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Multiresolution Analysis with Lattice Basis

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概要

In this note, we consider the multiresolution analysis of $L^2(\mathbb{R}^n)$ with lattice basis and wavelet basis associated with it. Our main results are Theorem 1,characterizing orthonormal basis of V_j and Theorem 2, , characterizing wavelet basis.

Let $A \in GL(n;\mathbb{R}^n)$ and define

$$\Gamma = \Gamma_A = \{Ak; k \in \mathbb{Z}^n\}.$$

Let $A = (\vec{a_1}, \vec{a_2}, \cdots, \vec{a_n})$, where $\vec{a_i} (i = 1,2,\ldots,n)$ are column vectors in $\mathbb{R}^n$.

We call $\vec{a_i}'s$ *a basis of the lattice.* Let $Q_n = [0,1]^n$ and

$$\Omega_A = \sum_{j=1}^{n} t_j \vec{a_j}; (t_1, \cdots, t_n) \in Q_n.$$

Let $A^* \in GL(n;\mathbb{R})$ be such that $A^t A^* = E_n$ and

$$\Gamma^* = \Gamma_{A^*} = \{A^* k; k \in \mathbb{Z}^n\}.$$

We call Γ^* the dual lattice of the lattice Γ_A.

Definition 1 *A multiresolution analysis of* $L^2(\mathbb{R}^n)$ *is a collection of closed subspaces* $V_j (j \in \mathbb{Z})$ *of* $L^2(\mathbb{R}^n)$ *such that*

(1) $V_j's$ *are increasing and* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$

(2) $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$, *where* $x \in \mathbb{R}^n$

(3) $f \in L^2(\mathbb{R}^n)$ *belongs to* V_0 *if and only if* $f(x-\gamma) \in V_0$ *for any* $\gamma \in \Gamma$

(4) There exists $g \in V_0$ *such that* $\{g(x-\gamma); \gamma \in \Gamma\}$ *is a Riesz basis of* V_0.

The above condition (4) means that *there exist constant,* $C_1, C_2, 0 < C_1 \leq C_2$ *such that for any sequence of scalars* $a(\gamma)$, $\gamma \in \Gamma$,

$$C_1 \sum_{\gamma\in\Gamma} |a(\gamma)|^2 \leq \| \sum_{\gamma\in\Gamma} a(\gamma) g(x-\gamma)\|^2 \leq C_2 \sum_{\gamma\in\Gamma} |a(\gamma)|^2.$$

The Fourier transform of $f(x-\gamma)$, $\gamma \in \Gamma$ is

$$\widehat{f(x-\gamma)}(\xi) = \exp(-\sqrt{-1}\xi\cdot\gamma)\widehat{f(\xi)}$$

For $\phi \in V_0$, let

$$(\frac{1}{2})^{\frac{n}{2}} \varphi(\frac{x}{2}) = \sum_{\gamma\in\Gamma} a(\gamma)\varphi(x-\gamma).$$

Then its Fourier transform is

$$2^{\frac{n}{2}} \widehat{\varphi(2\xi)} = [\sum_{\gamma\in\Gamma} a(\gamma) \exp(-\sqrt{-1}\xi\cdot\gamma)]\widehat{\varphi(\xi)} \; (\xi \in \mathbb{R}^n).$$

Define

$$m_0(\xi) = \sum_{\gamma\in\Gamma} a(\gamma) \exp(-\sqrt{-1}\xi\cdot\gamma) \tag{1}$$

The function $m_0(\xi)$ is $\underline{2\pi\Gamma^* - \textit{periodic}}$ and $\widehat{\varphi(2\xi)} = m_0(\xi)\widehat{\varphi(\xi)}$.

For $f_1(x), f_2(x) \in L^2(\mathbb{R}^n)$,and $\gamma_1, \gamma_2 \in \Gamma$, we have a formula

$$< f_1(x-\gamma_1), f_2(x-\gamma_2) >= (\frac{1}{2\pi})^n < \exp(-\sqrt{-1}\xi\cdot(\gamma_1-\gamma_2))\hat{f}_1, \hat{f}_2 >$$

Lemma 1 *For* $\gamma \in \Gamma$, *we have a formula*

$$(\frac{1}{2\pi})^n < \exp(-\sqrt{-1}\xi\cdot\gamma)\hat{f}_1, \hat{f}_2 >= \frac{1}{|\det(A)|}\int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi\cdot k)C(f_1,f_2)(\xi)d\xi$$

,where $\gamma = Ak$, $k \in \mathbb{Z}^n$, *and*

$$C(f_1,f_2)(\xi) = \sum_{\gamma^*\in\Gamma^*} f_1(2\pi\widehat{A^*\xi + 2}\pi\gamma^*)\overline{f_2(2\pi\widehat{A^*\xi} + 2\pi\gamma^*)}.$$

Proof. We have a formula

$$\begin{aligned}\langle\exp(-\sqrt{-1}\xi\cdot Ak)\hat{f}_1,\hat{f}_2\rangle &= \int_{\mathbb{R}^n} \exp(-\sqrt{-1}A^t\xi\cdot k)\widehat{f_1(\xi)}\overline{\widehat{f_2(\xi)}}d\xi \\ &= (2\pi)^n\int_{\mathbb{R}^n} \exp(-2\pi\sqrt{-1}\xi\cdot Ak)\widehat{f_1(2\pi\xi)}\overline{\widehat{f_2(2\pi\xi)}}d\xi \\ &= \frac{1}{|\det(A)|}\int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi\cdot k)C(f_1,f_2)(\xi)d\xi \qquad \square\end{aligned}$$

Note that the function $\widetilde{C(f_1,f_2)}(\xi) = C(f_1,f_2)(\frac{A^t\xi}{2\pi})$ is $2\pi\Gamma^* -$ *periodic.*

Theorem 1 *Let* $\varphi(x) \in L^2(\mathbb{R}^n)$. *Then a system*

$$\{2^{\frac{nj}{2}}\varphi(2^jx-\gamma);\gamma\in\Gamma\}$$

is an orthonormal basis of $V_j(j \in \mathbb{Z})$ *if and only if*

$$\widetilde{C(f_1,f_2)}(\xi) = |\det(A)| \quad a.a.\xi \in \mathbb{R}^n.$$

Proof. It is sufficient to prove for V_0. We have a formula,

$$\langle\varphi(x-\gamma_1),\varphi(x-\gamma_2)\rangle = (\frac{1}{2\pi})^n\langle\exp(-\sqrt{-1}\xi\cdot(\gamma_1-\gamma_2))\hat{\varphi},\hat{\varphi}\rangle \qquad (3)$$

With $\gamma_j = Ak_j(j=1,2)$, by Lemma 1,the right hand side of equation (3) is equal to

$$\frac{1}{|\det(A)|}\int_{\mathbb{R}^n/\mathbb{Z}^n} \exp(-2\pi\sqrt{-1}\xi\cdot(k_1-k_2))\widetilde{C(\varphi,\varphi)}(2\pi A^*\xi)d\xi.$$

If $\widetilde{C(\varphi,\varphi)}(\xi) = |\det(A)|$ *a.a.*, then the left hand side of equation (3) is equal to $\delta(\gamma_1,\gamma_2)$.

Conversely, let the left hand side of equation (3) $= \delta(\gamma_1,\gamma_2)$.

Put

$$\widetilde{C(\varphi,\varphi)}(2\pi A^*\xi) = \sum_{l\in\mathbb{Z}^n} a(l)\exp(2\pi\xi\cdot l)$$

then the right hand side of equation (3)

$$= \frac{1}{|\det(A)|}\sum_{l\in\mathbb{Z}^n}\int_{\mathbb{R}^n/\mathbb{Z}^n}\exp(-2\pi\sqrt{-1}\xi\cdot(k_1-k_2-l))d\xi = \frac{a(k_1-k_2)}{|\det(A)|}.$$

Hence, we get $a(0) = |\det(A)|$, *and* $a(l) = 0$, $l \neq 0$,

i.e. $\widetilde{C(\varphi,\varphi)}(\xi) = |\det(A)|$, *a.a.* $\xi \in \mathbb{R}^n$. □

Corollary 1 *When a system* $\{\varphi(x-\gamma);\gamma\in\Gamma\}$ *is an orthonormal basis of* V_0*, we have a formula*

$$\sum_{\eta\in E}|m_0(\xi+\pi A^*\eta)|^2 = 1, \quad \textit{for almost all}\ \xi\in\mathbb{R}^n, \tag{4}$$

, *where* $E = \{0,1\}^n$.

Proof.

$$\begin{aligned}
\widetilde{C(\varphi,\varphi)}(2\xi) &= |\det(A)| \\
&= \sum_{\gamma^*\in\Gamma^*}|\varphi(\widehat{2\xi+2\pi\gamma^*})|^2 \\
&= \sum_{\gamma^*\in\Gamma^*}|m_0(\xi+\pi\gamma^*)|^2|\varphi(\widehat{\xi+\pi\gamma^*})|^2 \\
&= \sum_{k\in\mathbb{Z}^n}|m_0(\xi+\pi A^*k)|^2|\varphi(\widehat{\xi+\pi A^*})|^2 \\
&= \sum_{\eta\in E}|m_0(\xi+\pi A^*k)|^2|\det(A)|\,.
\end{aligned}$$

□

Now let $\{g(x-\gamma);\gamma\}$ be a Riesz basis of V_0.

$$C_1|\det(A)| \leq \widetilde{C(g,g)}(\xi) \leq C_2|\det(A)|,\ a.a.\ \xi\in\mathbb{R}^n$$

,with $0 < C_1 \leq C_2$.

Define $\varphi(x)$ as

$$\widehat{\varphi(\xi)} = \sqrt{|\det(A)|}\frac{\widehat{g(\xi)}}{\sqrt{C(g,g)(\xi)}} \cdot$$

Then, the system $\langle 2^{\frac{nj}{2}}\varphi(2^j x - \gamma)\, \gamma \in \Gamma\rangle$ is an orthonormal basis of $V_j (j \in \mathbb{Z})$ by Theorem 1.

Our aim is to decompose V_{j+1} as $V_{j+1} = V_j \bigoplus W_j \; (j \in \mathbb{Z})$.

At first we consider the decomposition $V_1 = V_0 \bigoplus W_0$.

Let $\psi_\varepsilon(x) \in V_1,\, \varepsilon \in E$ and put

$$\widehat{\psi_\varepsilon(2\xi)} = m_\varepsilon(\xi)\widehat{\varphi(\xi)}.$$

, where $\psi_{(0,\dots,0)}(x) = \varphi(x)$ and $m_\varepsilon(\xi)\, (\varepsilon \in E)$ are $2\pi\Gamma^* - \mathit{periodic}$.

Put

$$m_\varepsilon(\xi) = \sum_{\eta \in E} \exp(-\sqrt{-1}\xi \cdot A\eta) m_{\varepsilon,\eta}(2\xi)$$

With these notations, we have

Theorem 2 *The system* $\langle \psi_\varepsilon(x-\gamma)\, \gamma \in \Gamma,\, \varepsilon \in E\rangle$ *is an orthonormal basis if and only if the matrix*

$$U(\xi) = 2^{\frac{n}{2}} m_{\varepsilon,\eta}(\xi))_{(\varepsilon.\eta)\in E^2} \tag{5}$$

is unitary for a.a. $\xi \in \mathbb{R}^n$.

In this case, define

$$W_{(0,\eta)} = \overline{\langle \psi_\eta(x-\gamma);\, \gamma \in \Gamma\rangle}\, (\eta \in E)\ ,$$

and

$$W_0 = \bigoplus_{\eta \in E\setminus\{0\}} W_{(0,\eta)} \cdot$$

Proof. By the Plancherel formula, for $\psi_\varepsilon(x-\gamma_1), \psi_\eta(x-\gamma_2)$ we have a formula

$$
\begin{aligned}
\langle \psi_\varepsilon(x-\gamma_1), \psi_\eta(x-\gamma_2)\rangle &= (\frac{1}{2\pi})^n \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1-\gamma_2))\hat{\psi_\varepsilon}, \hat{\psi_\eta}\rangle \\
&= (\frac{1}{2\pi})^n \int_{\mathbb{R}^n} \langle \exp(-\sqrt{-1}\xi \cdot (\gamma_1-\gamma_2)) m_\varepsilon(\frac{\xi}{2}) \overline{m_\eta(\frac{\xi}{2})} |\widehat{\varphi(\frac{\xi}{2})}|^2 d\xi \\
&= 2^n |\det(A)| (\frac{1}{2\pi})^n \int_{\mathbb{R}^n/2\pi\Gamma^*} \exp(-\imath\xi \cdot 2(\gamma_1-\gamma_2)) m_\varepsilon(\xi)\overline{m_\eta(\xi)} d\xi
\end{aligned}
$$

Thus it is sufficient to consider the integral $I := \int_{\mathbb{R}^n/2\pi\Gamma^*} m_\varepsilon(\xi)\overline{m_\eta(\xi)}d\xi$

Now, for the integral I, we have

$$
\begin{aligned}
I &= \sum_{\varepsilon'}\sum_{\eta'} \int_{\mathbb{R}^n/2\pi\Gamma^*} \exp(-\imath\xi \cdot A(\varepsilon' - \eta'))[m_{(\varepsilon,\varepsilon')}(2\xi)\overline{m_{(\eta,\eta')}(2\xi)}]d\xi \\
&= \sum_{\varepsilon'}\sum_{\eta'} \frac{1}{|\det(A)|} \int_{\mathbb{R}^n/2\pi\mathbb{Z}^n} \exp(-\imath\xi \cdot (\varepsilon' - \eta')[m_{(\varepsilon,\varepsilon')}(2A^*\xi)\overline{m_{(\eta,\eta')}(2A^*\xi)}]d\xi
\end{aligned}
$$

, whence we get the result. □

On the other hand, we have a formula

$$m_\varepsilon(\xi + \pi A^*\eta) = \sum_{\eta' \in E} \exp(-\imath\xi \cdot -\imath\pi\eta \cdot \eta') m_{(\varepsilon,\eta')}(2\xi),$$

where $\eta \in E$. Define the matrix Λ,

$$\Lambda = ((\exp(-\sqrt{-1}\pi\varepsilon \cdot \eta)),$$

then we have the identity $\Lambda\Lambda^* = 2^n$.

We have also the equation

$$U(\xi) = 2^{\frac{n}{2}} diag((\exp(\sqrt{-1}\xi \cdot A\varepsilon'))\Lambda^{-1}((m_\varepsilon(\xi + \pi A^*\eta)).$$

Thus, we get

Corollary 2 *The system* $\langle \psi_\varepsilon(x-\gamma); \gamma \in \Gamma, \varepsilon \in E\rangle$ *is an orthonormal basis of* V_1 *if and only if the matrix*

$$(m_\varepsilon(\xi + \pi A^*\eta))_{(\varepsilon,\eta)\in E^2}$$

is unitary for almost all $\xi \in \mathbb{R}^n$.

Example $\cdots\cdots$ Let $A=(a_{kj})\in GL(n;\mathbb{R})$ and $\overrightarrow{a_j}=(a_{kj})_{1\le k\le n}$ $(j=1,\dots,n)$. Let $g(x)$ be the characteristic function of the undamental domain Ω_A of the associated lattice Γ_A .
We have its Fourier transform $\widehat{g(\xi)}$;

$$\widehat{g(\xi)}=|\det(A)|\exp(-\frac{1}{2}\sqrt{-1}\sum_{k=1}^{n}\xi_k(\sum_{j=1}^{n}a_{jk}))\prod_{j=1}^{n}\frac{\sin(\frac{\sum_{k=1}^{n}a_{jk}\xi_k}{2})}{\frac{\sum_{k=1}^{n}a_{jk}\xi_k}{2}}.$$

Let $\varphi(x)=\frac{g(x)}{\sqrt{|\det(A)|}}$, then the system $\langle\varphi(x-\gamma)\,\gamma\in\Gamma\rangle$ is an orthonormal basis of V_0 . Here we have the expression

$$m_0(\xi)=\exp(-\frac{1}{2}\sqrt{-1}\sum_{k=1}^{n}\xi_k(\sum_{j=1}^{n}a_{jk}))\prod_{j=1}^{n}\cos(\frac{\sum_{k=1}^{n}a_{jk}\xi_k}{2}).$$

The factor $m_u(\xi)=\prod_{j=1}^{n}\cos(\frac{\sum_{k=1}^{n}a_{jk}\xi_k}{2})$ corresponds to the characteristic function of the domain obtained translating Ω_A to the origin $\delta_A/2$, where δ_A is the diagonal of Ω_A .
Let $n=2$, and

$$\eta_0=\begin{pmatrix}0\\0\end{pmatrix},\eta_1=\begin{pmatrix}0\\1\end{pmatrix},\eta_2=\begin{pmatrix}1\\0\end{pmatrix},\eta_3=\begin{pmatrix}1\\1\end{pmatrix}$$

Put $\xi=(\xi_1,\xi_2)^t$ and define

$$\begin{aligned}
m_0(\xi)&=m_0(\xi+\pi A^*\eta_0)\\
m_1(\xi)&=\sqrt{-1}\exp(-\sqrt{-1}\xi\cdot A\eta_3)m_0(\xi+\pi A^*\eta_1)\\
m_2(\xi)&=\sqrt{-1}\exp(-\sqrt{-1}\xi\cdot A\eta_2)m_0(\xi+\pi A^*\eta_2)\\
m_3(\xi)&=-\exp(-\sqrt{-1}\xi\cdot A\eta_1)m_0(\xi+\pi A^*\eta_3)
\end{aligned}$$

Then the system

$$\langle\psi_0(x-\gamma),\,\psi_1(x-\gamma)\,,\psi_2(x-\gamma),\,\psi_3(x-\gamma);\gamma\in\Gamma,\,x\in\mathbb{R}^2\rangle$$

defined as follows, is an orthonormal basis of V_1 , where $\psi_0(x)=\varphi(x)$ and

$$\langle\psi_1(x-\gamma)\,\psi_2(x-\gamma),\,\psi_3(x-\gamma);\gamma\in\Gamma,\,x\in\mathbb{R}^2\rangle$$

is an orthonormal basis of W_0 ; for $j = 0, 1, 2, 3$

$$\widehat{\psi_j(2\xi)} = m_j(\xi)\widehat{\varphi(\xi)}.$$

<u>*Remark*</u> : The factors $\exp(-\sqrt{-1}\xi \cdot A\eta_3)$, $\exp(-\sqrt{-1}\xi \cdot A\eta_2)$, and $\exp(-\sqrt{-1}\xi \cdot A\eta_1)$ in $m_1(\xi)$, $m_2(\xi)$, *and* $m_3(\xi)$ could be of forms

$$\exp(-\sqrt{-1}\xi \cdot Ak_1), \exp(-\sqrt{-1}\xi \cdot Ak_2), \ and \ \exp(-\sqrt{-1}\xi \cdot Ak_3)$$

(,respectively) as long as *column vectors* $k_j \in \mathbb{Z}^2$ $(j = 1, 2, 3)$.

With the form $m_j(\xi) = \exp(-\frac{\xi \cdot A\eta_3}{2})m_{u,j}(\xi)$ $(j = 0, 1, 2, 3)$, we describe some examples ;

<u>Type B_2</u> :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2})\cos(\frac{\xi_2}{2})$$

$$m_{u,1}(\xi) = -\sqrt{-1}\exp(-\sqrt{-1}\xi_2)\cos(\frac{\xi_1 - \xi_2}{2})\sin(\frac{\xi_2}{2})$$

$$m_{u,2}(\xi) = -\sqrt{-1}\exp(-\sqrt{-1}(\xi_1 - \xi_2))\sin(\frac{\xi_1 - \xi_2}{2})\cos(\frac{\xi_2}{2})$$

$$m_{u,3}(\xi) = -\exp(-\sqrt{-1}\xi_1)\sin(\frac{\xi_1 - \xi_2}{2})\sin(\frac{\xi_2}{2})$$

<u>Type C_2</u> :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2})\cos(\xi_2)$$

$$m_{u,1}(\xi) = -\sqrt{-1}\exp(-\sqrt{-1}(\xi_1 + \xi_2)\cos(\frac{\xi_1 - \xi_2}{2})\sin(\xi_2)$$

$$m_{u,2}(\xi) = -\sqrt{-1}\exp(-\sqrt{-1}(\xi_1 - \xi_2))\sin(\frac{\xi_1 - \xi_2}{2})\cos(\xi_2)$$

$$m_{u,3}(\xi) = -\exp(-2\sqrt{-1}\xi_2)\sin(\frac{\xi_1 - \xi_2}{2})\sin(\xi_2)$$

<u>Type D_2</u> :

$$m_{u,0}(\xi) = \cos(\frac{\xi_1 - \xi_2}{2})\cos(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,1}(\xi) = -\sqrt{-1}\exp(-2\sqrt{-1}\xi_1)\cos(\frac{\xi_1 - \xi_2}{2})\sin(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,2}(\xi) = -\sqrt{-1}\exp(-\sqrt{-1}(\xi_1 - \xi_2))\sin(\frac{\xi_1 - \xi_2}{2})\cos(\frac{\xi_1 + \xi_2}{2})$$

$$m_{u,3}(\xi) = -\exp(-\sqrt{-1}(\xi_1 + \xi_2))\sin(\frac{\xi_1 - \xi_2}{2})\sin(\frac{\xi_1 + \xi_2}{2})$$

Type G_2 :

$$
\begin{aligned}
m_{u,0}(\xi) &= \cos(\frac{\xi_1}{2})\cos(\frac{3\xi_1-\sqrt{3}\xi_2}{4})\\
m_{u,1}(\xi) &= \sqrt{-1}\exp(\sqrt{-1}\frac{\xi_1-\sqrt{3}\xi_2}{2})\cos(\frac{\xi_1}{2})\sin(\frac{3\xi_1-\sqrt{3}\xi_2}{4})\\
m_{u,2}(\xi) &= -\sqrt{-1}\exp(-\sqrt{-1}\xi_1)\sin(\frac{\xi_1}{2})\cos(\frac{3\xi_1-\sqrt{3}\xi_2}{4})\\
m_{u,3}(\xi) &= \exp(\sqrt{-1}\frac{3\xi_1-\sqrt{3}\xi_2}{2})\sin(\frac{\xi_1}{2})\sin(\frac{3\xi_1-\sqrt{3}\xi_2}{4})
\end{aligned}
$$

□

Let

$$
\begin{aligned}
V_0 &= \overline{\langle \psi_0(x-\gamma);\gamma\in\Gamma\rangle}\\
W_{(0,j)} &= \overline{\langle \psi_j(x-\gamma);\gamma\in\Gamma\rangle}\,(j=1,2,3)\\
W_0 &= \bigoplus_{j=1}^{3} W_{(0,j)}
\end{aligned}
$$

Then

$$V_1 = V_0 \oplus W_0 .$$

Now, in general , with the notation in Corollary 2, for $\eta \in \tilde{E}$: $E \setminus (0,\dots,0)$ define

$$W_{(j,\eta)} = \overline{\langle 2^{\frac{nj}{2}}\psi_\eta(2^j x-\gamma);\gamma\in\Gamma\rangle}\ (j\in\mathbb{Z}) .$$

Then we have an orthogonal decomposition

$$
\begin{aligned}
V_{j+1} &= V_j \bigoplus_{\eta\in\tilde{E}} W_{(j,\eta)}\\
L^2(\mathbb{R}^n) &= V_0 \bigoplus_{(j\geq 1,\eta\in\tilde{E})} W_{(j,\eta)}\\
L^2(\mathbb{R}^n) &= \bigoplus_{(j\in\mathbb{Z},\eta\in\tilde{E})} W_{(j,\eta)}
\end{aligned}
$$

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