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On Definitions of Chaos

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1 Introduction

Chaotic dynamical systems theory is an attractive and important branch of mathematics of interest to scientists in many disciplines. Although there has been no universally accepted mathematical definition of chaos, the most widely utilized definition of chaos is due to R. Devaney([Dev89]). He isolates three properties as beeing the essential features of chaos.

Definition of Choas (R. Devaney, 1989)
Let X be a meric space. A continuous function f from X to itself is said to be chaotic on X if f has the following three properties :

1. *f is topologically transitive; that is, for all non-empty open sets U and V of X, there exists a natural number k such that $f^k(U) \cap V$ is non-empty.*
2. *periodic points are dense in the space.*
3. *f has sensitive dependence on initial conditions; that is, there is a positive constant δ (sensitive constant) such that for every point x of X and every neighborhood N of x there exists a point y in N and a nonnegative integer n such that n-th iterates $f^n(x)$ and $f^n(y)$ are more than distance δ apart.*

In a certain sense, Condition 1. is an irreducibilty condition, Condition 2. an element of regularity, and Condition 3. an element of unpredictability.

Hereafter we call Condition 1. **transitivity**, Condition 2. **density**, and Condition 3. **sensitivity**.

It is proper for Chaos to be topological property and invarinat under the action of topological conjugate.

Transitivity(i.e.1.) and density (i.e.2.) are topological conditions, but unfortunately sensitivity(i.e.3.) is depend to a mertic.

On the otherhand, the sensitivity condition. which is famous as " butterfly effect", is considered as being the central idea in chaos. This condition captures the idea that in chaotic systems small errors in experimental readings eventually lead to large scale divergence. And it is easily understood.

Further sensitivity or expansiveness of a dynamical system are recognaized as formalizations of the notion of unpredictability in the presence of chaos. S. MacEachern and L. Berliner in [MB93], sharpen these two concepts under the case of a compact set of the real line.

The requirement of density is less intuitive than sensitivity, but it appeals to those looking for patterns or somewhat regularity within a seeming random system: density implies that *there is order in chaos.*

The irony of Devaney's definition is that the more understood each condition is, the more redundant it is in relationship to the other two.

Banks et al. pointed out in [BBCDS92] a redundancy : despite its popular appeal, sensitivity is mathematically redundant: transitivity and density imply sensitivity.

Their elegant result makes clear that chaos is a property relying only on the topological, and not the metric, properties of a space.

Vellekoop and Berglund showed in [VB94] a more redundancy: namely on intervals, transitivity is equal to chaos.

Crannell points out that this result of Banks et al. yields a somewhat less intuitive definition of chaos, and asks in [Cran95] " why transitivity? – why not something else? ", and suggested a slightly more natural concept, **blending**, as an alternative to transitivity.

Recently Touhey introduced in [Tou97] another definition of chaos, equivalent to Devney's one.

The subject of this paper is to survey these papers on a redundancy in the definition of chaos by Devaney in his text [Dev89], and ones on other definitions of chaos.

2 Redundancey in the definitions of Chaos

Banks et al. point out in [BBCDS92] a redundancy in the definition of chaos by Devaney:

Theorem *Let X be a meric space and f a continuous function on X. If f is topologically transitive and has dense periodic points, then f has sensitive dependence on initial conditions.*

Namely, transitivity and density of Periodic points imply sensitivity.

And in the same volume, Assaf IV and Gadbois [AG92] show that this is only redundancy for a general map:

Theorem *(A) transitivity and sensitivity do not imply density, and (B) density and sensitivity do not imply transitivity.*

For the case of (A), there is a following example ([AG92]):
Example A: Let $X = S^1 \setminus \{\exp(i2\pi p/q) : p, q \in \mathbb{Z}\}$ and $f(\exp(i\theta) = \exp(i2\theta)$.
The set of periodic points of f is empty.

For the case of (B), there are following examples B1, and B2 on an interval :
Example B1 ([AG92]): Let X be a cylinder $S^1 \times [0\ 1]$ with the induced " taxicab"-metric and $g(\exp(\ i\theta, t) = (\exp(i2\theta),\ t)$.
If we take $U = S^1 \times [0,\ \frac{1}{2})$ and $V = S^1 \times [\frac{1}{2},\ 1])$,then U can not intersect with V under iterations of g.

Example B2 ([VB94]): Let $X = \mathbb{R}^1$ and f be

$$f(x) = \begin{cases} 3x, & \text{if } 0 \leq x < \frac{1}{3} : \\ -3x+2 & \text{if } \frac{1}{3} \leq x < \frac{2}{3} : \\ 3x-2 & \text{if } \frac{2}{3} \leq x < 1 : \\ f(x-1) & \text{if } 1 \leq x \leq 2 . \end{cases}$$

f has sensitivity, because of its expansivenss $|\frac{df}{dx}| = 3$. If we take $U = [0,\ 1)$ and $V = (1, 2]$,then U can not intersect with V under iterations of f.

2.1 On an interval, transitivity = Chaos

If we restrict our attension to maps on an interval, a stronger result can be obtained by M. Vellekoop and R. Berglund [VB94]:
Proposition
Let I be a , not necessarily finite, interval and f a continuous function from I to itself. If f has (topological) transitivity, then f has (1) density and (2) sensitivity.

Namely, for maps on an interval, both sensitivity and density are redundant conditions in the definition of chaos.

And further they note that there are no other trivialities in Devaney's definition when restricted to intervals: on intervals, neither density nor sensitivity is enough to ensure any of the other conditions of chaos.

We have an example B2 on an interval showing that density and sensitivity do not imply transitivity.

And in [VB94], they give an example on an interval satisfying that sensitivity does not imply density:

Example C ([VB94]): Let $I = [0, \ \frac{3}{4}]$ and f be

$$f(x) = \begin{cases} \frac{3}{2}x, & \text{if } 0 \leq x < \frac{1}{2} : \\ \frac{3}{2}(1-x) & \text{if } \frac{1}{2} \leq x < \frac{3}{4} . \end{cases}$$

And in [VB94], they note the identity map on an interval as an example satisfying that density does not imply sensitivity. However we are thirsty for a non-linear example on an interval of this case.

3 Another definitions of Chaos

For one dimensional dynamical systems, the result by M. Vellekoop and R. Berglund leaves us only the study of the tansitivity.

Crannell points out that these redundant results yields a somewhat less intuitive definition of chaos, and asks in [Cran95] " why transitivity? – why not some thing else? ". Of the three conditions (transitivity, density, and sensitivity), sensitivity is clearly the most easily understood. In order to restore this rost sense of intuitiveness, she suggests a slightly more natural concept, **blending**, as an alternative to transitivity.

3.1 Transitivity or Blending

Definition([Cran95]):
Let M be a subset of $\mathbb{R}^n$ and f a continuous function on M. f is weakly blending if, for any pair of non-empty open sets U and V in M, there is some $k > 0$ so that $f^k(U) \cap f^k(V) \neq \emptyset$. And f is strongly blending if, for any pair of non-empty open sets U and V in M, there is some $k > 0$ so that $f^k(U) \cap f^k(V)$ contains a non-empty open subset.

It is clear that blending functions are not necessarily transitive, and transitive functions are not necessarily blending. She gives examples as follows:

Example D([Cran95]): Let $X = S^1$ and $f : S^1 \longrightarrow S^1$, given by $f(\theta) = \theta + k$, where $\frac{k}{\pi}$ is irrational.

In this case, f is transitive but not strongly or weakly blending.

Example E([Cran95]): Let $X = [-1 \ \ 1]$ and $f : X \longrightarrow X$ satisfying:

- $|f'(x)| > 2$, on except at the vertices of f; and
- each vertex of the graph of f likes alternately on the line $y = \frac{x}{2}$ and $y = -\frac{x}{2}$.

This function is strongly blending but not transitive.

And she gives an example on an interval satisfying that weakly blending does not imply transitivity, and nor strongly blending :

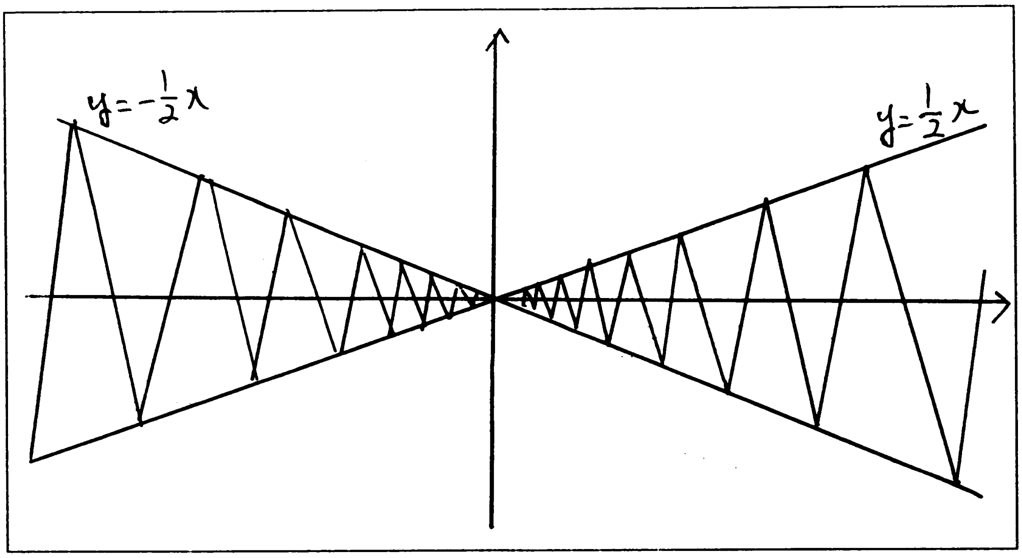

図 1: Graph of a function of EX. F which is strongly blending but not transitive

Example F ([Cran95]): Let $I = [-1,\ 1]$ and f be

$$f(x) = \begin{cases} -2x-2, & \text{if } -1 \le x \le -\frac{1}{2} : \\ 2x, & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} : \\ 2-2x, & \text{if } \frac{1}{2} \le x \le 1 \ . \end{cases}$$

She remarks that these two examples(i.e., D, E) have not density. Therefore, adding density condition to blending conditions, she obtains the following theorem:

Theorem 1[Cran95]:

Let M be a subset of $\mathbb{R}^n$ and f a continuous function on M with dense periodic points. Then if f is strongly blending, f is also transitive.

Namely, density and strongly bending condition imply chaos.

And if M is resticted to a compact set in $\mathbb{R}^1$, she gives another theorem:

Theorem 2[Cran95]:

Let I be a compact subset of $\mathbb{R}^1$ and f a continuous function on I with transitivity and a reppeling fixed point. Then f is weakly blending.

3.2 Yet another definition of chaos

P. Touhey proposes a new and natural definition of chaos in [Tou97], equivalent to Devaney's. He reformulates the two topological conditions of transitivity and density of periodic points as a single condition that yields a simple, concice definition of chaos. He use his definition to give a charcterization of chaos that restores the lost sense of intuitiveness: a map $f : X \to X$ is chaotic on X if and only if it mixes together, via periodic cycles, any finite number of non-empty open subsets in infinitely many ways.

Definition of Choas[Tou97]:

Let X be a meric space. A continuous function f from X to itself is said to be "CHAOTIC" on X if given U and V, non-empty open sets in X, there exists a periodic point $p \in U$ and a non-negative integer k such that $f^k(p) \in V$.

Namely, every pair of non-empty open subsets of X shares a periodic point.

Theorem[Tou97] :

Let X be a meric space, f a continuous function on X. The following are equivalent:

- *f is "CHAOTIC",*
- *f is chaotic,*
- *any finite collection of non-empty open subsets of X shares a periodic point,*
- *any finite collection of non-empty open subsets of X* shares infinitely many periodic orbits.

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