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LARGE TIME BEHAVIOR FOR COMPRESSIBLE EULER EQUATIONS WITH DAMPING AND VACUUM

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ABSTRACT. We introduce some new results of [17,19-20] on the asymptotic behavior of compressible isentropic flow through porous medium with vacuum. The model system is the compressible Euler equation with frictional damping. As $t \to \infty$, the density is conjectured to obey to the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. Here we give a definite answer to this conjecture without any assumptions on smallness or regularity for the initial data. We proved that the L^∞ weak entropy solutions to the Cauchy problem of damped Euler equations converge strongly in L^p with decay rates to the self similar solutions of porous medium equation. Furthermore, we prove the density function tends to the Barenblatt's solution of porous medium equation while the momentum is described by the Darcy's law provided that the initial mass is finite.

1. Introduction.

We study the asymptotic behavior of compressible isentropic flow through porous media when vacuum occurs. As $t \to \infty$, the density is conjectured to obey to the well-known porous medium equation and the momentum is expected to be formulated by Darcy's law. Although, many contributions are made for the small smooth solutions or piecewise smooth Riemann solutions away from vacuum since the pioneer work of Nishida [34], some key problems in this topic remain open. Among them, the large time asymptotic behavior for the solutions with vacuum has been a long-standing open problem. The main difficulties come from that such problem involves three mechanism: nonlinear convection, lower order dissipation of damping and the resonance from vacuum. Since any result on this problem will help us to understand the interaction of the effects of these three mechanism, the evolution of vacuum boundary, singularity development and other complicated phenomena caused by vacuum, it is of mathematical significance and physical importance, in view of the strong physical background of vacuum. Besides, this study may present useful information for the design of effective numerical schemes to capture the vacuum boundary. Here we shall introduce recent works of Marcarti, Pan and myself

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in [17-20] which will give a complete answer to this problem. In fact, we showed that the L^∞ weak entropy solutions with vacuum selected by the physical entropy-flux pairs, converge strongly in L^p with decay rates to the self similarity solutions of porous medium equation, determined by the end-states of the initial data and initial mass. New approaches are developed to deal with the nonlinear convection, nonlinear coupling and the singularity near vacuum based on the conservation of mass, the structure of the convection and the existence of mechanical energy function. This approach seems remarkable since we do not need smallness assumptions on the solutions.

We now formulate our results. Consider the compressible Euler equation with frictional damping

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\alpha\rho u, \end{cases} \tag{1.1}$$

with the following initial data

$$\rho(x,0) = \rho_0(x) \geq 0,\ m(x,0) = m_0(x). \tag{1.2}$$

Such a system occurs in the mathematical modelling of compressible flow though porous medium. Here ρ, u and P denote respectively the density, velocity, and pressure, $m = \rho u$ is the momentum and the constant $\alpha > 0$ models friction. Assuming the flow is polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma$, $1 < \gamma < 3$, with P_0 a positive constant, and γ the adiabatic gas exponent. Without loss of generality, α and P_0 are normalized to be 1 throughout this paper.

(1.1) is hyperbolic with two characteristic speeds $\lambda_1 = u - \sqrt{P'(\rho)}$ and $\lambda_2 = u + \sqrt{P'(\rho)}$. Furthermore, (1.1) is strictly hyperbolic at the point away from vacuum where two characteristics coincide. Thus, this simple system involves three mechanisms: nonlinear convection, lower order dissipation of damping and the resonance due to vacuum. The interaction of these mechanisms lead to the big difference in qualitative behaviors of solutions from those of strictly hyperbolic conservation laws. For instance, the long time behavior of the solutions to Cauchy problem for strictly hyperbolic conservation laws were known to be the corresponding Riemann solutions, while one should expect the nonlinear diffusive phenomena in the large time behavior of solutions to (1.1)-(1.2).

In fact, in the applications, Darcy's law is used to approximate the momentum equation in system (1.1), and thus one obtains

$$\begin{cases} \rho_t = P(\rho)_{xx}, \\ m = -P(\rho)_x. \end{cases} \tag{1.3}$$

Where the second equation is the famous Darcy's law and the first equation is the well-known porous medium equation. So, it is natural to expect some relationship between system (1.1) and system (1.3). Actually, we have the following conjecture.

Conjecture. *As $t \to \infty$, the system (1.1) is equivalent to the system (1.3).*

In the case away from vacuum, system (1.1) can be transferred to the p-system with damping by changing to the Lagrangian coordinates; see [smoller]. The conjecture has been justified by Hsiao and Liu [10, 11] for small smooth solutions away from vacuum based on the energy estimates for derivatives. Since then, this problem attracts considerable attentions; see [9], [12], [13], [27], [32-34], [37] and [39]. However, all of these results are away from vacuum and/or require small smooth initial data. For more references on the p-system with damping, we refer to [4], [14], [15], [24] and [41].

When a vacuum occurs in the solution, the difficulty of the problem is greatly increased. The main difficulties come from the interaction of nonlinear convection, lower order dissipation of damping and the resonance due to vacuum. It is known that the nonlinearity is the reason for shock formation in finite time in a hyperbolic system. For hyperbolic conservation laws, the self-similarity is an important feature in constructing fundamental Riemann solutions and in describing the large time behaviors of solutions. The damping presents weak dissipation, it prevents the formation of singularity if the data is small and smooth. However, it breaks the self-similarity of the system. This is crucial for the large solutions. Another effects of difficulties is due to the resonance near vacuum which develop a new singularity. In fact, Liu and Yang [25, 26] observed that the local smooth solutions of (1.1) blow up in finite time *before shock formation*. This implies the moving of the interface between the vacuum and the gas. Due to this new singularity, it is very difficult to obtain the solutions with any degree of regularity. This makes (1.1) difficult to understand analytically and makes the construction of effective numerical methods for computing solutions a highly non-trivial problem. Indeed, the only global weak solutions are constructed in L^∞ space by using the method of compensated compactness; see Ding, Chen and Luo [6] for $1 < \gamma \leq \frac{5}{3}$ and Pan and myself [18] for $1 \leq \gamma < 3$. Thus, to study the large time behavior of solution of (1.1)–(1.2) with vacuum, it is suitable to consider the L^∞ weak solution.

Definition 1. *We call $(\rho, m)(x,t) \in L^\infty$ an entropy weak solution of (1.1)–(1.2), if it holds, for any non-negative test function $\phi \in \mathcal{D}(\mathbf{R}^2_+)$, that*

$$\begin{cases} \iint_{t>0}(\rho\phi_t + m\phi_x)\, dxdt + \int_{\mathbf{R}} \rho_0(x)\phi(x,0)\, dx = 0, \\ \iint_{t>0}[m\phi_t + (\frac{m^2}{\rho} + P(\rho))\phi_x - m\phi]\, dxdt + \int_{\mathbf{R}} m_0(x)\phi(x,0)\, dx = 0, \\ \iint_{t>0}(\eta_e\phi_t + q_e\phi_x - \rho u^2\phi)\, dxdt + \int_{\mathbf{R}} \eta_e(x,0)\phi(x,0)\, dx \geq 0. \end{cases} \tag{1.4}$$

Here, the entropy-flux pair (η_e, q_e) is associating with mechanical energy:

$$\begin{cases} \eta_e = \frac{1}{2}\rho u^2 + \frac{1}{(\gamma-1)}\rho^\gamma \\ q_e = \frac{1}{2}\rho u^3 + \frac{\gamma}{\gamma-1}\rho^\gamma u. \end{cases} \tag{1.5}$$

As the compensated compactness theory does not give any information on the regularity of the solutions, the methods for the case away from vacuum are not applicable here. Recently, some essential progress are made by Pan and myself. In [18], the authors followed the rescaling argument due to Serre and Hsiao [40] and obtained the first justification to the conjecture for vacuum case. It showed that the density in the L^∞ weak entropy solutions of (1.1) (1.2) converge to the similarity solution of porous medium equation along the level curve of the diffusive similarity profiles provided that one of the initial end-states is nonzero. The long time behavior of the momentum is not known however. This is far from satisfactory. Here we introduce new technique based on the conservation of mass and mechanical entropy analysis to prove this conjecture. We showed that the L^∞ weak entropy solutions with vacuum converge strongly in $L^p(R)$ with decay rates to the similarity solution of the porous medium equation determined uniquely by the end-states and the mass distribution of the initial data.

Let us explain the basic ideas. Two main difficulties are the lack of regularity and the singularity near vacuum. Our ideas are based on the nature of the system: the conservation of mass, the structure of the pressure law, the dissipation of damping and the existence of a convex entropy (the mechanical energy). We want to explore these features to control the singularity and nonlinearity. Our first observation is that the mechanical energy will give a uniform estimate for the solutions (ρ, m). However, this estimate is not useful in the proof of the long time behavior. We thus construct the proper functions by expanding the entropy around the self similar solutions $(\bar{\rho}, \bar{m})$ of Porous Media equations, this might give the estimate for the difference $(\rho - \bar{\rho}, m - \bar{m})$. In order to obtain the large time convergence, higher order estimates are necessary. One may perform the energy estimates for the derivatives if the solutions are smooth. However, our solutions are rather rough. It is possible to introduce anti-derivative $y(x, t)$ for $(\rho - \bar{\rho})(x, t)$. Thus, our entropy estimate becomes the derivatives estimates for y. Furthermore, the equation of y is wave equation with source term. Thus, the normal energy method will give some kind of estimate on y and its derivatives. Coupling these two estimates in a clever way, the uniform estimates for both y and its derivatives are possible. However, the life is not so easy. The singularity near vacuum makes our goal much further to reach. In order to control the singularity near vacuum, we explore the structure of the convection and found some useful inequality near vacuum. With the help of these inequality, the careful analysis on our two estimates gives the desired estimates. Then a weighted entropy estimates will give the decay rates. Our proof is somehow tricky and technical, this is due to the difficulties of the problem. Our argument becomes neat and simple when it was applied to the case away from vacuum.

Since (1.1) is hyperbolic, the entropy estimates is much more nature than the parabolic type energy estimate used in [10]. One may compare our proof with the proof by Liu and Hsiao [10] for smooth small solutions away from vacuum. In [10], the estimates were obtained by normal parabolic type energy method for wave equations. To weaken and decouple the nonlinearity, smallness and the third order estimates are necessary in order to close the arguments. One may check that such a method is not applicable for our case. The nonlinear terms in convection can not be controlled without higher order derivative estimates. Here, we succeed to close our argument in first order estimates for large rough solutions. This is one of the remarkable advantages of our approach.

2. Main theorems.

There are three subcases:

1). $\rho_+ > 0, \rho_- > 0$;

2). $\rho_+\rho_- = 0$, $\max\{\rho_+, \rho_-\} > 0$;

3). $\rho_- = \rho_+ = 0$ and $\int_{-\infty}^{+\infty} \rho_0(x)dx = M > 0$.

Let us first consider the case 1). Let us denote $\bar{\rho}$ the nonlinear wave of the first equation of (1.3) with boundary conditions

$$\bar{\rho}(\pm\infty) = \rho_\pm, \quad \rho_\pm > 0.$$

It is known that there exists a unique similarity solution $\bar{\rho}(\eta), \eta = \frac{x}{\sqrt{t+1}}$ with the boundary condition $\bar{\rho}(\eta) = \rho_\pm$, as $\eta \to \pm\infty$.

On the other hand, we define a constant x_0 by the following equation

$$\int_{-\infty}^{\infty} (\rho_0(x) - \bar{\rho}(x + x_0))dx = m_+ - m_-. \tag{2.1}$$

It is obvious that $\bar{\rho}(\frac{x+x_0}{\sqrt{t+1}})$ is the similarity solution of $(1.3)_1$, i.e. the following holds

$$\begin{cases} \bar{\rho}_t = P(\bar{\rho})_{xx} \\ \bar{\rho}|_{t=0} = \bar{\rho}(x + x_0). \end{cases} \tag{2.2}$$

Let

$$\bar{m} = -P(\bar{\rho})_x. \tag{2.3}$$

We shall compare the solution (ρ, m) of (1.1) with initial condition (1.2) to the functions $\bar{\rho} + \hat{\rho}(x, t), \bar{m} + \hat{m}(x, t)$, where the functions $\hat{\rho}(x, t), \hat{m}(x, t)$ are defined

$$\begin{cases} \hat{\rho}(x, t) = (m_+ - m_-)e^{-t}\theta(x), \\ \hat{m}(x, t) = m_- e^{-t} + (m_+ - m_-)e^{-t} \int_{-\infty}^{x} \theta(\xi)d\xi, \end{cases} \tag{2.4}$$

here $\theta(x)$ is a smooth function with compact support such that

$$\int_{-\infty}^{\infty} \theta(x)dx = 1.$$

Let

$$\begin{cases} y(x, t) = \int_{-\infty}^{x} \rho(\xi, t) - \bar{\rho}(\xi + x_0, t) - \hat{\rho}(\xi, t)d\xi, \\ z(x, t) = m(x, t) - \bar{m}(x + x_0, t) - \hat{m}(x, t). \end{cases} \tag{2.5}$$

Define

$$\begin{cases} y_0 = \int_{-\infty}^{x} \rho_0(\xi) - \bar{\rho}(\xi + x_0) - \hat{\rho}(\xi, 0)d\xi, \\ z(x, t) = m_0(x) - \bar{m}(x + x_0) - \hat{m}(x, 0). \end{cases} \tag{2.6}$$

Then we have the following results

Theroem 1 (see [19]). *Suppose that* $y_0(x) \in H^1$, $\rho_0(x) - \bar{\rho}(x + x_0) - \hat{\rho}(x, 0)$, $m_0(x) - \hat{m}(x, 0) \in L^2 \cap L^\infty$, *then there exists a global weak entropy solution of (1.1) such that*

$$\int_{-\infty}^{\infty} y^2 + y_x^2 + y_t^2 dx + \int_0^{\infty} \int_{-\infty}^{\infty} y_x^2 + y_t^2 dx dt \leq C \tag{2.7}$$

$$\int_{-\infty}^{\infty} |y_x|^p + |y_t|^p dx \leq C(1+t)^{-\alpha}, \; 2 \leq p < \infty, \tag{2.8}$$

where $\alpha < \frac{1}{4}$ *and the constant* $C > 0$ *only depends on the initial data. Furthermore, if* $\rho_- = \rho_+$, *then*

$$\int_{-\infty}^{\infty} |y_x|^p + |y_t|^p dx \leq C(1+t)^{-1}, \; 2 \leq p < \infty. \tag{2.9}$$

Remark 1.

(1) *Theorem 1 implies that the weak entropy solution* $\rho(x, t)$ *converges strongly in* $L^P(R)$ *towards the nonlinear diffusive profile* $\bar{\rho}$ *as* $t \to \infty$ *when* $\rho_-, \rho_+ > 0$. *Furthermore, theorem 1 also infers the strong convergence* $m(x, t)$ *to* $\bar{m}$, *while the results of [18] only show the weak convergence of* $m(x, t)$.

(2) *Theorem 1 claims the uniqueness of the asymptotic behavior for the solutions to (1.1), if the initial data has the same end-states. Hence, the asymptotic behavior of the solutions to (1.1) are uniquely determined by the end-states of initial data.*

Now we consider the case 2). we have the following stability theorem.

Theorem 2 (see [20]). *Suppose that* $y_0(x) \in H^1$, $y_x(x, 0), y_t(x, 0) \in L^2 \cap L^\infty$, *then there is a positive constant* C *independent of time such that, for any* $t \geq 0$, *it holds*

$$\begin{aligned} &\|y_x(\cdot, t)\|_{L^{\gamma+1}}^{\gamma+1} + \|y_t(\cdot, t)\|^2 + \int_0^t \|y_x(\cdot, \tau)\|_{L^{\gamma+1}}^{\gamma+1} \, d\tau \\ &+ \int_0^t \|y_t(\cdot, \tau)\|^2 \, d\tau \leq C. \end{aligned} \tag{2.10}$$

Remark 2. *Theorem 2 implies that the weak entropy solution* $\rho(x, t)$ *converges strongly in* $L^{\gamma+1}(R)$ *towards the nonlinear diffusive profile* $\bar{\rho}$ *as* $t \to \infty$ *when* $\rho_- \rho_+ = 0$ *and* $\max\{\rho_-, \rho_+\} > 0$. *It would be interest to prove the decay rates for this case.*

Now we consider the case 3) which is an important open problem. This case has particular interest since the asymptotic behavior is expected to be the famous Barenblatt's solution of porous media equation. We give a definite answer to this expectation when $\frac{1+\sqrt{5}}{2} < \gamma < 1 + \sqrt{2}$. We first give some properties of Barenblatt's solution.

By the results of [1], the solution of

$$\begin{cases} \bar{\rho}_t = \bar{\rho}^{\gamma}_{xx}, \\ \bar{\rho}(-1, x) = M\delta(x), \; M > 0. \end{cases} \tag{2.11}$$

should take the form

$$\bar{\rho}(x,t) = (t+1)^{-\frac{1}{\gamma+1}}\{(A - B\xi^2)_+\}^{\frac{1}{\gamma-1}}, \tag{2.12}$$

with $\xi = x(t+1)^{-\frac{1}{\gamma+1}}$, $(f)_+ = \max\{0, f\}$, $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$ and A determined by

$$2A^{\frac{\gamma+1}{2(\gamma-1)}}B^{-\frac{1}{2}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}\,d\theta = M. \tag{2.13}$$

$\bar{\rho}$ is a weak solution to (2.11) such that

$$\int_{-\infty}^{\infty}\bar{\rho}\,dx = M, \tag{2.14}$$

and

$$\bar{\rho} = 0, \ if \ |\xi| \geq \sqrt{A/B}. \tag{2.15}$$

Hence, for any finite time $T > 0$, $\bar{\rho}$ has compact support. This is the properties of finite speed of propagation for porous media equation. Furthermore, the derivatives of $\bar{\rho}$ is not continuous across the interface between the gas and vacuum. This is because the porous media equation is parabolic away from vacuum and is not at vacuum. For the definition of the weak solution to (2.11), we refer to [1], [2] and [38].

Kamin proved in [38] that (2.11) admits at most one solution. Here, we addressed the initial data at $t = -1$ to avoid the singularity at $t = 0$. Thus, we have the following lemmas from (2.11)–(2.15).

Lemma 2.1. *If M is finite, then there is one and only one solution $\bar{\rho}(x,t)$ to (2.11). Furthermore, the follows hold.*

(1) $\bar{\rho}(x,t)$ is continuous on R.

(2) There is a number $b = (\frac{A}{B})^{\frac{1}{2}} > 0$, such that $\bar{\rho}(x,t) > 0$ if $|x| < bt^{\frac{1}{\gamma+1}}$ and $\bar{\rho}(x,t) = 0$ if $|x| \geq bt^{\frac{1}{\gamma+1}}$.

(3) $\bar{\rho}(x,t)$ is smooth if $|x| < bt^{\frac{1}{\gamma+1}}$.

In terms of the explicit form of $\bar{\rho}$, it is easy to check the following estimates.

Lemma 2.2. *For $\bar{\rho}$ defined in (2.12) and $t > 0$, we have*

$$\begin{cases} |\bar{\rho}| \leq C(1+t)^{-\frac{1}{\gamma+1}}, \\ |(\bar{\rho}^{\gamma-1})_x| \leq C(1+t)^{-\frac{\gamma}{\gamma+1}}, \ |(\bar{\rho}^{\gamma-1})_t| \leq C(1+t)^{-\frac{2\gamma}{\gamma+1}}, \\ |(\bar{\rho}^{\gamma})_x| \leq C(1+t)^{-1}, \ |(\bar{\rho}^{\gamma})_t| \leq C(1+t)^{-\frac{2\gamma+1}{\gamma+1}}, \end{cases} \tag{2.16}$$

and

$$\begin{cases} \int_{-\infty}^{\infty} \bar\rho^2 dx \le C(1+t)^{-\frac{1}{\gamma+1}} \\ \int_{-\infty}^{\infty} \bar\rho^\gamma dx \le C(1+t)^{-\frac{\gamma-1}{\gamma+1}} \\ \int_{-\infty}^{\infty} (\bar\rho^{\gamma-1})_x^2 \, dx \le C(1+t)^{-\frac{2\gamma-1}{\gamma+1}} \\ \int_{-\infty}^{\infty} (\bar\rho^{\gamma-1})_t^2 \, dx \le C(1+t)^{-\frac{4\gamma-1}{\gamma+1}} \\ \int_{-\infty}^{\infty} (\bar\rho^{\gamma})_x^2 \, dx \le C(1+t)^{-\frac{2\gamma+1}{\gamma+1}} \\ \int_{-\infty}^{\infty} (\bar\rho^{\gamma})_t^2 \, dx \le C(1+t)^{-\frac{4\gamma+1}{\gamma+1}}. \end{cases} \tag{2.17}$$

Theorem 3 (see [17]). *Suppose $\rho_0(x) \in L^1(\mathbf{R})$, and*

$$M = \int_{-\infty}^{\infty} \rho_0(x) \, dx.$$

Let (ρ, m) be an L^∞ entropy weak solution of the Cauchy problem (1.1)–(1.2), satisfying the following estimates

$$0 \le \rho(x,t) \le C, \quad |m(x,t)| \le C\rho(x,t), \tag{2.18}$$

and let $\bar\rho$ be the Barenblatt's solution of (1.3) with mass M and $\bar m = -P(\bar\rho)_x$. Then

$$\begin{cases} \|\bar\rho\|_{L^2}^2 \le C(1+t)^{-\frac{1}{\gamma+1}}, \\ \|\bar\rho\|_{L^\gamma}^\gamma \le C(1+t)^{-\frac{\gamma-1}{\gamma+1}}. \end{cases} \tag{2.19}$$

Define $y = -\int_{-\infty}^{x} (\rho - \bar\rho)(r,t) \, dr$. If $y \in L^2(\mathbf{R})$, then there exist positive constants $k_1 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{\gamma-1}{\gamma}\}$, $k_2 = \min\{\frac{\gamma^2}{(\gamma+1)^2}, \frac{1}{\gamma}\}$ and C such that for any $\varepsilon > 0$,

$$\begin{cases} \|(\rho-\bar\rho)(x,t)\|_{L^2}^2 \le C(1+t)^{-k_1+\varepsilon}, if \ 1 < \gamma \le 2, \\ \|(\rho-\bar\rho)(x,t)\|_{L^\gamma}^\gamma \le C(1+t)^{-k_2+\varepsilon}, if \ \gamma \ge 2. \end{cases} \tag{2.20}$$

Furthermore,

$$\begin{cases} k_1 > \frac{1}{\gamma+1}, if \ \frac{1+\sqrt5}{2} < \gamma \le 2, \\ k_2 > \frac{\gamma-1}{\gamma+1}, if \ 2 \le \gamma < 1+\sqrt2. \end{cases} \tag{2.21}$$

Remark 3. *(1) Condition (2.16) is fulfilled if the solutions are in the physical region initially. The invariant region theory verifies (2.16).*

(2) Since Barenblatt's solution $\bar\rho$ decays itself, it is necessary to compare the decay rate of $\bar\rho$ with that of $\rho-\bar\rho$. (2.21) shows that $\|\rho-\bar\rho\|_{L^2}$ decays fast than $\|\bar\rho\|_{L^2}$ when $\frac{1+\sqrt5}{2} < \gamma \le 2$ and $\|\rho-\bar\rho\|_{L^\gamma}$ decays fast than $\|\bar\rho\|_{L^\gamma}$ when $2 \le \gamma < 1+\sqrt2$. It is noted that $\frac{5}{3} \in (\frac{1+\sqrt5}{2}, 1+\sqrt2)$. Thus Theorem 3 states that any L^∞ entropy weak solutions of (1.1)–(1.2) satisfying the conditions in Theorem 3 must converge to the

related Barenblatt's solution of (1.3) with the same mass when $\gamma \in (\frac{1+\sqrt{5}}{2}, 1+\sqrt{2})$. Although there is not uniqueness for the solutions, our results indicated the unique asymptotic profile determined by the initial mass.

(3) It would be interest to prove that there exists some constant p such that $\|\rho - \bar{\rho}\|_{L^p}$ decays fast than $\|\bar{\rho}\|_{L^p}$ when $\gamma \in (1, \frac{1+\sqrt{5}}{2})$ and $\gamma \in (1+\sqrt{2}, \infty)$.

(4) The decay rates in Theorem 3 seem not to be optimal, it would be interest to find the optimal decay rates.

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