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On the order completeness in partially ordered linear spaces

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§1 INTRODUCTION AND BASIC RESULTS

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

(P1) $E = P - P$,

(P2) $P \cap (-P) = \{0\}$.

An order relation in E can be defined by $x \le y \Longleftrightarrow y - x \in P$. We call a linear space E equipped with such a positive cone P a (partially) ordered linear space, and denote it by (E, P).

For a subset A of E, the generalized supremum $\operatorname{Sup} A$ is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bounds of A. In other words,

$U(A) = \{x \in E \mid y \le x,\ \forall y \in A\}$,

$\operatorname{Sup} A = \{a \in U(A) \mid b \le a,\ \ b \in U(A) \Longrightarrow a = b\}$.

The generalized infimum $\operatorname{Inf} A$ and the set of all lower bounds $L(A)$ are defined similarly. The basic properties of the generalized supremum has been investigated in [3],[4],[5], and a remarkable result is that this notion gives us a method to construct an order completion of E, when it is not order complete. In constructing this theory, the condition

$$U(A) = (\operatorname{Sup} A) + P \qquad (\text{ for every subset } A \subset E\,) \tag{1}$$

is extremely important. In many cases, the generalized supremum $\operatorname{Sup} A$ can be empty, even if $U(A) \neq \emptyset$. In the space $C[0,1]$ with the natural positive cone $P = \{f \in C[0,1] \mid f(x) \ge 0\ (x \in [0,1])\}$ for example, it is easy to find a subset $A \subset C[0,1]$ such that $U(A) \neq \emptyset$ and $\operatorname{Sup} A = \emptyset$. This means that the space $(C[0,1], P)$ does not satisfy the condition (1). For another example, let X be the space of all $n \times n$ symmetric matrices with real coeffcients, and we adopt the positive cone $P = \{A \in X \mid (Ax, x) \ge 0,\ x \in \mathbb{R}^d\}$. Then (X, P) satisfies the condition (1) while it is neither order complete nor a lattice ([4]). In this paper, we will consider the sequence spaces l_1, l_2 with typical positive cones, and investigate the condition (1) for each case.

An ordered linear space (E, P) is said to be monotone order complete (m.o.c. for short) if every totally ordered subset A of E with $U(A) \neq \emptyset$ has the least upper bound $\operatorname{lub} A$ in E. In the case $E = \mathbb{R}^d$, (E, P) is m.o.c. if and only if P is closed ([5]). In the case when E is a Banach space with a closed positive cone P satisfying $P^* - P^* = E^*$, (E^*, P^*) is m.o.c. where E^* is the topological dual of E and $P^* = \{x^* \in E^* \mid x^*(x) \ge 0,\ x \in P\}$. The proofs of these facts can be seen in a previous paper [6].

Theorem 1. *Suppose that an ordered linear space (E, P) is m.o.c., then (E, P) satisfies the condition* (1). *In particular,* $\operatorname{Sup}\{a, b\} \neq \emptyset$ *for every $a, b \in E$, and* $U(a, b) = (\operatorname{Sup}\{a, b\}) + P$.

The proof of this theorem can be seen in [4]. A convex subset C of E is said to be algebraically closed if every straight line of E meets C by a closed interval. A point x of a convex subset $C \subset E$ is called an algebraic interior point of C if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in C$. Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of C by int$C$ (ext$C$). Moreover, $\partial C =$ (int$C$ $\cup$ ext$C)^c$ is called the algebraic boundary of C. Let (E, P) be an ordered linear space and suppose that P is algebraically closed with nonempty algebraic interior. A convex subset F of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $F = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P. For $F \in \mathfrak{F}(P)$, $\dim F$ is defined as the dimension of aff$F$ where aff$F$ denotes the affine hull of F. The proof of the following theorem can be seen in [5].

Theorem 2. *Let (E, P) be an ordered linear space and suppose that P is algebraically closed and* $\operatorname{int} P \neq \emptyset$. *If* $\dim F < \infty$ *for every* $F \in \mathfrak{F}(P)$, *then* (E, P) *satisfies the condition* (1).

In [5], it is proved that the algebraic closedness of the positive cone P is a necessary condition for the monotone order completeness of (E, P). The following result is considered to be an improvement of this fact.

Theorem 3. *If an ordered linear space* (E, P) *satisfies the condition* (1) *and* $\operatorname{int} P \neq \emptyset$, *then the positive cone P is algebraically closed.*

For two distinct points x, $y \in E$, we denote the closed segment between x and y by $[x, y] = \{(1-t)x + ty \mid 0 \leq t \leq 1\}$. Also, the half open segment is defined by $(x, y] = \{(1-t)x + ty \mid 0 < t \leq 1\}$. $[x, y)$ and open segments (x, y) are defined analogously. For a convex subset C of E,

$$C^a = C \cup \{x \in E \mid (x, y] \subset C \text{ for some } y \in C\}$$

is called the algebraic closure of C. Clearly, C is algebraically closed, if and only if $C = C^a$. We note that C^a is not always algebraically closed, in other words, $C^a = (C^a)^a$ does not always hold.

Lemma 1. *Let P be a convex cone in a linear space E. Then P^a is also a convex cone.*

proof. Let x be an arbitrary point of P^a, and take $y \in P$ such that $(x, y] \subset P$. Since P is a cone, $(1-\lambda)\mu x + \lambda \mu y = \mu((1-\lambda)x + \lambda y) \in P$ for every $0 < \mu$, and $0 < \lambda \leq 1$. This means that $(\mu x, \mu y] \subset P$ and $\mu x \in P^a$. Hence it is sufficient to show that $x_1 + x_2 \in P^a$ for every $x_1, x_2 \in P^a$. Let y_1, y_2 be such that $(x_1, y_1], (x_2, y_2] \subset P$ respectively. Since P is a convex cone, $(1-\lambda)(x_1 + x_2) + \lambda(y_1 + y_2) = (1-\lambda)x_1 + \lambda y_1 + (1-\lambda)x_2 + \lambda y_2 \in P$ for every $0 < \lambda \leq 1$. This means that $(x_1 + x_2, y_1 + y_2] \in P$ and $x_1 + x_2 \in P^a$.

proof of Theorem 3. Suppose that the positive cone P is not algebraically closed. Then there exists $x \in P^a \setminus P$. We define a subset $A \subset E$ by

$$A = -P^a.$$

Since $0 \not\geq -x$, we have $0 \notin U(A)$, and clearly $U(A) \subset U(-P) = P$. Moreover, we can conclude that

$$\operatorname{int} P \subset U(A) \subset P \setminus \{0\}, \tag{2}$$

and $U(A) \neq \emptyset$ in particular. To prove (2), we take $z \in \operatorname{int} P$, and $-x \in -P^a$. Then there exists $y \in P$ such that $(x, y] \subset P$. Since $z \in \operatorname{int} P$, we can choose a positive number $\mu > 0$ such that $z + \mu(x - y) \in P$. It is easy to see that

$$\begin{aligned}(x, z] &\subset \operatorname{conv}\{(x, y] \cup [z, z + \mu(x - y)]\} \\ &\subset P\end{aligned}$$

by the convexity of P. Hence, $z - (-x) = 2(\frac{x+z}{2}) \in P$. Since $z \in \operatorname{int} P$ and $-x \in -P^a$ can be taken arbitrarily, we obtain $\operatorname{int} P \subset U(A)$. Now we take $u \in U(A)$ and $a \in A = -P^a$ arbitrarily. By Lemma 1, we see $2a \in -P^a$ and hence $\frac{1}{2}u - a = \frac{1}{2}(u - 2a) \in P$. This means that $\frac{1}{2}u \in U(A)$. By (2), $u \neq 0$ and $u - \frac{1}{2}u \in P$ This means that u is not a minimal element of $U(A)$. Since $u \in U(A)$ is arbitrary, we have obtained that $\operatorname{Sup} A = \emptyset$ and the condition (1) fails.

Remark. Algebraic closedness of P is obviously a necessary condition for the order completeness. However, we cannot say P is algebraically closed when (E, P) is only a lattice. The two dimensional space $\mathbb{R}^2$ with lexicographical order is an example.

Corollary 1. *If* $\dim E < \infty$, *then* (E, P) *satisfies the condition* (1) *if and only if* P *is closed.*

proof. In finite dimensional cases, (E, P) is m.o.c. if P is closed([5]). Hence by Theorem 1, it satisfies the condition (1). The converse follows directly from Theorem 3.

Next we consider the family of the generalized suprema $\{\operatorname{Sup} A \mid A \subset E\}$, and construct an order completion of (E, P) in the case $E = \mathbb{R}^d$ and P is closed. By Corollary 1, the condition (1) holds in such cases. Let $\mathfrak{B}$ and $\mathfrak{B}'$ be the family of all upper bounded subset and lower bounded subset in $\mathbb{R}^d$ respectively, i.e.

$$\mathfrak{B} = \{A \subset \mathbb{R}^d \mid A \neq \emptyset,\ U(A) \neq \emptyset\},$$
$$\mathfrak{B}' = \{B \subset \mathbb{R}^d \mid B \neq \emptyset,\ L(B) \neq \emptyset\}.$$

We define an equivalence relation $\sim$ in $\mathfrak{B}$ by

$$A \sim B \Longleftrightarrow U(A) = U(B) \quad (A, B \in \mathfrak{B}).$$

Let $\tilde{E}$ be the quotient set $\mathfrak{B}/\sim\ = \{[A] \mid A \in \mathfrak{B}\}$ where $[A]$ denotes the equivalence class of A.

For every $[A] \in \tilde{E}$, two operations $u([A]) = U(A)$ and $l([A]) = L(U(A))$ are well defined. By virtue of (1), $\tilde{E}$ can be identified with the set $\{U(A) \mid A \in \mathfrak{B}\}$ or the set $\{\operatorname{Sup} A \mid A \in \mathfrak{B}\}$. We now define an order relation in $\tilde{E}$ by

$$[A] \leq [B] \Longleftrightarrow u([B]) \subset u([A]) \quad ([A], [B] \in \tilde{E}).$$

Definition. *For every* $[A]$, $[B] \in \tilde{E}$ *and* $\lambda \in \mathbb{R}$,

$$[A] + [B] = [l([A]) + l([B])]$$
$$\lambda[A] = \begin{cases} [\lambda l([A])] & (\lambda > 0) \\ [0^+ l([A])] = [-P] & (\lambda = 0) \\ [\lambda u([A])] & (\lambda < 0), \end{cases}$$

where 0^+C *denotes the resession cone of a convex set* C.([7])

We define two subsets $\tilde{P}$ and $\tilde{E}_1$ of $\tilde{E}$ as follows.

$$\begin{aligned} \tilde{P} &= \{[A] \in \tilde{E} \mid [A] \geq [-P]\} \\ &= \{[A] \in \tilde{E} \mid u([A]) \subset P\} \\ \tilde{E}_1 &= \{[A] \in \tilde{E} \mid u([A]) = a + P \text{ for some } a \in \mathbb{R}^d\}. \end{aligned}$$

Note that the correspondence which assigns $a \in \mathbb{R}^d$ to $[A] \in \tilde{E}_1$ such that $u([A]) = a + P$ is one to one.

Theorem 4. *$\tilde{E}$ is an order complete vector lattice with the order '$\leq$', and the vector operation defined above. Moreover,*

(*a*) *$\tilde{P}$ is a convex cone in $\tilde{E}$ and satisfies* (P1), (P2), *and* $[A] \leq [B] \Longleftrightarrow [B]-[A] \in \tilde{P}$.

(*b*) *$\tilde{E}_1$ is a subspace which is order isomorphic to $(\mathbb{R}^d, P)$ by the correspondence $\mathbb{R}^d \ni a \longleftrightarrow [A] \in \tilde{E}_1$ where $u([A]) = a + P$.*

The proof of this theorem can be seen in [2], and [3].

§2 Examples in sequence spaces

We say that an ordered linear space (E, P) satisfies the condition (F) if it satisfies all the hypotheses in Theorem 2. In finite dimensional cases, (E, P) obviously satisfies the condition (F) whenever P is closed. In this section we consider some sequence spaces and investigate the relation among the monotone order completeness, the condition (1) and the condition (F). We denote $l_0 = \{x = (x_0, x_1, x_2, \cdots) \mid x_n = 0$ except for finitely many $n \in \mathbb{N} \cup \{0\}\}$, $l_1 = \{x = (x_0, x_1, x_2, \cdots) \mid \Sigma_{n=0}^{\infty} |x_n| < \infty\}$, $l_2 = \{x = (x_0, x_1, x_2, \cdots) \mid \Sigma_{n=0}^{\infty} x_n^2 < \infty\}$ and define two typical positive cones as follows.

$$P_1 = \{x = (x_0, x_1, x_2, \cdots) \in l_1 \mid x_0 \geq \sum_{n=1}^{\infty} |x_n|\},$$

$$P_2 = \{x = (x_0, x_1, x_2, \cdots) \in l_2 \mid x_0 \geq (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}}\}.$$

It is easy to see that P_1 and P_2 are both algebraically closed and $\operatorname{int} P_1 \neq \emptyset$, $\operatorname{int} P_2 \neq \emptyset$.

2.1 (l_1, P_1), (l_2, P_2)

The space (l_1, P_1) does not satisfy the condition (F). Indeed, $H = \{(x_0, x_1, x_2, \cdots) \in l_1 \mid x_0 = \Sigma_{n=1}^{\infty} x_n\}$ is a supporting hyperplane of P_1 and the face $F = H \cap P_1$ is infinite dimensional. In contrast, (l_2, P_2) satisfies the condition (F) ([4]).

Proposition 1. *(l_1, P_1) is m.o.c., and it satisfies the condition* (1) *in particular.*

For the proof of this proposition, we offer the following.

Definition. *An ordered linear space (E, P) is said to be sequentially monotone order complete* (s.m.o.c. for short) *if every totally ordered countable subset A of E with $U(A) \neq \emptyset$ has the least upper bound* lub *A in E.*

This condition is slightly weaker than the monotone order completeness in general.

Lemma 2. *For every upper bounded totally ordered subset A in (l_1, P_1), there exsits a countable subset $\{a_n\}_{n=1}^{\infty}$ of A such that $U(A) = U(\{a_n\})$.*

proof. We write $A = \{a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \mid \lambda \in \Lambda\}$, and let $(b_0, b_1, b_2, \cdots)$ be an upper bound of A. Since $a_{\lambda 0} \leq b_0 \quad (\lambda \in \Lambda)$, there exists $a_0 = \sup a_{\lambda 0}$. If there exists $a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \in A$ such that $a_{\lambda 0} = a_0$, then a_λ is the maximum of A and the lemma is trivial. Hence we assume that $a_{\lambda 0} < a_0 \quad (\lambda \in \Lambda)$. We can choose a sequence $\lambda_1, \lambda_2, \cdots$ such that $\{a_{\lambda_n}\}_{n=1}^{\infty}$ is nondecreasing and $a_{\lambda_n} \longrightarrow a_0$. For arbitrary $a_\lambda = (a_{\lambda 0}, a_{\lambda 1}, a_{\lambda 2}, \cdots) \in A$, there exists $n \in \mathbb{N}$ such that $a_\lambda \leq a_{\lambda_n}$, and this means that $U(A) = U(\{a_{\lambda_n}\})$.

proof of Proposition 1. By Lemma 2, it suffices to show that (l_1, P_1) is s.m.o.c. Let $a_m = (a_{m0}, a_{m1}, a_{m2}, \cdots)$ $(m = 1, 2, 3, \cdots)$ be an upper bounded increasing sequence in (l_1, P_1), and let $(b_0, b_1, b_2, \cdots)$ be an upper bound of $\{a_m\}$. Since $\{a_{m0}\}_{m=1}^{\infty}$ is nondecreasing and $a_{m0} \leq b_0$ $(m = 1, 2, \cdots)$, it is a convergent sequence. Moreover, $a_m \leq a_n$ $(1 \leq m \leq n)$ implies

$$(3) \qquad a_{n0} - a_{m0} \geq \sum_{i=1}^{\infty} |a_{ni} - a_{mi}| \quad (1 \leq m \leq n).$$

Hence, for each $i = 1, 2, \cdots$, $\{a_{ni}\}_{n=1}^{\infty}$ is a convergent sequence. Thus we can define $a_0 = (a_{00}, a_{01}, a_{02}, \cdots)$ by $a_{0i} = \lim_{n \to \infty} a_{ni}$ $(i = 0, 1, 2 \cdots)$. By (3), we have for each $N = 1, 2, \cdots$, $a_{n0} - a_{m0} \geq \Sigma_{i=1}^{N} |a_{ni} - a_{mi}|$ $(1 \leq m \leq n)$. Hence we obtain by letting $n \longrightarrow \infty$ that $a_{00} - a_{m0} \geq \Sigma_{i=1}^{N} |a_{0i} - a_{mi}|$ $(m, N \in \mathbb{N})$. Since $N \in \mathbb{N}$ is arbitrary and $a_m \in l_1$, this inequality yields that $a_0 \in l_1$ and

$$a_{00} - a_{m0} \geq \sum_{i=1}^{\infty} |a_{0i} - a_{mi}| \quad (m \in \mathbb{N}).$$

This means $a_0 \geq a_m$ $(m \in \mathbb{N})$, and $a_0 \in U(\{a_m\})$. It remains to prove that a_0 is the minimum of $U(\{a_m\})$. For $b = (b_0, b_1, b_2, \cdots) \in U(\{a_m\})$, we have

$$b_0 - a_{m0} \geq \sum_{i=1}^{N} |b_i - a_{mi}| \quad (m, N \in \mathbb{N}).$$

Letting $m \longrightarrow \infty$, we obtain $b_0 - a_{00} \geq \sum_{i=1}^{N} |b_i - a_{0i}|$ $(N \in \mathbb{N})$. Since $N \in \mathbb{N}$ is arbitrary we also have $b_0 - a_{00} \geq \sum_{i=1}^{\infty} |b_i - a_{0i}|$. This means $b \geq a_0$ and the proof is complete.

The monotone order completeness of (l_2, P_2) can be proved by analogy. We remark that there is a different way to prove the monotone order completeness of these sapces by using the fact mentioned in §1.

2.2 (l_0, P_2), (l_1, P_2)

We rewrite $P_2 \cap l_0$ and $P_2 \cap l_1$ by P_2 in these spaces. In both spaces, $\operatorname{int} P_2 \neq \emptyset$, and the condition (F) holds. Consequently the condition (1) also holds. Moreover, (l_0, P_2) is not m.o.c.([4]).

Proposition 2. *(l_1, P_2) is not m.o.c.*

proof. We consider the convergent series $\Sigma_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. First we show that there is a subsequence $\{n_k\}_{k=0}^{\infty}$ of the sequence $1, 2, 3, \cdots$ such that $n_0 = 1$ and

$$\begin{aligned} S &= \sqrt{A_1} + \sqrt{A_2} + \sqrt{A_3} + \cdots \\ &< +\infty, \end{aligned}$$

$$\text{where } A_k = \frac{1}{(n_{k-1}+1)^2} + \frac{1}{(n_{k-1}+2)^2} + \cdots + \frac{1}{{n_k}^2} \qquad (k = 1,\ 2,\ 3,\ \cdots).$$

Indeed, if we choose the subsequence $\{n_k\}_{k=0}^{\infty}$ by $n_k = 2^k \ \ (k = 0, 1, 2, 3, \cdots)$, then

$$\begin{aligned} A_k &= \frac{1}{(2^{k-1}+1)^2} + \frac{1}{(2^{k-1}+2)^2} + \cdots + \frac{1}{2^{2k}} \\ &\le \frac{1}{2^{2(k-1)}} + \frac{1}{2^{2(k-1)}} + \cdots + \frac{1}{2^{2(k-1)}} = \frac{1}{2^{k-1}}. \end{aligned}$$

Hence we have

$$\sum_{k=1}^{\infty} \sqrt{A_k} \leqq \sum_{k=1}^{\infty} \sqrt{\frac{1}{2^{k-1}}} < +\infty.$$

Now we define a sequence $\{a_n\}_{n=0}^{\infty}$ in l_1 by

$$\begin{aligned} a_0 &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots\cdots), \\ a_1 &= (S_1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ 0,\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots\cdots\cdots), \\ a_2 &= (S_2,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ \frac{1}{n_1+1},\ \cdots,\ \frac{1}{n_2},\ 0,\ 0,\ 0,\ \cdots\cdots\cdots\cdots), \\ a_3 &= (S_3,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_1},\ \frac{1}{n_1+1},\ \cdots,\ \frac{1}{n_2},\ \frac{1}{n_2+1},\ \cdots,\ \frac{1}{n_3},\ 0,\ 0,\ \cdots\cdots\cdots), \\ &\vdots \\ b_0 &= (2S,\ 0,\ 0,\ 0,\ 0\ \cdots\cdots\cdots), \end{aligned}$$

where $S_n = \sum_{k=1}^{n} \sqrt{A_k} \ \ (n = 1, 2, \cdots)$. Since $\sum_{k=1}^{n} A_k \le S_n^2$ for every n, we see that

$$\frac{\pi^2}{6} - 1 < S^2. \tag{4}$$

By the definition of A_k, we have $\sqrt{A_k} = (\frac{1}{(n_{k-1}+1)^2} + \frac{1}{(n_{k-1}+2)^2} + \cdots + \frac{1}{{n_k}^2})^{\frac{1}{2}} \quad (k = 1,\ 2,\ 3,\ \cdots)$. Therefore,

$$\begin{aligned} a_k - a_{k-1} &= (\sqrt{A_k},\ 0,\ \cdots\ 0,\ \frac{1}{n_{k-1}+1},\ \cdots,\ \frac{1}{n_k},\ 0,\ \cdots) \\ &\in P_2 \quad (k = 1,\ 2,\ 3,\ \cdots). \end{aligned}$$

Moreover, by (4), $(2S-S_k)^2 - 1^2 - (\frac{1}{2})^2 - \cdots - (\frac{1}{n_k})^2 = (2S-S_k)^2 - A_1 - A_2 - \cdots - A_k \geqq S^2 - A_1 - A_2 - \cdots - A_k > \frac{\pi^2}{6} - 1 - A_1 - A_2 - \cdots - A_k > 0$, it follows that

$$b_0 - a_k = (2S - S_k,\ 1,\ \frac{1}{2},\ \cdots,\ \frac{1}{n_k},\ 0,\ \cdots) \in P_2,$$

for every $k \in \mathbb{N}$. Hence the sequence $\{a_k\}$ is increasing and upper bounded in (l_1, P_2). Let $b = (b_0, b_1, b_2, \cdots)$ be an arbitrary element in $U(\{a_k\})$. Since $b \in l_1$, there is at least a number $n \in \mathbb{N}$ such that $b_n \neq \frac{1}{n}$. We define

$$b' = (b_0,\ b_1,\ b_2,\ \cdots,\ b_{n-1},\ \frac{1}{n},\ b_{n+1},\ \cdots),$$

then $b - b' = (0,\ 0,\ \cdots,\ b_n - \frac{1}{n},\ 0,\ 0,\ \cdots) \notin P_2 \cup (-P_2)$. This means that b and b' are not comparable with respect to the order of P_2. Moreover, it follows from the relation $b \geq a_k$ $(k = 0, 1, 2, \cdots)$ that

$$\begin{aligned}
0 &\leqq (b_0 - S_k)^2 - (b_1 - 1)^2 - (b_2 - \frac{1}{2})^2 \\
&\quad - \cdots - (b_{n-1} - \frac{1}{n-1})^2 - (b_n - \frac{1}{n})^2 - (b_{n+1} - \frac{1}{n+1})^2 - \cdots \\
&\leqq (b_0 - S_k)^2 - (b_1 - 1)^2 - (b_2 - \frac{1}{2})^2 \\
&\quad - \cdots - (b_{n-1} - \frac{1}{n-1})^2 - (b_{n+1} - \frac{1}{n+1})^2 - \cdots,
\end{aligned}$$

for sufficiently large k. This means $b' \geq a_k$ $(k = 0, 1, 2, \cdots)$. Thus we find that b is not the minimum of $U(\{a_k\})$, and since b is arbitrary it follows that $\mathrm{lub}\{a_k\}$ does not exist.

2.3 (l_0, P_1)

We rewrite $P_1 \cap l_0$ by P_1. P_1 is still algebraically closed in l_0. Indeed, we can easily see that $(1, 0, 0, 0, \cdots) \in \mathrm{int}\, P_1$. Let H be the subspace of l_0 defined by $H = \{x = (x_0, x_1, x_2, \cdots) \mid x_0 = \sum_{n=1}^{\infty} x_n\}$. Then H is a supporting hyperplane of P_1. The face $F = H \cap P_1$ contains the elements $(1, 1, 0, 0, \cdots)$, $(1, 0, 1, 0, 0, \cdots)$, $(1, 0, 0, 1, 0, \cdots), \cdots$ and they are affinely independent. Hence $\dim F = \infty$ and (l_0, P_1) does not satisfy the condition (F).

Proposition 3. (l_0, P_1) *does not satisfy the condition* (1), *and hence it is not m.o.c.*

proof. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in l_0 defined by

$$a_n = (\frac{1}{2^n},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \cdots,\ \frac{1}{2^n},\ 0,\ 0,\ \cdots) \quad (n = 1, 2, 3, \cdots).$$

Since $a_n - a_{n-1} = (-\frac{1}{2^n}, 0,\ \cdots,\ 0,\ \frac{1}{2^n},\ 0,\ 0,\ \cdots) \in -P_1$ for every $n = 1, 2, 3, \cdots$, $\{a_n\}$ is a decreasing sequence in (l_0, P_1). Also, we can see that $a_0 = (-1,\ 0,\ 0,\ 0,\ \cdots)$ is a lower bound of $\{a_n\}$. For an arbitrary lower bound $b = (b_0, b_1, b_2, \cdots)$ of $\{a_n\}$, and we define

$$b' = (b_0 + \frac{1}{2^{m+1}},\ b_1,\ b_2,\ \cdots,\ b_m,\ \frac{1}{2^{m+1}},\ 0,\ 0,\ \cdots).$$

Obviously, $b' \geq b$ holds and for sufficiently large n,

$$a_n - b' = (\frac{1}{2^n} - b_0 - \frac{1}{2^{m+1}},\ \frac{1}{2} - b_1,\ \cdots,\ \frac{1}{2^m} - b_m,\ 0,\ \frac{1}{2^{m+2}},\ \cdots,\ \frac{1}{2^n},\ 0,\ 0,\ \cdots).$$

Since $a_n \geq b$, we have

$$\begin{aligned}
\frac{1}{2^n} - b_0 - \frac{1}{2^{m+1}} &\geq |\frac{1}{2} - b_1| + \cdots + |\frac{1}{2^m} - b_m| + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^n} - \frac{1}{2^{m+1}} \\
&= |\frac{1}{2} - b_1| + \cdots + |\frac{1}{2^m} - b_m| + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n}.
\end{aligned}$$

It follows that b' is also a lower bound of $\{a_n\}$ while $b' \geq b$. Since $b \in L(\{a_n\})$ is arbitrary, $L(\{a_n\})$ has no maximal element. This means that $\mathrm{Inf}\{a_n\} = \emptyset$ while $L(\{a_n\}) \neq \emptyset$, in other words, (l_0, P_1) does not satisfy the condition (1).

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