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Remarks on a New Existence Theorem for Generalized Vector Equilibrium Problems and its Applications

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We consider a generalized vector equilibrium problem, which is the following set-valued vector version of Ky Fan's minimax inequality:

$$\text{Find } \overline{x} \in C \text{ such as to satisfy } \varphi(\overline{x}, y) \nsubseteq K(\overline{x}) \text{ for all } y \in C, \qquad \text{(GVEP)}$$

where

- X and E are topological vector spaces,
- C is a nonempty closed convex subset of X,
- $\varphi : C \times C \to 2^E$ is a set-valued map, and
- K is a set-valued map from C to E.

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Using a particular case of the extended version of Fan-KKM theorem [6, Theorem 2.1], we can formulate the following general existence theorem for $(GVEP)$ in topological vector spaces.

First, we need to recall the following definitions. Let $\psi : C \times C \to 2^E$ and $L : C \to 2^E$ be two other set-valued maps, and denote by $\mathcal{F}(C)$ the set of all finite subsets of C.

Definition 1. *We say that ψ is diagonally quasi convex in its first argument relatively to L, in short L-diagonally quasi convex in x, if for any A in $\mathcal{F}(C)$ and any y in $co(A)$, we have $\psi(A,y) \not\subseteq L(y)$.*

Definition 2. *We say that φ is K-transfer semicontinuous in y if for any $(x,y) \in C \times X$ with $\varphi(x,y) \subset K(y)$, there exist an element $x' \in C$ and an open $V \subset X$ containing y such that $\varphi(x',v) \subset K(v)$ for all $v \in V$.*

Theorem 1. *([7, Theorem 2.1]) Suppose that*

(A0) $\psi(x,y) \not\subseteq L(y) \implies \varphi(x,y) \not\subseteq K(y)\ \forall x,y \in C$;

(A1) *ψ is L-diagonally quasi-convex in x;*

(A2) *φ is K-transfer semicontinuous in y;*

(A3) *there is a nonempty compact subset B in X such that for each $A \in \mathcal{F}(C)$ there is a compact convex $B_A \subset X$ containing A such that, for every $y \in B_A \setminus B$, there exists $x \in B_A \cap C$ with*

$$y \in int_X \{v \in X : \psi(x,v) \subseteq L(v)\}.$$

Then there exists $\overline{y} \in B$ such that $\varphi(x,\overline{y}) \not\subseteq K(\overline{y})$ for all $x \in C$.

Theorem 1 generalizes [2, Theorem 2.1], which is proved by means of a Fan-Browder fixed point theorem - an immediate consequence of the Fan-KKM theorem. As we will mention in the 'Assumptions analysis' subsection, our hypotheses are more general than those used in [2]. Besides, the scalar version of this result extends [10, Theorem 4] (we take $C_A = co(A \cup R) \cap X$ where R is the convex compact which contains C in [10, Theorem 4, (4iii)]). Other particular cases are [1, Theorem 2], [12, Theorem 2.1], [13, Theorem 2.11], [11, Theorem 1], [8, Corollary 2.4], [9, Lemma 2.1] and [3, Theorem 2]. The origin of this kind of results goes back to Ky Fan [5]. His classical minimax inequality can be deduced from our result by setting $E = \mathbb{R}$, $K(x) = \mathbb{R}^*_+$ and $\varphi(x,y) = \psi(x,y) = f(x,y) - \sup_{x \in C} f(x,x)$ for all $x, y \in C$.

Let us turn to Theorem 1 and analyze its requirements by presenting different situations where assumptions (A0)-(A3) hold true. Let $(P(y))_{y \in C}$ a family of proper convex closed cones on E with $int\, P(y) \neq \emptyset$ for all $y \in C$.

- **Pseudomonotonicity**

Remark 1. *(A0) holds provided one of the following statements is satisfied.*

(a) $\varphi = \psi$ *and* $K = L$.

(b) $X = C$, $K(y) = -L(y) = -int\, P(y)$, $\psi(x,y) = \varphi(y,x)$ *for all* $x, y \in C$, *and* φ *is* P_x*-pseudomonotone, that is,*

$$\varphi(x,y) \not\subseteq int\, P(x) \Longrightarrow \varphi(y,x) \not\subseteq -int\, P(x)\ \forall x, y \in C.$$

- **Convexity.**

Remark 2. *(A1) holds provided that, for every* $y \in C$, *one has either*

(a) $\psi(y,y) \not\subseteq L(y)$, *and*

(b) *the set* $\{x \in C : \psi(x,y) \subseteq L(y)\}$ *is convex,*

or

(i) $L(y) = -int\, P(y)$ *and* $P(y)$ *is w-pointed* [1],

(ii) $\psi(y,y) \subseteq P(y)$, *and*

(iii) ψ *is left* P_y*-quasiconvex, that is, for all* $x_1, x_2, y \in C$ *and all* $\lambda \in [0,1]$, *one has either*

$$\psi(x_1,y) \subseteq \psi(\lambda x_1 + (1-\lambda)x_2, y) + P(y)$$
$$or$$
$$\psi(x_2,y) \subseteq \psi(\lambda x_1 + (1-\lambda)x_2, y) + P(y).$$

- **Continuity**

[1] A cone P is w-pointed if $P \cap -int\, P = \emptyset$.

Remark 3. *(A2) holds provided that one of the following statements is satisfied.*

(a) φ *is (transfer) u.s.c in* y *with compact values and if* K *has an open graph.*

(b) φ *is (transfer) u.s.c in* y *and* $K(x) = O$ *for all* $x \in C$, *where* O *is an open subset of* E.

(c) For each $x \in C$, *the set* $\{y \in X : \varphi(x,y) \not\subseteq K(y)\}$ *is closed in* C.

- Coercivity.

Remark 4. *(A3) holds if one of the following statements is satisfied.*

(a) C *is compact.*

(b) There is $x_0 \in C$ *such that* $\psi(x_0,.)$ *is* K*-compact.*

(c) There is a nonempty compact subset B *in* C *such that for each* $y \in C \setminus B$ *there exists* $x \in B \cap C$ *such that* $\psi(x,y) \subseteq L(y)$.

(d) There is a nonempty compact subset B *of* C *and a compact convex subset* $B' \in C$ *such that for each* $y \in C \setminus B$ *there exists* $x \in B' \cap C$ *with*

$$y \in int \{v \in X : \psi(x,v) \subseteq L(v)\}.$$

Besides, when the classical assumption (c) of Remark 3 is satisfied, (A3) holds provided that

(e) there is a nonempty compact subset B *in* X *such that for each* $A \in \mathcal{F}(C)$ *there is a compact convex* $B_A \subset X$ *containing* A *such that, for every* $y \in C \setminus B$, *there exists* $x \in B_A \cap C$ *with* $\varphi(x,y) \subseteq K(y)$.

Applications

a) Generalized vector variational like-inequalities

Let us consider a set-valued operator T from C into $L(X,E)$, and a bifunction η from C to itself. We write for $\Pi \subset L(X,E)$ and $x \in C$, $\langle \Pi, x \rangle =$

$\{\langle\pi, x\rangle : \pi \in \Pi\}$, where $\langle\pi, x\rangle$ denotes the evaluation of the linear mapping π at x which is supposed to be continuous on $L(X, E) \times X$[2].

The generalized vector variational inequality problem $(GVVLIP)$ takes the following form:

$$\text{Find } \overline{x} \in C \text{ such that }, \langle T\overline{x}, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x})\ \forall y \in C.$$

Thus $(GVVLIP)$ is a particular case of $(GVEP)$ if we take

$$\varphi(x, y) = \{\langle t, \eta(y, x)\rangle : t \in Tx\}.$$

For the reader's convenience, we recall the following definitions.

Definition 3. *1) T is said to be η-pseudomonotone if, for all $x, y \in C$,*

$$\langle Tx, \eta(y, x)\rangle \not\subseteq -int\, P(x) \Rightarrow \langle Ty, \eta(y, x)\rangle \not\subseteq -int\, P(x).$$

2) T is said to be V-hemicontinuous if for any $x, y \in C$ and $t \in]0, 1[$ $T(tx + (1-t)y) \to T(y)$ as $t \to 0_+$ (i.e. for any $z_t \in T(tx + (1-t)y)$ there exists $z \in Ty$ such that for any $a \in C$, $\langle z_t, a\rangle \to \langle z, a\rangle$ as $t \to 0_+$).

It has to be observed that when T is single-valued, we recover the hemicontinuity used in [4]. if $\eta(x, y) = x - y$ for all $x, y \in C$, η is dropped from the definition of pseudomonotonicity.

The linearization lemma plays a significant role in variational inequalities. Chen [4] extended this lemma to the single-valued vector case. For our need in this subsection, we state it in the set-valued case by using standard Minty's argument. Consider the following problem, which may be seen as a dual problem of $(GVVLIP)$,

$$\text{Find } \overline{x} \in C \text{ such that } \langle Ty, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x})\ \forall y \in C. \qquad \text{(GVVLIP*)}$$

Lemma 1. *Suppose that $\eta(\cdot, x)$ is affine and $\eta(x, x) = 0$ for each $x \in C$. If T is η-pseudomonotone and V-hemicontinuous then $(GVVLIP)$ and $(GVVLIP*)$ are equivalent.*

As an application of Theorem 1, we are now in position to formulate the following existence result for $(GVVLIP)$.

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[2]A typical situation when X is a reflexive Banach and E is a Banach

Theorem 2. *Suppose that*

(i) the mapping $int\, P(\cdot)$ *has an open graph in* $C \times L(X,E)$*;*

(ii) for each $x \in C$, $\eta(\cdot, x)$ *is affine,* $\eta(x, \cdot)$ *is continuous and* $\eta(x,x) = 0$*;*

(iii) T *is compact valued,* η*-pseudomonotone and V-hemicontinuous;*

(iv) there is a nonempty compact subset B *in* C *such that for each* $A \in \mathcal{F}(C)$ *there is a compact convex* $B_A \subset C$ *containing* A *such that, for every* $y \in B_A \setminus B$*, there exists* $x \in B_A \cap C$ *with*

$$y \in int_C \{v \in C : \langle Tv, \eta(x,v)\rangle \subseteq -int\, P(v)\}.$$

Then $(GVVLIP)$ *has at least one solution, which is in* B.

Proof. Set $\varphi(x,y) = \langle Tx, \eta(x,y)\rangle$, $\psi(x,y) = \langle Ty, \eta(x,y)\rangle$ and $K(x) = -int\, P(x)$ for all $x, y \in C$. We can show that the assumptions of Theorem 1 are satisfied; see the proof of Theorem 4.1 in [7]. Therefore, from Theorem 1, there exists $\overline{x} \in B$ such that

$$\langle Ty, \eta(y, \overline{x})\rangle \not\subseteq -int\, P(\overline{x})\ \forall y \in C.$$

Hence $(GVVLIP*)$ has a solution in B, which completes the proof of the theorem according to Lemma 1. ∎

b) Vector complementarity problems

A natural extension of the classical nonlinear complementarity problem, (CP) for short, is considered as follows. Let T be a single-valued operator from C, which is supposed to be a convex closed cone, to $L(X,E)$. The vector complementarity problem considered in this subsequent, (VCP) for short, is to find $\overline{x} \in C$ such that

$$\langle T(\overline{x}), \overline{x}\rangle \notin int\, P(\overline{x}), \text{ and } \langle T(\overline{x}), y\rangle \notin -int\, P(\overline{x}) \text{ for all } y \in C.$$

This problem collapses to (CP) when $E = \mathbb{R}$ and $P(x) = \mathbb{R}_+$ for all $x \in C$.

By means of vector variational inequalities, we can formulate the following existence theorem for (VCP).

Theorem 3. *Suppose that*

(i) *the set-valued map* $int\, P(\cdot)$ *has an open graph in* $C \times L(X, E)$;

(ii) T *is pseudomonotone and hemicontinuous;*

(iv) *there is a nonempty compact subset* B *in* C *such that for each* $A \in \mathcal{F}(C)$ *there is a compact convex* $B_A \subset C$ *containing* A *such that, for every* $y \in B_A \setminus B$, *there exists* $x \in B_A \cap C$ *with*

$$y \in int_C \{v \in C : \langle Tv, x - v\rangle \in -int\, P(v)\}.$$

Then (VCP) *has at least one solution, which is in* B.

Proof. It is clear that all the assumptions of Theorem 2 are satisfied with $\eta(x, y) = x - y$ for all $x, y \in C$. Therefore there exists $\overline{x} \in B$ such that

$$\langle T\overline{x}, z - \overline{x})\rangle \notin -int\, P(\overline{x})\ \forall z \in C. \tag{1}$$

Since C is a convex cone, then setting in (1), $z = 0$ and $z = y + \overline{x}$ for an arbitrary $y \in C$, we get respectively

$$\langle T\overline{x}, \overline{x})\rangle \notin int\, P(\overline{x}) \text{ and } \langle T\overline{x}, y)\rangle \notin -int\, P(\overline{x}).$$

Hence we conclude that $\overline{x}$ is also a solution to (VCP). ■

c) Vector optimization

Here, to convey an idea about the use of vector variational-like inequalities in vector optimization which involves smooth vector mappings, we prove the existence of solutions to weak vector optimization problems, $(WVOP)$ for short, by considering the concept of invexity. Let us state the problem as follows.

$$\text{Find } \overline{x} \in C \text{ such that } \phi(y) - \phi(\overline{x}) \notin -int\, P \text{ for all } y \in C, \tag{WVOP}$$

where $\phi : C \to E$ is a given vector-valued function and P is a given convex cone in E.

Let $\eta : C \times C \to X$ be a given function, and denote by $\nabla\phi$ the Fréchet derivative of ϕ once the latter is assumed to be Fréchet differentiable.

Theorem 4. *Suppose that P is a convex cone in E with $\operatorname{int} P \neq \emptyset$, and let $\phi : C \to E$ be a Fréchet differentiable mapping. Assume that*

(i) $\langle \nabla\phi(x), \eta(y,x)\rangle \notin -\operatorname{int} P$ *implies* $\langle \nabla\phi(y), \eta(y,x)\rangle \notin -\operatorname{int} P$ *for all* $x, y \in C$;

(ii) ϕ *is P-invex with respect to η, that is,*

$$\phi(y) - \phi(x) - \langle \nabla\phi(x), \eta(y,x)\rangle \in P \quad \forall x, y \in C.$$

(iii) $\nabla\phi$ *is hemicontinuous;*

(iv) *for each $x \in C$, $\eta(.,x)$ is affine, $\eta(x,.)$ is continuous and $\eta(x,x) = 0$;*

(v) *there is a compact subset B in C such that for every finite subset A in C there is a compact convex $C_A \subset X$ containing A such as to satisfy, for every $y \in C \backslash B$, there exists $x \in C_A \cap C$ with $\langle \nabla\phi(x), \eta(x,y)\rangle \in -\operatorname{int} P$.*

Then (WVOP) has at least one solution.

Proof. First, by virtue of Theorem 2 with $T := \nabla\phi$, we get

$$\langle \nabla\phi(\overline{x}), \eta(y,\overline{x})\rangle \notin -\operatorname{int} P \ \ \forall y \in C.$$

Then the P-invexity of ϕ allows us to conclude. ∎

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