Method II (minimal polynomials)
The minimal polynomial of $X \in A_{\mathfrak{p},\lambda}$ is $q(x) = (x - \lambda_1)(x - \lambda_2)$ and if $\epsilon = 0$ and $\lambda_1 \ne \lambda_2$, we have

$$I^0_{\mathfrak{p}}(\lambda) = \langle q(X)_{ij},\ \mathrm{Trace}\, X - k\lambda_1 - (n-k)\lambda_2;\ 1 \le i \le n,\ 1 \le j \le n\rangle.$$

Theorem 2. Putting $q^\epsilon(x) = (x - \lambda_1)(x - \lambda_2 - k\epsilon)$, we have

$$I^\epsilon_{\mathfrak{p}}(\lambda) = \langle q^\epsilon(F)_{ij},\ \mathrm{Trace}\, F - k\lambda_1 - (n-k)\lambda_2;\ 1 \le i \le n,\ 1 \le j \le n\rangle$$

$$\text{if } \begin{cases} M^\epsilon_{\mathfrak{p}}(\lambda) \text{ has a regular infinitesimal character} & (\epsilon \ne 0), \\ \lambda_1 \ne \lambda_2 & (\epsilon = 0). \end{cases}$$

Here $F = (E_{ij}) \in M(n; \mathfrak{g}) \subset M(n; U^\epsilon(\mathfrak{g}))$.

Remark 2. This theorem is generalized in the case of any generalized Verma module $M^\epsilon_{\mathfrak{p}}(\lambda)$ of the scalar type for any reductive Lie algebra $\mathfrak{g}$ (with some condition for λ). The minimal polynomials are explicitly given by [OO] attached to any finite dimensional faithful completely reducible representation of $\mathfrak{g}$.
In the case of the classical Lie algebra $\mathfrak{g}$ defined by $\mathfrak{g} = \mathfrak{gl}^\sigma_N$ with an involution σ, we may naturally choose $F = (E_{ij} + \sigma(E_{ij})) \in M(N; \mathfrak{g})$ (natural representation), which is studied by [O5].

Theorem 3. *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ with $\mathfrak{b} \subset \mathfrak{p}$.*
Put $J^\epsilon(\lambda) := \sum_{X\in\mathfrak{b}} U^\epsilon(\mathfrak{g})(X - \lambda(X))$. *Then*

$$J^\epsilon_{\mathfrak{p}}(\lambda) = J^\epsilon(\lambda) + I^\epsilon_{\mathfrak{p}}(\lambda) \qquad \textbf{(GAP)}$$

if λ is regular with $\epsilon \ne 0$ (or the centralizer of λ is contained in $\mathfrak{p}$ with $\epsilon = 0$).

Remark 3. i) $M(\lambda) := U(\mathfrak{g})/J(\lambda)$ is the usual Verma module and $M_{\mathfrak{p}}(\lambda)$ is its quotient.
ii) The necessary and sufficient condition for (GAP) is obtained for any $\mathfrak{p}$ of $\mathfrak{gl}_n$ by [O3].
iii) Theorem 3 holds for the two-sided ideal constructed by the minimal polynomial

for the natural representation if $\mathfrak{g}$ is classical (with some more (possible) exceptional values for λ if $\mathfrak{g}$ is exceptional). See [O5] and [OO].

2. Applications to Integral Transformations

2.1. Penrose Transformations.
$G_{\mathbb{C}}$: a complex reductive Lie group with a real from G
$P_{\mathbb{C}}$: a parabolic subgroup of $G_{\mathbb{C}}$
V: a G-orbit in $G_{\mathbb{C}}/P_{\mathbb{C}}$
$\mathcal{O}_\lambda$: a holomorphic line bundle over $G_{\mathbb{C}}/P_{\mathbb{C}}$
The image of any G-map: $H^*_V(\mathcal{O}_\lambda) \to \{\text{functions}\}$ satisfies the system of differential equations which we constructed because the ideal is two-sided.

2.2. Poisson transformations.
G: a connected real semisimple Lie group with finite center
K: a maximal compact subgroup of G
P: a parabolic subgroup containing a minimal parabolic subgroup P_{min}

$$\begin{array}{rcll} \mathcal{P}_\lambda : \mathcal{B}(G/P;L_\lambda) & \to & (\subset \mathcal{B}(G/P_{min};L_\lambda) \xrightarrow{\mathcal{P}^{min}_\lambda}) & \mathcal{A}(G/K;\mathcal{M}_\lambda). \\ f \mapsto & & (\mathcal{P}_\lambda f)(g) = \int_K f(gk)dk & \end{array}$$

Here $\mathcal{M}_\lambda$ is a maximal ideal of the ring of invariant differential operators on G/K and $\mathcal{A}(G/K;\mathcal{M}_\lambda)$ is its solution space.

Then if $\mathcal{P}^{min}_\lambda$ is bijective ($\Leftrightarrow e(\lambda) \neq 0$ by [K–]: this condition due to [H1]) and (GAP) is valid, the image of $\mathcal{P}_\lambda$ is characterized by our system because (GAP) assures that the function in $\mathcal{B}(G/P_{min};L_\lambda)$ satisfying our system is in $\mathcal{B}(G/P;L_\lambda)$. In paticular, these assumptions are satisfiled for any G and P if $\lambda = 0$.

Remark 4. i) If G is classical and P is maximal, then our system defined from the minimal polynomial is of order two or three.
ii) The images of $\mathcal{D}'$, $\mathcal{C}^\infty$ and L^p are also characterized. The case of L^p is studied by [BOS].

3. Radon Transformations on Grassmanians

Integrate the functions on ℓ-subspace in $\mathbb{F}^n$ over k-subspaces ($k > \ell$):

$$\mathcal{R}^\ell_k : \mathcal{B}(\mathbb{X}^n_\ell) \ni \phi \mapsto (\mathcal{R}^\ell_k\phi)(g) = \int_{O(k)/O(\ell)\times O(k-\ell)} \phi(gk)dk \in \mathcal{B}(\mathbb{X}^n_k).$$

This is studied by Funk(1916), Gelfand, Helgason, Grinberg, Gonzalez, Kakehi etc.(cf. Helgason's Book [H2]).
For simplicity we assume $\mathbb{F} = \mathbb{R}$.

Fact: $\mathcal{R}^\ell_k$ is lifted up a G-map: $\mathcal{R}^\ell_k : \mathcal{B}(G/P_{\ell,n};L_{(k,0)}) \to \mathcal{B}(G/P_{k,n};L_{(\ell,0)})$

Theorem 4 ([O3]). *Suppose $0 < \ell < k < n$ and $\ell + k < n$. Then $\mathcal{R}^\ell_k$ is a toplogical G-isomorphism onto the solution space of the system:*

$$\Big\{\Phi\big((x_{ij})_{\substack{1\le i\le n\\ 1\le j\le k}}\big) \in \mathcal{B}(M^0(n,k;\mathbb{R}));$$
$$\Phi(xg) = |\det g|^{-\ell}\Phi(x) \quad \text{for } g \in GL(k,\mathbb{R}),$$
$$\det\Big(\frac{\partial}{\partial x_{i_\mu j_\nu}}\Big)_{\substack{1\le\mu\le\ell+1\\ 1\le\nu\le\ell+1}}\Phi(x) = 0 \quad (\textit{Capelli type})$$
$$\text{for } 1 \le i_1 < \cdots < i_{\ell+1} \le n,\ 1 \le j_1 < \cdots < j_{\ell+1} \le k\Big\}.$$

Remark 5. **i)** $\mathcal{B}$ can be replaced by $\mathcal{D}'$, $\mathcal{C}^\infty$ etc.
ii) The above det($\cdot$) can be replaced by a single higer order equation (cf. [Ka]).

Lemma (Generalized Capelli identity, [O3]). If $\left[\frac{\partial}{\partial x_{ij}}, x_{st}\right] = \epsilon\delta_{is}\delta_{jt}$, then

$$\det\Big(\sum_{\nu=1}^{n} x_{\nu i_k}\frac{\partial}{\partial x_{\nu j_\ell}} + \epsilon(m-\ell)\delta_{i_k j_\ell}\Big)_{\substack{1\le k\le m\\ 1\le\ell\le m}}$$
$$= \sum_{1\le\nu_1<\cdots<\nu_m\le n} \det\Big(x_{\nu_p i_q}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}} \cdot \det\Big(\frac{\partial}{\partial x_{\nu_p i_q}}\Big)_{\substack{1\le p\le m\\ 1\le q\le m}}$$

for $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_m\}$.

Remark 6. If $m > n$, the above equals 0. ($m = n \Rightarrow$ Capelli identity [Ca]).

4. Hypergeometric Functions

P: a parabolic subgroup of a real reductive Lie group G
Q_j: closed subgroups of G so that $Q_j\backslash G/P$ have open cosets ($j = 1, 2$)
λ, μ_j: characters of P, Q_j, respectively.
ϕ_1: a function on G with $\phi_1(q_1xp) = \mu_1(q_1)\lambda(p)\phi_1(x)$ for $q_1 \in Q_1$ and $p \in P$
ϕ_2: a function on G with $\phi_2(q_2xp) = \mu_1(q_2)\lambda^*(p)\phi_2(x)$ for $q_2 \in Q_2$ and $p \in P$ ($\lambda^* = -\lambda - 2\rho_P$)

Definition (Hypergeometric functions, [O3]).

$$\Phi_{\phi_1,\phi_2}(x) := \int_K \phi_1(xk)\phi_2(k)dk \ \left(= \int_K \phi(k)\phi_2(x^{-1}k)dk\right)$$

Remark 7. **i)** Equations satsified by $\Phi(x)$: left action of Q_1, right action of Q_2 and our system for $\mathcal{B}(G/P; L_\lambda)$. $\Rightarrow$ the total system of differential equations.
ii) $Q_1 = Q_2 = K \Rightarrow$ (zonal) spherical functions (ex. Lauricella's F_D etc.)
$P = P_{\ell,n}$, $Q_2 = P_{k,n}$, $\lambda = (k, 0)$, $\mu_2 = (-\ell, 0)$, ϕ_2 is the kernel function of $\mathcal{R}_k^\ell$.

Theorem 5. *Let H be a connected subgroup of $GL(n,\mathbb{R})$ such that $\big(H_{\mathbb{C}} \times GL(\ell,\mathbb{C}), \mathbb{C}^n \otimes \mathbb{C}^\ell\big)$ is a prehomogeneous vector space ($\Leftrightarrow \exists$ an open orbit). Then the soutions of our total system are our hypergeometric functions (i.e. integral transforms) of the relative invariants on $M^o(n,\ell;\mathbb{R})$.*

Example (Gelfand's Hypergeometric functions, [Ge2]). $\ell = 1$.
$H = GL(1,\mathbb{R})_+ \times \cdots \times GL(1,\mathbb{R})_+$ and $\mu_1 = (\alpha_1, \cdots, \alpha_n)$ with $\sum_{j=1}^n \alpha_j = -k$.

$$\Phi(\alpha, x) = \int_{t_1^2+\cdots+t_k^2=1} \prod_{j=1}^{n} \Big|\sum_{\nu=1}^{k} t_\nu x_{j\nu}\Big|_{\pm}^{\alpha_j} \omega \quad \text{(Hypergeometric functions).}$$

Our total system is ($\Phi(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ is an even function for column vectors $\mathbf{x}_\nu$)

$$\sum_{j=1}^{k} x_{ij}\frac{\partial\Phi}{\partial x_{ij}} = \alpha_j\Phi \quad \text{for } 1 \le i \le n \quad (H\text{-action})$$

$$\sum_{\nu=1}^{n} x_{\nu i}\frac{\partial\Phi}{\partial x_{\nu j}} = -\ell\delta_{ij}\Phi \quad \text{for } 1 \le i,\ j \le k \quad (GL(k,\mathbb{R})\text{-action})$$

$$\frac{\partial^2\Phi}{\partial x_{i_1j_1}x_{i_2j_2}} = \frac{\partial^2\Phi}{\partial x_{i_2j_1}x_{i_1j_2}} \quad \text{for } 1 \le i_1 < i_2 \le n,\ 1 \le j_1 < j_2 \le k \quad \text{(Capelli type).}$$

Remark 8. **i)** $n = 4$ and $k = 2$ in Example $\Rightarrow$ Gauss Hypergeometric functions.
ii) If the prehomogeneous vector space has *finite orbits* in $M^o(n,\ell;\mathbb{C})$, then the total system is holonomic (cf. [Ta]).

5. Twisted Radon Transformations

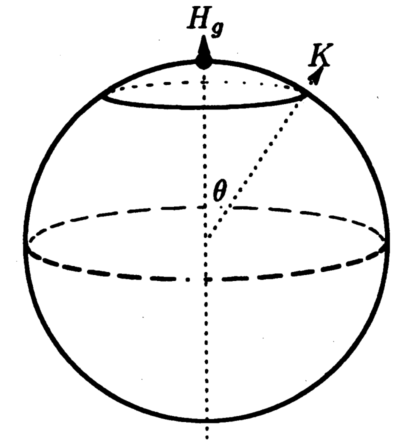
Consider

$$\mathcal{R}_{G/H}^{G/K} : \mathcal{F}(G/K) \to \mathcal{F}(G/H),\ f \mapsto (\mathcal{R}_{G/H}^{G/K} f)(x) = \int_{H/K\cap H} f(xh)dh$$

with a certain function space $\mathcal{F}$.

What happens if $H_g := gHg^{-1}$ (is twisted) with $g \in G$?

Example.
$G = SO(3)$ and $K = H = SO(2)$.
If θ corresponds to the zero of a zonal spherical function, the twisted Radon transform has a kernel corresponding to the spectral parameter of the zonal spherical function.



Other $\theta \Rightarrow$ Small divisor, Diophantine approximation...

What is the good value of θ ?

Problem. $H_g/K \cap H_g$ is totally geodesic submanifold of $G/K \Rightarrow$?

We have an affirmative answer in some examples:
$\mathbb{X}_2^n = G/K,\quad G = U(n,\mathbb{F}),\ K = U(2,\mathbb{F}) \times U(n-2,\mathbb{F})$
$\mathbb{X}_k^n = G/H,\quad G = U(n,\mathbb{F}),\ H = \big(U(k,\mathbb{F}) \times U(n-k,\mathbb{F})\big)'$

$$I_{2,n-2} := \begin{pmatrix} I_2 & \\ & -I_{n-2} \end{pmatrix},\quad I_{1,1,k-1,n-k-1} := \begin{pmatrix} I_1 & & & \\ & -I_1 & & \\ & & I_{k-1} & \\ & & & -I_{n-k-1} \end{pmatrix},$$

$K = \{g \in G;\ g = I_{2,n-2}\, g\, I_{2,n-2}\}$
$H = \{g \in G;\ g = I_{1,1,k-1,n-k-1}\, g\, I_{1,1,k-1,n-k-1}\}$
$\quad\Rightarrow\ H/H \cap K = \mathbb{P}^{k-1}(\mathbb{F}) \times \mathbb{P}^{n-k-1}(\mathbb{F}) \subset \mathbb{X}_2^n$ (totally geodesic).

Theorem 6 [Kurita-O, in preparation]. *Suppose $2 \le k \le n-2$. The above twisted Radon transformation*

$$\mathcal{R} : \mathcal{B}(G/K) \ni f \mapsto (\mathcal{R}f)(x) = \int_{H/K\cap H} f(xh)dh \in \mathcal{B}(G/H)$$

is a topological isomorphism onto the image of the non-twisted Radon transformation if $n \ne 2k$. If $n = 2k$, the <u>odd</u> functions are in the kernel and the same result holds on the <u>even</u> functions.

Remark 9. $n = 4$ and $k = 2$ and $\mathbb{F} = \mathbb{C} \Rightarrow$ the simplest non-trivial case, $H/K \cap H = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and $\mathcal{R} : \mathcal{B}(\mathbb{X}_2^4/\mathbb{Z}_2) \simeq \mathcal{B}(\mathbb{X}_2^4/\mathbb{Z}_2)$.

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