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A missing term in the energy inequality for weak solutions to the Navier-Stokes equations

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1 Introduction

Let Ω be a domain in $\mathbf{R}^3$, and we denote the set of smooth solenoidal vectors with the compact support in Ω by $\mathcal{V}$. The spaces H and V are respectively the completion of $\mathcal{V}$ in the topology of $L^2(\Omega)$ and $H^1(\Omega)$. V' is the dual space of V with respect to the $L^2(\Omega)$-paring. For given $\boldsymbol{u}_0 \in H$ and $\boldsymbol{f} \in L^2(0,T;V')$, it is well-known that the initial(-boundary) value problem of the Navier-Stokes equations

$$\begin{cases} \boldsymbol{u}_t - \Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in} \quad \Omega\times(0,\infty), \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \boldsymbol{u}|_{\partial\Omega\times(0,\infty)} = \boldsymbol{o} & (\text{if} \quad \partial\Omega\neq\emptyset), \\ \boldsymbol{u}|_{\Omega\times\{t=0\}} = \boldsymbol{u}_0 & \end{cases} \tag{1.1}$$

has a weak solution $\boldsymbol{u}$ in the sense of Leray-Hopf, which satisfies the energy inequality

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}|^2 dx + \int_0^t\int_\Omega |\nabla\boldsymbol{u}|^2 dx d\tau \le \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2 dx + \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, dx d\tau.$$

It is uncertain that $\boldsymbol{u}$ satisfies the energy identity

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}|^2 dx + \int_0^t\int_\Omega |\nabla\boldsymbol{u}|^2 dx d\tau = \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2 dx + \int_0^t\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\, dx d\tau.$$

Furthermore it is still unsolved that every weak solution satisfies the energy inequality.

The author has been investigated the energy inequality or identity with extra term in [1, 2]. In particular we had the following result.

Theorem 1.1 ([1]) *Assume that* Ω *is bounded. There exists a weak solution satisfying an energy inequality with an extra term*

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}|^2 dx + \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2 dx d\tau$$
$$+\frac{1}{2}\limsup_{h\downarrow 0}\int_h^t \int_\Omega \left|\frac{\boldsymbol{u}(x,\tau)-\boldsymbol{u}(x,\tau-h)}{h^{\frac{1}{2}}}\right|^2 dx d\tau$$
$$\leq \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2 dx + \int_0^t \int_\Omega \boldsymbol{f}\cdot \boldsymbol{u}\, dx d\tau.$$

It is a still inequality. In the paper [2], we discuss the energy of weak solutions satisfying a posteriori estimate

$$\limsup_{h\downarrow 0}\int_h^t \int_\Omega \left|\frac{\boldsymbol{u}(x,\tau)-\boldsymbol{u}(x,\tau-h)}{h^{\frac{1}{2}}}\right|^2 dx d\tau = 0, \qquad (1.2)$$

and got an energy identity with an extra term.

In this note we shall give a similar result without (1.2). Further we shall also improve the result in [2] under the assumption (1.2).

The energy identity is formally derived from the inner product between the both sides of the Navier-Stokes equations and the solution $\boldsymbol{u}$ itself. However, the paring $\int_\Omega \boldsymbol{u}_t \cdot \boldsymbol{u}\, dx$ is not integrable in t, because of $\boldsymbol{u}_t \in L^{\frac{4}{3}}(0,T;V')$ and of $\boldsymbol{u} \in L^2(0,T;V)$. This obstracts the validity of the relation

$$\int_0^t \int_\Omega \boldsymbol{u}_t \cdot \boldsymbol{u}\, dx dt = \frac{1}{2}\int_0^t \frac{d}{dt}\int_\Omega |\boldsymbol{u}|^2 dx dt = \frac{1}{2}\int_\Omega |\boldsymbol{u}(t)|^2 dx - \frac{1}{2}\int_\Omega |\boldsymbol{u}_0|^2 dx.$$

To aboid this difficulty we use the following idea. Put

$$\boldsymbol{U}(t) = \begin{cases} \varphi(t)\boldsymbol{u}(t) & \text{for} \quad t > 0, \\ \boldsymbol{o} & \text{for} \quad t \leq 0, \end{cases}$$

Here φ is an arbitrarily fixed function in $C_0^\infty(\mathbf{R};\mathbf{R})$ with supp $\varphi \subset (0,\infty)$. We consider the paring $\int_\Omega \boldsymbol{U}_t(t)\cdot \boldsymbol{U}(t-s)\, dx$ instead of $\int_\Omega \boldsymbol{U}_t(t)\cdot\boldsymbol{U}(t)\, dx$. Then by virtue of the Hausdorff-Young's inequality, $\int_\Omega \boldsymbol{U}_t(t)\cdot \boldsymbol{U}(t-s)\, dx$ is integrable on $\mathbf{R}^2$ as a function of t and s. Therefore by Funini's theorem, $\int_\Omega \boldsymbol{U}_t(\cdot)\cdot\boldsymbol{U}(\cdot - s)\,dx$ is in $L^1(\mathbf{R})$ for almost all s. By passing the limit

$$\lim_{s\to 0}\int_0^t\int_\Omega \boldsymbol{U}_t(t)\cdot\boldsymbol{U}(t-s)\,dxdt$$

in some sense, we can get the energy identity with a extra term. The expresion of the extra term depends on the regularity of weak solution (of course the extra term vanishes provided the solution is smooth enough). Therefore we should give the expression under the condition as weak as possible. For that we label the following consitions as [C1] – [C4] in the sequel.

[C1] $\boldsymbol{u} \in L^\infty(0,T;H)\cap L^2(0,T;V)$ with $\boldsymbol{u}_t \in L^{\frac{4}{3}}(0,T;V')$ satisfies (1.1) in the sense of Leray-Hopf.

[C2] $\boldsymbol{u}$ satisfies [C1] and

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbf{R}}\|\boldsymbol{U}(t)-\boldsymbol{U}(t-h)\|_H^2 dt = 0.$$

[C3] $\boldsymbol{u}$ satisfies [C1] and

$$\frac{1}{h}\int_{\mathbf{R}}\|\boldsymbol{U}(t)-\boldsymbol{U}(t-h)\|_H^2 dt \le \omega(|h|)^2, \quad \int_0^{|h|}\frac{\omega(\rho)^2}{\rho}\,d\rho < 0.$$

[C4] $\boldsymbol{u}$ satisfies [C1] and

$$\frac{1}{h}\int_{\mathbf{R}}\|\boldsymbol{U}(t)-\boldsymbol{U}(t-h)\|_H^2 dt \le \omega(|h|)^2, \quad \int_0^{|h|}\frac{\omega(\rho)}{\rho}\,d\rho < 0.$$

From now we denote the paring between the elements of V' and V by $\langle\cdot,\cdot\rangle_{V',V}$; and the inner product on H by $\langle\cdot,\cdot\rangle_H$. And the operators A and B from $V'\to V$ are defined by

$$\langle A\boldsymbol{u},\boldsymbol{v}\rangle_{V',V} = -\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx,$$

$$\langle B\boldsymbol{v},\boldsymbol{v}\rangle_{V',V} = \int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{v}\,dx.$$

Then we can write (1.1) as

$$\begin{cases} \boldsymbol{u}_t + A\boldsymbol{u} + B\boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad V' \quad \text{a.e. } t, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases}$$

Theorem 1.2 *Let $\boldsymbol{u}$ be a weak solution. Then the identity*

$$-\frac{1}{2}\int_0^\infty \|\boldsymbol{u}(t)\|_H^2 \frac{d\varphi(t)}{dt}\,dt + \int_0^\infty \langle A\boldsymbol{u}(t), \boldsymbol{u}(t)\rangle_{V',V}\varphi(t)\,dt$$

$$+ \text{“}\lim_{s\to 0}\text{”} \int_0^\infty \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\,dt$$

$$= \int_0^\infty \langle \boldsymbol{f}(t), \boldsymbol{u}(t)\rangle_{V',V}\varphi(t)\,dt$$

holds for any $\varphi \in C_0^\infty(0,\infty)$ in the sense of

$$\text{“}\lim_{s\to 0}\text{”}\cdot = \begin{cases} \lim_{\varepsilon\to+0} \dfrac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{\varepsilon} \cdot\, ds & \textit{for the case } [\mathrm{C1}], \\ \lim_{h\to 0} \dfrac{1}{h}\displaystyle\int_0^h \cdot\, ds & \textit{for the case } [\mathrm{C2}], \\ \mathrm{ap}\lim_{s\to 0}\cdot & \textit{for the case } [\mathrm{C3}]. \end{cases}$$

Here ap lim *is the approximate limit.*

Remark 1.1 In [2] we have proved a similar result for the case [C3] with

$$\text{“}\lim_{s\to 0}\text{”}\cdot = \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \cdot\, ds.$$

This is improved as above.

Theorem 1.3 *Assume that a weak solution $\boldsymbol{u}$ satisfies* [C4]. *For given t, s $(t > s > 0)$, we take ε and δ so small that $0 < \varepsilon < s - \delta$, and $s + \delta < t - \delta$. Let $\varphi_{\delta,s,t} \in C_0^\infty(0,\infty)$ satisfy* $\mathrm{supp}\,\varphi_{\delta,s,t} \subset [s-\delta, t+\delta]$, *$\varphi_{\delta,s,t}(\tau) \equiv 1$ on $[s+\delta, t-\delta]$, and $\left|\dfrac{d\varphi_{\delta,s,t}(\tau)}{d\tau}\right| \le C\delta^{-1}$ Then $\boldsymbol{u}$ belongs to $C([0,\infty); H)$, and the identity*

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_H^2 + \int_s^t \langle A\boldsymbol{u}(\tau), \boldsymbol{u}(\tau)\rangle_{V',V}d\tau$$

$$+ \lim_{\delta\downarrow 0} \mathrm{ap}\lim_{\tau\to 0} \int_0^\infty \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-\tau)\rangle_{V',V}\varphi_{\delta,s,t}(t)\,dt$$

$$= \frac{1}{2}\|\boldsymbol{u}(s)\|_H^2 + \int_s^t \langle \boldsymbol{f}(\tau), \boldsymbol{u}(\tau)\rangle_{V',V}d\tau$$

Passing to the limit as $s \downarrow 0$, we have

Corollary 1.1 *If $\boldsymbol{u}$ satisfies* [C4], *then it holds that*

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_H^2 + \int_0^t \langle A\boldsymbol{u}(\tau), \boldsymbol{u}(\tau)\rangle_{V',V} d\tau$$

$$+ \lim_{s\downarrow 0} \lim_{\delta\downarrow 0} \mathrm{ap} \lim_{\tau\to 0} \frac{1}{2\varepsilon} \int_0^\infty \int_{-\varepsilon}^{\varepsilon} \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-\tau)\rangle_{V',V} \varphi_{\delta,s,t}(t)\, dt$$

$$= \frac{1}{2}\|\boldsymbol{u}_0\|_H^2 + \int_0^t \langle \boldsymbol{f}(\tau), \boldsymbol{u}(\tau)\rangle_{V',V} d\tau.$$

2 Proofs

The energy identity is reduced to

$$\int_{\mathbf{R}} \langle \boldsymbol{U}_t, \boldsymbol{U}\rangle_{V',V}\, dt = 0,$$

if $\boldsymbol{u}$ is sufficiently smooth. Indeed if so, then

$$\begin{aligned} 0 &= \int_{\mathbf{R}} \langle \boldsymbol{U}_t, \boldsymbol{U}\rangle_{V',V}\, dt \\ &= \int_0^\infty \langle \varphi \boldsymbol{u}_t + \varphi_t \boldsymbol{u}, \varphi \boldsymbol{u}\rangle_{V',V}\, dt \\ &= \int_0^\infty \langle \boldsymbol{u}_t, \boldsymbol{u}\rangle_{V',V}\, \varphi^2 dt + \frac{1}{2}\int_0^\infty \|\boldsymbol{u}\|_H^2 \frac{d}{dt}\varphi^2 dt \\ &= -\int_0^\infty \langle A\boldsymbol{u} + B\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{u}\rangle_{V',V}\, \varphi^2 dt + \frac{1}{2}\int_0^\infty \|\boldsymbol{u}\|_H^2 \frac{d}{dt}\varphi^2 dt. \end{aligned}$$

Taking $\varphi = \varphi_j \in C_0^\infty(0,\infty)$ such that

$$\varphi_j^2 \to \chi_{[0,t]}, \quad \frac{d}{dt}\varphi_j^2 \to -\delta_t + \delta_0$$

as $j \to \infty$, we get the desired identity. Here χ_K is the characteristic function of the set K, and δ_p is the Direc mass at p.

Consequently the proof of Theorems is reduced to showing

$$\int_{\mathbf{R}} \langle \boldsymbol{U}_t(t), \boldsymbol{U}(t-s)\rangle_{V',V}\, dt \to 0 \quad \text{as} \quad s \to 0$$

in some sense.

Proposition 2.1 *We have*

$$\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds=0 \qquad \textit{for the case } [\mathrm{C1}],$$

$$\lim_{h\to 0}\frac{1}{h}\int_{0}^{h}\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds=0 \qquad \textit{for the case } [\mathrm{C2}],$$

$$\mathrm{ap}\lim_{s\to 0}\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dt=0 \qquad \textit{for the case } [\mathrm{C3}].$$

Proof. **Case [C1].** Since $\boldsymbol{U}\in L^2(\mathbf{R};H)$ with compact support, we have

$$\begin{aligned}
&\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s-h),\boldsymbol{U}(s)\rangle_H\,ds\\
&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h),\boldsymbol{U}(s)\rangle_H\,ds-\int_{\mathbf{R}}\langle \boldsymbol{U}(s-h),\boldsymbol{U}(s)\rangle_H\,ds\\
&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h),\boldsymbol{U}(s)\rangle_H\,ds-\int_{\mathbf{R}}\langle \boldsymbol{U}(s),\boldsymbol{U}(s+h)\rangle_H\,dt=0.
\end{aligned}$$

Therefore it holds that

$$\begin{aligned}
0&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s-h),\boldsymbol{U}(s)\rangle_H\,ds\\
&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s-h),\boldsymbol{U}(s)\rangle_{V',V}\,ds\\
&=\int_{\mathbf{R}}\left\langle\int_{s-h}^{s+h}\boldsymbol{U}_t(t)\,dt,\boldsymbol{U}(s)\right\rangle_{V',V}ds.
\end{aligned}$$

Since $\boldsymbol{U}_t\in L^1(\mathbf{R};V')$ and $\boldsymbol{U}\in L^2(\mathbf{R};V)$ with compact support, the above integral has meaning. Using Fubini's theorem, we have

$$\begin{aligned}
0&=\int_{\mathbf{R}}\left\langle\boldsymbol{U}_t(t),\int_{t-h}^{t+h}\boldsymbol{U}(s)\,ds\right\rangle_{V',V}dt\\
&=\int_{\mathbf{R}}\left\langle\boldsymbol{U}_t(t),\int_{-h}^{h}\boldsymbol{U}(t-s)\,ds\right\rangle_{V',V}dt\\
&=\int_{-h}^{h}\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds.
\end{aligned}$$

Consequently we get

$$\left(\lim_{h\to 0}\frac{1}{2h}\right)\int_{-h}^{h}\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds=0.$$

Case [C2]. Since

$$\begin{aligned}0&=\int_{\mathbf{R}}\left(\|\boldsymbol{U}(s+h)\|_H^2-\|\boldsymbol{U}(s)\|_H^2\right)ds\\&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s+h)+\boldsymbol{U}(s)\rangle_H\,ds\\&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s+h)-\boldsymbol{U}(s)+2\boldsymbol{U}(s)\rangle_H\,ds\\&=\int_{\mathbf{R}}\|\boldsymbol{U}(s+h)-\boldsymbol{U}(s)\|_H^2\,ds+2\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s)\rangle_H\,ds.\end{aligned}$$

we have

$$\lim_{h\to 0}\frac{1}{h}\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s)\rangle_H\,ds=0$$

by [C2]. By Fubini's theorem we get

$$\begin{aligned}&\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s)\rangle_H\,ds\\&=\int_{\mathbf{R}}\langle \boldsymbol{U}(s+h)-\boldsymbol{U}(s),\boldsymbol{U}(s)\rangle_{V',V}\,ds\\&=\int_{\mathbf{R}}\left\langle\int_s^{s+h}\boldsymbol{U}_t(t)\,dt,\boldsymbol{U}(s)\right\rangle_{V',V}ds\\&=\int_{\mathbf{R}}\left\langle \boldsymbol{U}_t(t),\int_{t-h}^{t}\boldsymbol{U}(s)\,ds\right\rangle_{V',V}dt\\&=\int_{\mathbf{R}}\left\langle \boldsymbol{U}_t(t),\int_0^{h}\boldsymbol{U}(t-s)\,ds\right\rangle_{V',V}dt\\&=\int_0^h\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds.\end{aligned}$$

Consequently we get

$$\lim_{h\to 0}\frac{1}{h}\int_0^h\int_{\mathbf{R}}\langle \boldsymbol{U}_t(t),\boldsymbol{U}(t-s)\rangle_{V',V}\,dtds=0.$$

Case [C3]. If $\boldsymbol{u}$ satisfies [C3], then

$$\int_{\mathbf{R}} (1+|\tau|)\|\hat{\boldsymbol{U}}(\tau)\|_H^2 d\tau < \infty, \quad \hat{\boldsymbol{U}}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-it\tau}\boldsymbol{U}(t)\,dt$$

(by refinement of the argument of J. Simon [3]; see also [2]). Put

$$U(s) = \int_{\mathbf{R}} \langle \boldsymbol{U}_t(t), \boldsymbol{U}(t-s)\rangle_{V',V}\,dt.$$

Then $\hat{U}(\tau) = -\sqrt{2\pi} i\tau \|\hat{\boldsymbol{U}}(\tau)\|_H^2$ is an odd function, and belongs to $L^1 \cap L^2(\mathbf{R})$. Therefore $\mathcal{F}^{-1}[\hat{U}]$ is continuous, $\mathcal{F}^{-1}[\hat{U}](0) = 0$, and

$$U(s) = \mathcal{F}^{-1}[\hat{U}](s) \quad \text{a. e.} \quad s \in \mathbf{R}.$$

For $\varepsilon > 0$ put

$$E_\varepsilon = \{ s \in (-r,r) \mid |U(s)| > \varepsilon \}.$$

Since $\mathcal{F}^{-1}[\hat{U}]$ is continuous,

$$\mathcal{L}^1(E_\varepsilon) = \mathcal{L}^1\left(\left\{ s \in (-r,r) \,\middle|\, \left|\mathcal{F}^{-1}[\hat{U}](s)\right| > \varepsilon \right\}\right),$$

is zero for small $\varepsilon > 0$. Therefore we have

$$\lim_{r \to +0} \frac{\mathcal{L}^1(E_\varepsilon)}{2r} = 0.$$

Proof of Theorem 1.2. Put

$$\begin{aligned}
&\int_{\mathbf{R}} \langle \boldsymbol{U}_t(t), \boldsymbol{U}(t-s)\rangle_{V',V}\,dt \\
= &\int_0^\infty \langle \varphi(t)\boldsymbol{u}_t(t) + \varphi_t(t)\boldsymbol{u}(t), \varphi(t-s)\boldsymbol{u}(t-s)\rangle_{V',V} dt \\
= &\int_0^\infty \langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)^2 dt \\
&+ \frac{1}{2}\int_0^\infty \|\boldsymbol{u}\|_H^2 \frac{d}{dt}\varphi(t)^2 dt \\
&+ \int_0^\infty \langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)(\varphi(t-s)-\varphi(t))\,dt \\
&+ \int_0^\infty \|\boldsymbol{u}(t)\|_H^2 \varphi_t(t)(\varphi(t-s)-\varphi(t))\,dt \\
&+ \int_0^\infty \langle \boldsymbol{u}(t), \boldsymbol{u}(t-s) - \boldsymbol{u}(t)\rangle_H \varphi_t(t)\varphi(t-s)\,dt \\
= &\int_0^\infty (J_1(t,s) + J_2(t,s) + J_3(t,s) + J_4(t,s) + J_5(t,s))\,dt,
\end{aligned}$$

$$I_i(s) = \int_0^\infty J_i(t,s)\,dt.$$

Assume that supp $\varphi \subset [t_0, t_1]$ for t_0 and t_1 satisfying $0 \le t_0 - 2\varepsilon_0 < t_1 + 2$ T,. There exists $C > 0$ such that $\sup_{|s|\le\varepsilon} |\varphi(t-s) - \varphi(t)| \le C\varepsilon$, and $|\varphi|$ hold. Then it follows from $\boldsymbol{u}_t \in L^1(0,T;V')$ and $\boldsymbol{u} \in L^2(0,T;V)$ that

$$\begin{aligned}
\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I_3(s)|\,ds = {} & C\int_{-\varepsilon}^{\varepsilon}\int_{t_0-\varepsilon}^{t_1+\varepsilon} |\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}|\,dtds \\
\le {} & C\int_{t_0-\varepsilon}^{t_1+\varepsilon} \|\boldsymbol{u}_t(t)\|_{V'} \left(\int_{-\varepsilon}^{\varepsilon} \|\boldsymbol{u}(t-s)\|_V ds\right) dt \\
\le {} & C\int_0^T \|\boldsymbol{u}_t(t)\|_{V'}\sqrt{2\varepsilon}\left(\int_0^T \|\boldsymbol{u}(s)\|_V^2 ds\right)^{\frac{1}{2}} dt \\
\le {} & C\sqrt{\varepsilon} \to 0
\end{aligned}$$

as $\varepsilon_0 > \varepsilon \downarrow 0$. In particular for any $\delta > 0$

$$\frac{\mathcal{L}^1(\{s \in (-\varepsilon,\varepsilon) \mid |I_3(s)| > \delta\})}{2\varepsilon} \le \frac{1}{\delta}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I_3(s)|\,ds \to 0$$

as $\varepsilon_0 > \varepsilon \downarrow 0$. This means

$$\operatorname{ap}\lim_{s\to 0} I_3(s) = 0$$

Similarly we have

$$\lim_{h\to 0}\int_0^h |I_3(s)|\,ds = 0.$$

In a similar manner we get

$$\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I_4(s)|\,ds = \lim_{h\to 0}\frac{1}{h}\int_0^h |I_4(s)|\,ds = \operatorname{ap}\lim_{s\to 0} I_4(s) = 0.$$

Since $\boldsymbol{u} \in L^2 \cap L^\infty(0,T;H)$, it holds that

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I_5(s)|\,ds$$

$$\le C\| \sup_{t\in(0,T)} \|\boldsymbol{u}(t)\|_H \int_{\min\{t_0,t_0+s\}}^{\max\{t_1,t_1+s\}} \|\boldsymbol{u}(t)\|_H \|\boldsymbol{u}(t-s) - \boldsymbol{u}(t)\|_H dtds \to 0$$

as $\varepsilon_0 > \varepsilon \downarrow 0$. Therefore we can get

$$\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I_5(s)|\,ds = \lim_{h\to 0}\frac{1}{h}\int_0^h |I_5(s)|\,ds = \operatorname{ap}\lim_{s\to 0} I_5(s) = 0.$$

Combining these estimates with Fubini's theorem and Proposition 2.1, we obtain

$$\frac{1}{2}\int_0^\infty \|\boldsymbol{u}(t)\|_H^2 \frac{d}{dt}\varphi^2 dt$$

$$= \begin{cases} -\lim\limits_{\varepsilon\downarrow 0}\dfrac{1}{2\varepsilon}\displaystyle\int_0^\infty\int_{-\varepsilon}^{\varepsilon}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)^2 dsdt & \text{in the case [C1]},\\ -\lim\limits_{h\to 0}\dfrac{1}{h}\displaystyle\int_0^\infty\int_0^h\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)^2 dsdt & \text{in the case [C2]},\\ -\mathrm{ap}\lim\limits_{s\to 0}\displaystyle\int_0^\infty\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)^2 dt & \text{in the case [C3]}.\end{cases}$$

We can replace φ^2 by φ. Indeed, for $\varphi \in C_0^\infty(2\varepsilon_0, \infty)$ and $\psi \in C_0^\infty(2\varepsilon_0, \infty)$ we have in [C1], taking $\varepsilon < \varepsilon_0$,

$$\frac{1}{2}\int_0^\infty \|\boldsymbol{u}(t)\|_H^2 \frac{d}{dt}(\varphi+\psi)^2 dt$$

$$= -\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^\infty\int_{-\varepsilon}^{\varepsilon}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}(\varphi+\psi)^2 dsdt,$$

which reduces to

$$\frac{1}{2}\int_0^\infty \|\boldsymbol{u}(t)\|_H^2 \left(\frac{d\varphi}{dt}\psi + \varphi\frac{d\psi}{dt}\right) dt$$

$$= -\lim_{\varepsilon_0>\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^\infty\int_{-\varepsilon}^{\varepsilon}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi\psi\, dsdt.$$

Take ψ such that $\operatorname{supp}\varphi \subset \operatorname{supp}\psi$, and $\psi \equiv 1$ on $\operatorname{supp}\varphi$. Then we have

$$\frac{1}{2}\int_0^\infty \|\boldsymbol{u}\|_H^2 \frac{d\varphi}{dt} dt = -\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\, dsdt.$$

Other cases are proved in the same way. This shows that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_H^2 = \begin{cases} \lim\limits_{\varepsilon\downarrow 0}\dfrac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{\varepsilon}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V} ds & \text{in the case [C1]},\\ \lim\limits_{h\to 0}\dfrac{1}{h}\displaystyle\int_0^h\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V} ds & \text{in the case [C2]},\\ \mathrm{ap}\lim\limits_{s\to 0}\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V} ds & \text{in the case [C3]}\end{cases}$$

in $\mathcal{D}'(0,\infty)$. We now prove

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}(t-s)\,ds \to \boldsymbol{u}(t) \quad \text{as} \quad \varepsilon_0 > \varepsilon \downarrow 0$$

in $L^2(\varepsilon_0, T; V)$. Indeed, it holds that

$$\left\|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} (\boldsymbol{u}(t-s) - \boldsymbol{u}(t))\,ds\right\|_V^2 \le \frac{1}{2}\int_{-1}^{1} \|\boldsymbol{u}(t-\varepsilon s) - \boldsymbol{u}(t)\|_V^2 ds,$$

and therefore

$$\int_{\varepsilon_0}^{T}\left\|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} (\boldsymbol{u}(t-s) - \boldsymbol{u}(t))\,ds\right\|_V^2 dt \le \sup_{|s|\le 1}\int_{\varepsilon_0}^{T} \|\boldsymbol{u}(t-\varepsilon s) - \boldsymbol{u}(t)\|_V^2 dt \to 0$$

as $\varepsilon_0 > \varepsilon \to 0$. Consequently we have for $\varphi \in C_0^\infty(2\varepsilon_0, \infty)$

$$\int_0^\infty \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \langle -A\boldsymbol{u}(t) + \boldsymbol{f}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\,dsdt$$

$$\to \int_0^\infty \langle -A\boldsymbol{u}(t) + \boldsymbol{f}(t), \boldsymbol{u}(t)\rangle_{V',V}\varphi(t)\,dt$$

as $\varepsilon_0 > \varepsilon \to 0$.

Assume that supp $\varphi \subset (2\varepsilon_0, T)$. By using of Hausdorff-Young's inequality we have

$$\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t) \in L^2((0,T)\times(0,T)),$$
$$\langle -A\boldsymbol{u}(t) - B\boldsymbol{u}(t) + \boldsymbol{f}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t) \in L^2((0,T)\times(0,T)),$$

and

$$\langle \boldsymbol{u}_t(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t) = \langle -A\boldsymbol{u}(t) - B\boldsymbol{u}(t) + \boldsymbol{f}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)$$

for almost all t, and for all s. Therefore we have verified the existence of the limit

$$\lim_{\varepsilon_0 > \varepsilon \downarrow 0} \frac{1}{2\varepsilon}\int_0^\infty \int_{-\varepsilon}^{\varepsilon} \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\,dsdt.$$

In the same way we can see the existence of

$$\begin{cases} \lim_{h\to 0} \dfrac{1}{h}\displaystyle\int_0^\infty \int_0^h \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\,dsdt & \text{in the case [C2]}, \\ \operatorname{ap}\lim_{s\to 0} \displaystyle\int_0^\infty \langle B\boldsymbol{u}(t), \boldsymbol{u}(t-s)\rangle_{V',V}\varphi(t)\,dt & \text{in the case [C3]}. \end{cases}$$

Consequently the proof is complete.

Proof of Theorem 1.3 (sketch). If $\boldsymbol{u}$ satisfies [C4], then it belongs to $C([0,T); H)$ (also using the argument of J. Simon [3]; see also [2]). Inserting $\varphi = \varphi_{\delta,s,t}$ in the identity in Theorem 1.2, and passing to the limit $\delta \downarrow 0$, we get the identity in Theorem 1.3.

References

[1] Nagasawa, T., *A new energy inequality and partial regularity for weak solutions of Navier-Stokes equations*, J. Math. Fluid Mech. 3 (2001), 40–56.

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[3] Simon, J., *Sobolev, Besov and Nikolskii fractional spaces: Imbeddings and comparisons for vector valued spaces on an interval*, Ann. Mat. Pura Appl. (4) **157** (1990), 117–148.