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ON THE STEFAN PROBLEM WITH SURFACE TENSION IN A VISCOUS INCOMPRESSIBLE FLUID FLOW

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Abstract. A solidification/melting process with the supercooling near the interface in the case where the fluid is flowing is described by the Stefan problem with Gibbs-Thompson law at the interface and the initial-boundary value problem for the incompressible Navier-Stokes equations. This paper is devoted to prove that the set of classical solutions of the problem mentioned above converges to the solution of the problem without the supercooling as the surface tension coefficient tends to zero.

1. Introduction. Let a region Ω with outer boundary Σ be separated by a moving boundary Γ_t into the liquid region $\Omega_t^{(1)}$ and the solid region $\Omega_t^{(2)}$. Let $\boldsymbol{v}$, p, and $\theta^{(1)}$ be the velocity , the pressure and the temperature of the liquid, respectively. They are assumed to satisfy the following equations:

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1.1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p - \nu\Delta\boldsymbol{v} = \boldsymbol{f}(\theta^{(1)}), \tag{1.2}$$

$$\begin{aligned}&\frac{\partial \theta^{(1)}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\theta^{(1)} - \frac{1}{\rho C_p^{(1)}}\nabla\cdot\left(\kappa^{(1)}(\theta^{(1)})\nabla\theta^{(1)}\right)\\ &= \frac{2\nu}{C_p^{(1)}}\mathbf{D}(\boldsymbol{v}):\mathbf{D}(\boldsymbol{v}) \qquad \text{in} \bigcup_{0<t\leq T}\left(\Omega_t^{(1)}\times\{t\}\right),\end{aligned} \tag{1.3}$$

These are the Navier-Stokes equations and the heat equation with the transport and viscous dissipation terms, where ν, ρ, $C_p^{(1)}$ and $\kappa^{(1)}$ are a kinematic viscosity, the density, the specific heat at the constant pressure and the heat conductivity of the liquid, respectively. In $\Omega_t^{(2)}$, we consider only the heat transfer:

$$\frac{\partial \theta^{(2)}}{\partial t} - \frac{1}{\rho_e C_p^{(2)}}\nabla\cdot\left(\kappa^{(2)}(\theta^{(2)})\nabla\theta^{(2)}\right) = 0 \qquad \text{in} \bigcup_{0<t\leq T}\left(\Omega_t^{(2)}\times\{t\}\right), \tag{1.4}$$

where ρ_e, $C_p^{(2)}$ and $\kappa^{(2)}$ are the density, the specific heat at the constant pressure and the heat conductivity of the solid, respectively. On the liquid-solid interface Γ_t, we impose the following conditions:

$$\boldsymbol{v}\cdot\boldsymbol{n} = \left(1 - \frac{\rho_e}{\rho}\right)V, \tag{1.5}$$

$$2\nu\Pi\mathbf{D}(\boldsymbol{v})\boldsymbol{n} = \Pi[\boldsymbol{v}(\boldsymbol{v} - V\boldsymbol{n})^*]\boldsymbol{n}, \tag{1.6}$$

$$l\rho_e V = -\left(\kappa^{(1)}(\theta^{(1)})\nabla\theta^{(1)} - \kappa^{(2)}(\theta^{(2)})\nabla\theta^{(2)}\right)\cdot\boldsymbol{n}, \tag{1.7}$$

(1.8) $$\theta^{(1)} = \theta^{(2)} = \theta_1 \left(1 - \frac{\sigma}{l} H\right),$$

or

(1.9) $$\theta^{(1)} = \theta^{(2)} = \theta_1 \quad \text{on} \quad \bigcup_{0<t\leq T} (\Gamma_t \times \{t\}).$$

These conditions are derived by applying conservation laws of mass, momentum and energy across the interface. But here we impose thermal equillibrium conditions (1.8) or (1.9) instead of the normal component of momentum. Especially condition (1.8) is called the Gibbs-Thompson's law. Here Π, $\mathbf{D}(\boldsymbol{v})$, and H are a projection operator on Γ_t, the velocity deformation tensor and the twice mean curvature of Γ_t, respectively. l, θ_1 and σ are the latent heat, the equilibrium temperature and the surface tension, respectively. To complete the problem, we further impose the initial and boundary conditions on the rigid boundary Σ :

(1.10) $$\begin{cases} \boldsymbol{v} = \boldsymbol{v}_{\sigma,0} \quad \text{or} \quad \boldsymbol{v} = \boldsymbol{v}_0, \\ \theta^{(1)} = \theta^{(1)}_{\sigma,0} \quad \text{or} \quad \theta^{(1)} = \theta^{(1)}_0 \qquad \text{on} \quad \bar{\Omega}^{(1)} \equiv \bar{\Omega}^{(1)}_0, \end{cases}$$

(1.11) $$\theta^{(2)} = \theta^{(2)}_{\sigma,0} \quad \text{or} \quad \theta^{(2)} = \theta^{(2)}_0 \qquad \text{on} \quad \bar{\Omega}^{(2)} \equiv \bar{\Omega}^{(2)}_0,$$

(1.12) $$\begin{cases} \boldsymbol{v} = 0, \\ \theta^{(1)} = \theta_2 \qquad \text{on} \quad \Sigma_T. \end{cases}$$

In the sequal, by (P_σ) we mean problem (1.1)-(1.8), (1.10)-(1.12), and by (P) problem (1.1)-(1.7),(1.9)-(1.12). $(\boldsymbol{v}_{0,\sigma}, \theta^{(1)}_{\sigma,0}, \theta^{(2)}_{\sigma,0})$ and $(\boldsymbol{v}_0, \theta^{(1)}_0, \theta^{(2)}_0)$ are initial data imposed on problems (P_σ) and (P), respectively.

In [5] and [7], we have proved the unique classical solvability of problems (P_σ) and (P), respctively. In this paper, we prove that the problem (P_σ) is uniquely solvable on a certain finite time interval independent of $\sigma \in (0, \sigma^*)$, $\sigma^* << 1$, and that problem (P) is the limit case of problem (P_σ) as σ tends to zero. This is done on the basis of a uniform estimate of the solution of problem (P_σ) with respect to σ which is obtained in some wider space of functions than the space defined in [5]. Bazaliĭ and Degtyarev [1] also studied such a limit problem of the Stefan problem with Gibbs-Thompson's law involving only the process of heat transfer. They showed the convergence in a class that the space of the limit functions is compactly embedded. We prove this convergence holds in the same class of the limit functions.

We study the above problem in the function spaces defined as follows. Let D_T be a cylindorical domain $D \times (0,T)$, where D is a domain in $\mathbf{R}^n$, and $T > 0$. Let l be a non-negative integer and $\alpha \in (0,1)$. By $C^{l+\alpha, \frac{l+\alpha}{2}}(D_T)$ we denote anisotropic Hölder space of functions whose norm are defined by

$$|f|^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T} \equiv \sum_{2r+|m|=0}^{l} |\partial_t^r \partial_x^m f|^{(0)}_{D_T} + \langle f \rangle^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T},$$

where

$$\langle f \rangle^{(l+\alpha, \frac{l+\alpha}{2})}_{D_T} \equiv \sum_{2r+|m|=l-1}^{l} |\partial_t^r \partial_x^m f|^{(\frac{l+\alpha-(2r+|m|)}{2})}_{t,D_T} + \sum_{2r+|m|=l} |\partial_t^r \partial_x^m f|^{(\alpha)}_{x,D_T}$$

$$\begin{cases} |f|^{(0)}_{D_T} \equiv \sup\limits_{(x,t)\in D_T} |f(x,t)|, \\ |f|^{(\frac{\alpha}{2})}_{t,D_T} \equiv \sup\limits_{\substack{(x,t),(x,t')\in D_T,\\ t\neq t'}} \dfrac{|f(x,t)-f(x,t')|}{|t-t'|^{\frac{\alpha}{2}}}, \\ |f|^{(\alpha)}_{x,D_T} \equiv \sup\limits_{\substack{(x,t),(x',t)\in D_T,\\ x\neq x'}} \dfrac{|f(x,t)-f(x',t)|}{|x-x'|^{\alpha}}, \end{cases}$$

and

$$|m| = \sum_{i=1}^{n} m_i, \quad \partial_x^m = \frac{\partial^{|m|}}{\partial_{x_1}^{m_1}\cdots\partial_{x_n}^{m_n}}$$

for a multi index

$$m = (m_i) \quad (m_i \geq 0, i = 1, \cdots, n).$$

By $\tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_T)$ and $C_\sigma^{l+\alpha,\frac{l+\alpha}{2}}(D_T)$ we denote the function spaces

$$\left\{f \in C^{l+\alpha,\frac{l+\alpha}{2}}(D_T) \mid \partial_t f \in C^{l-1+\alpha,\frac{l-1+\alpha}{2}}(D_T)\right\}$$

equipped with the norm

$$\|f\|^{(l+\alpha,\frac{l+\alpha}{2})}_{D_T} \equiv |f|^{(l+\alpha,\frac{l+\alpha}{2})}_{D_T} + |\partial_t f|^{(l-1+\alpha,\frac{l-1+\alpha}{2})}_{D_T}$$

and

$$\left\{f \in \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_T) \mid \partial_x^m f \in C^{l+\alpha,\frac{l+\alpha}{2}}(D_T), \quad |m| = 2\right\}$$

equipped with the norm

$$|f|^{(l+\alpha,\frac{l+\alpha}{2})}_{\sigma,D_T} \equiv \|f\|^{(l+\alpha,\frac{l+\alpha}{2})}_{D_T} + \sigma \sum_{|m|=2} |\partial_x^m f|^{(l+\alpha,\frac{l+\alpha}{2})}_{D_T} \quad (\sigma > 0),$$

respectively. By $C_0^{l+\alpha,\frac{l+\alpha}{2}}(D_T)$, $\tilde{C}_0^{l+\alpha,\frac{l+\alpha}{2}}(D_T)$ and $C_{\sigma,0}^{l+\alpha,\frac{l+\alpha}{2}}(D_T)$ we denote the function spaces

$$\left\{f \in C^{l+\alpha,\frac{l+\alpha}{2}}(D_T) \mid \partial_t^k f|_{t=0} = 0, \quad k = 0, 1, \cdots, \left[\frac{l+\alpha}{2}\right]\right\},$$

$$\left\{f \in \tilde{C}^{l+\alpha,\frac{l+\alpha}{2}}(D_T) \mid \partial_t^k f|_{t=0} = 0, \quad k = 0, 1, \cdots, \left[\frac{l+1+\alpha}{2}\right]\right\}$$

and

$$\left\{f \in C_\sigma^{l+\alpha,\frac{l+\alpha}{2}}(D_T) \mid \partial_t^k f|_{t=0} = 0, \quad k = 0, 1, \cdots, \left[\frac{l+1+\alpha}{2}\right]\right\},$$

respectively. By $C^{l+\alpha}(D)$, we define the space of functions $f(x), x \in D$, with the norm

$$|f|^{l+\alpha}_D \equiv \sum_{|m|\leq l} |D_x^m f|^{(0)}_D + \langle f\rangle^{(l+\alpha)}_D, \quad |f|^{(0)}_D \equiv \sup_{x\in D} |f(x)|,$$

$$\langle f\rangle_D^{(l+\alpha)} \equiv \sum_{|m|=l} \langle D_x^m f\rangle_D^{(l+\alpha)} \equiv \sup_{\substack{x,y\in D,\\ x\neq y}} \sum_{m=|l|} \frac{|D_x^m f(x) - D_y^m f(y)|}{|x-y|^\alpha}.$$

We also need the following seminorm:

$$\langle\langle f\rangle\rangle_{D_T}^{(1+\alpha,\gamma)} \equiv \sup_{\substack{\tau,t\in(0,T),\\ \tau\neq t}} \frac{\langle f(x,t)-f(x,\tau)\rangle_D^{(\gamma)}}{|t-\tau|^{\frac{1+\alpha-\gamma}{2}}},$$

where $\alpha,\gamma \in (0,1)$. Furthermore, by $\mathcal{H}^{l+\alpha}$, $\mathcal{X}_{\sigma,T}^{l+\alpha}$ and $\mathcal{X}_T^{l+\alpha}$ we mean function spaces $C^{l+2+\alpha}(\bar\Omega^{(1)}) \times C^{l+3+\alpha}(\bar\Omega^{(1)}) \times C^{l+3+\alpha}(\bar\Omega^{(2)})$,
$C^{l+2+\alpha,\frac{l+2+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Omega_t^{(1)}\times\{t\})\right) \times C^{l+\alpha,\frac{l+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Omega_t^{(1)}\times\{t\})\right)$
$\times C^{l+3+\alpha,\frac{l+3+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Omega_t^{(1)}\times\{t\})\right) \times C^{l+3+\alpha,\frac{l+3+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Omega_t^{(2)}\times\{t\})\right)$
$\times C_\sigma^{l+3+\alpha,\frac{l+3+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Gamma_t\times\{t\})\right)$ and the space such that
$C_\sigma^{l+3+\alpha,\frac{l+3+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Gamma_t\times\{t\})\right)$ is replaced by $\tilde C^{l+3+\alpha,\frac{l+3+\alpha}{2}}\left(\bigcup_{0<t\le T}(\Gamma_t\times\{t\})\right)$ in the definition of $\mathcal{X}_{\sigma,T}^{l+\alpha}$, respectively.

Now let us describe our main result.

Theorem 1.1. *Assume that*

$$\Gamma\equiv\Gamma_0\in C^{5+\alpha},\quad \Sigma\in C^{4+\alpha},$$

$$\boldsymbol{f}\in C^{1+\alpha}(0,\infty),\quad \kappa^{(i)}\in C^{3+\alpha}(0,\infty),\quad \boldsymbol{v}_{\sigma,0}\in C^{3+\alpha}(\bar\Omega^{(1)}),$$

$$\theta_{\sigma,0}^{(i)}\in C^{4+\alpha}(\bar\Omega^{(i)}),\quad \theta_1\in C^{4+\alpha,\frac{4+\alpha}{2}}(\mathbf{R}^3\times(0,T]),\quad \theta_2\in C^{4+\alpha,\frac{4+\alpha}{2}}(\Sigma_T),$$

and the inequalities

$$\kappa_0<\kappa^{(i)}(\theta)<\kappa_0^{-1},\quad \left|\left.\sum_{i=1,2}\left((-1)^{i-1}\kappa^{(i)}(\theta_{\sigma,0}^{(i)})\nabla\theta_{\sigma,0}^{(i)}\cdot\boldsymbol{n}\right)\right|_\Gamma\right|>a_0,$$

$$|\rho-\rho_0|\le b_0,\quad \left|\sum_{i=1,2}\left((-1)^{i-1}\kappa^{(i)}(\theta_{\sigma,0}^{(i)})\right)\nabla\theta_{\sigma,0}^{(1)}\cdot\boldsymbol{\tau}\right|_\Gamma^{(0)}\le b_0$$

hold for some positive constants $\kappa_0(\le 1)$, $a_0$and $b_0<1/(4C_3)$, C_3 in (4.1), where $\boldsymbol{\tau}$ is a tangential vector to Γ. Moreover we assume that the compatibility conditions up to order 1 hold. Then problem (P_σ) has a unique solution $(\boldsymbol{v},\nabla p,\theta^{(1)},\theta^{(2)},\Gamma_t)\in\mathcal{X}_{\sigma,T_0}^\alpha$ for some $T_0>0$ which is independent of σ.

Furthermore, let $\{(\boldsymbol{v}_\sigma,\nabla p_\sigma,\theta_\sigma^{(1)},\theta_\sigma^{(2)},\Gamma_{\sigma,t})\}$ be a set of solutions of problem (P_σ) in the space $\mathcal{X}_{\sigma,T}^{2+\alpha}$, $(\boldsymbol{v},\nabla p,\theta^{(1)},\theta^{(2)},\Gamma_t)$ be a solution of problem (P) in the space $\mathcal{X}_T^\alpha$ and $(\boldsymbol{v}_{\sigma,0},\theta_{\sigma,0}^{(1)},\theta_{\sigma,0}^{(2)})$ converge to $(\boldsymbol{v}_0,\theta_0^{(1)},\theta_0^{(2)})$ in the space $\mathcal{H}^\alpha$ as σ tends to 0, then $(\boldsymbol{v}_\sigma,\nabla p_\sigma,\theta_\sigma^{(1)},\theta_\sigma^{(2)},\Gamma_{\sigma,t})$ converges to $(\boldsymbol{v},\nabla p,\theta^{(1)},\theta^{(2)},\Gamma_t)$ in the space $\mathcal{X}_T^\alpha$ as σ tends to 0 on some interval $[0,T]$ which is independent of σ.

2. Reduction of the problem. Let $\mathcal{M}$ be a 2-dimensional manifold which is isometric to Γ, $\omega = (\omega_1, \omega_2)$ be a generic point on $\mathcal{M}$ and

$$X_0 : \mathcal{M} \longrightarrow \Gamma$$

be a $C^{5+\alpha}$-diffeomorphism. We define a mapping X from $\mathcal{M} \times [-\gamma_0, \gamma_0]$ to a neighborhood N_0 of Γ in the form

$$X(\omega, \lambda) = X_0(\omega) + \boldsymbol{n}(X_0(\omega))\lambda$$

where $\boldsymbol{n}(X_0(\omega))$ is a unit normal to Γ at $X_0(\omega)$ directing into $\Omega^{(1)}$. Here a positive number γ_0 is assumed to be chosen so small that the mapping X is regular and one-to-one. Let $(\omega(x), \lambda(x))$ be the inverse mapping of X, and introduce the following notation:

$$\phi^{(i)}(\omega, \lambda) = \nabla_x \omega_i(x)|_{x=X(\omega,\lambda)}, \quad i = 1, 2,$$

$$\phi^{(3)}(\omega, \lambda) = \nabla_x \lambda(x)|_{x=X(\omega,\lambda)},$$

$$M^{(k)} = \left(\frac{\partial^2}{\partial x_i \partial x_j} \omega_k(x) \right)_{i,j=1,2,3} \Bigg|_{x=X(\omega,\lambda)}, \quad k = 1, 2,$$

$$M^{(3)} = \left(\frac{\partial^2}{\partial x_i \partial x_j} \lambda(x) \right)_{i,j=1,2,3} \Bigg|_{x=X(\omega,\lambda)}.$$

Now, for some $T > 0$, let us assume that the interface $\Gamma_t, t \in [0, T]$, is represented by $X_0(\omega(x)) + \boldsymbol{n}(X_0(\omega(x)))d(\omega(x), t)$ with some function $d(\omega, t)$ satisfying $d(\omega, 0) = 0$. Then $\bigcup_{0 \le t \le T}(\Gamma_t \times \{t\})$ can be represented as

$$\{(x, t) \in N_0 \times [0, T] | \quad \Phi_d(x, t) \equiv \lambda(x) - d(\omega(x), t) = 0\}.$$

Accordingly, the Stefan condition (1.7) can be written as

$$l\rho_e \frac{\partial \Phi_d}{\partial t} - \kappa^{(1)}(\theta^{(1)})(\nabla \Phi_d \cdot \nabla \theta^{(1)}) + \kappa^{(2)}(\theta^{(2)})(\nabla \Phi_d \cdot \nabla \theta^{(2)}) = 0$$

and the twice mean curvature of Γ_t as

$$H(\omega, t) = -\frac{1}{|\nabla_x \Phi_d|} \left(\sum_{i,j=1,2} a_{ij}(\omega, d, \nabla_\omega d) \frac{\partial^2 d}{\partial \omega_i \partial \omega_j} + b(\omega, d, \nabla_\omega d) \right).$$

with

$$a_{ij}(\omega, d, p_1, p_2) = (\phi^{(i)} \cdot \phi^{(j)}) - \left[\sum_{k,l=1,2} (p_k \phi^{(k)} \cdot \phi^{(i)})(p_l \phi^{(l)} \cdot \phi^{(j)}) \right]$$
$$\times \left(1 + \left| \sum_{k=1,2} p_k \phi^{(k)} \right|^2 \right)^{-1},$$

$$b(\omega,d,p_1,p_2)=\sum_{k=1,2}p_k\mathrm{Tr}(M^{(k)})-\mathrm{Tr}(M^{(3)})$$
$$-\left[\sum_{k,l=1,2}p_kp_l(\phi^{(k)})^TM^{(3)}\phi^{(l)}-\sum_{k,l,m=1,2}p_kp_lp_m(\phi^{(k)})^TM^{(m)}\phi^{(l)}\right]$$
$$\times\left(1+\left|\sum_{k=1,2}p_k\phi^{(k)}\right|^2\right)^{-1},$$

where $p_k=\partial d/\partial\omega_k, k=1,2$ (see [2]). Here we denote by $(\boldsymbol{a}\cdot\boldsymbol{b})$, $\mathrm{Tr}A$ and $\boldsymbol{a}^T$ the scalar product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbf{R}^3$, the trace of the matrix A and the transposed vector of $\boldsymbol{a}$, respectively.

Next we introduce a transformation e_d (see [4]). Let X_T and Y_T be two coordinates (x_1,x_2,x_3,t) and (y_1,y_2,y_3,t) in $\mathbf{R}^3\times[0,T]$ such that $x=X(\omega,\lambda)$, $y=X(\omega,\eta)$. Then the mapping $e_d:Y_T\longrightarrow X_T$ is defined by

$$e_d(X(\omega,\eta),t)=\begin{cases}(X(\omega,\eta+\chi(\eta)d(\omega,t)),t) & \text{if}\quad (x,t)\in N_0\times[0,T],\\ (X(\omega,\lambda),t) & \text{if}\quad (x,t)\in {N_0}^c\times[0,T],\end{cases}$$

where $\chi(\lambda)\in C^\infty(-\infty,+\infty)$ is a cut-off function satisfying

$$\chi(\lambda)=\begin{cases}1 & \text{for}\quad |\lambda|\le\dfrac{\gamma_0}{4},\\ 0 & \text{for}\quad |\lambda|\ge\dfrac{3\gamma_0}{4},\end{cases}\qquad |\chi'(\lambda)|\le\frac{4}{3\gamma_0}.$$

It is obvious that $Q_T^{(1)}=\Omega^{(1)}\times(0,T]$, $Q_T^{(2)}=\Omega^{(2)}\times(0,T]$ and $\Gamma_T=\Gamma\times(0,T]$ are transformed onto $\bigcup_{0<t\le T}(\Omega_t^{(1)}\times\{t\})$, $\bigcup_{0<t\le T}(\Omega_t^{(2)}\times\{t\})$ and $\bigcup_{0<t\le T}(\Gamma_t\times\{t\})$, respectively by e_d. By denoting simply the transformed functions $\theta^{(1)}\circ e_d$, $\theta^{(2)}\circ e_d$, $\boldsymbol{v}\circ e_d$ and $p\circ e_d$ by $\theta^{(1)}$, $\theta^{(2)}$, $\boldsymbol{v}$ and p, respectively, problem (P_σ) can be rewritten in the fixed domain $Q_T^{(1)}\bigcup Q_T^{(2)}$ of the variables (y,t).

$$(2.1)\quad\begin{cases}\dfrac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{h}_d\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla_d)\boldsymbol{v}-\nu{\nabla_d}^2\boldsymbol{v}+\nabla_dp=\boldsymbol{f}(\theta^{(1)}) & \text{in}\quad Q_T^{(1)},\\ \nabla_d\cdot\boldsymbol{v}=0\quad\text{in}\quad Q_T^{(1)}, & \\ \boldsymbol{v}|_{t=0}=\boldsymbol{v}_{\sigma,0}\quad\text{on}\quad\bar{\Omega}^{(1)}, & \\ \boldsymbol{v}=0\quad\text{on}\quad\Sigma_T, & \\ \begin{cases}\boldsymbol{v}\cdot\boldsymbol{n}_d=\left(1-\dfrac{\rho_e}{\rho}\right)V,\\ 2\nu\Pi_d\mathbf{D}_d(\boldsymbol{v})\boldsymbol{n}_d=\Pi_d[\boldsymbol{v}(\boldsymbol{v}-V\boldsymbol{n}_d)^*]\boldsymbol{n}_d\end{cases} & \text{on}\quad\Gamma_T,\end{cases}$$

$$
(2.2)\quad \begin{cases}
\dfrac{\partial\theta^{(1)}}{\partial t} + (\boldsymbol{h}_d\cdot\nabla)\theta^{(1)} + (\boldsymbol{v}\cdot\nabla_d)\theta^{(1)} - \dfrac{1}{\rho C_p^{(1)}}\nabla_d\cdot\left(\kappa^{(1)}(\theta^{(1)})\nabla_d\theta^{(1)}\right) \\
\qquad = \dfrac{2\nu}{C_p^{(1)}}\mathbf{D}_d(\boldsymbol{v}):\mathbf{D}_d(\boldsymbol{v}) \quad \text{in} \quad Q_T^{(1)}, \\
\dfrac{\partial\theta^{(2)}}{\partial t} + (\boldsymbol{h}_d\cdot\nabla)\theta^{(2)} - \dfrac{1}{\rho_e C_p^{(2)}}\nabla_d\cdot\left(\kappa^{(2)}(\theta^{(2)})\nabla_d\theta^{(2)}\right) = 0 \quad \text{in} \quad Q_T^{(2)}, \\
\theta^{(1)}|_{t=0} = \theta_{\sigma,0}^{(1)} \quad \text{on} \quad \bar\Omega^{(1)}, \\
\theta^{(2)}|_{t=0} = \theta_{\sigma,0}^{(2)} \quad \text{on} \quad \bar\Omega^{(2)}, \\
\theta^{(1)} = \theta_2 \quad \text{on} \quad \Sigma_T, \\
\begin{cases}
\dfrac{\partial d}{\partial t} + \dfrac{1}{l\rho_e}\kappa^{(1)}(\theta^{(1)})(\nabla_d\Phi_d\cdot\nabla_d\theta^{(1)}) \\
\qquad -\dfrac{1}{l\rho_e}\kappa^{(2)}(\theta^{(2)})(\nabla_d\Phi_d\cdot\nabla_d\theta^{(2)}) = 0, \\
\theta^{(1)} = \theta^{(2)} = \theta_1\left(1 - \dfrac{\sigma}{l}H\right) \quad \text{on} \quad \Gamma_T.
\end{cases}
\end{cases}
$$

Here we set

$$
\begin{cases}
\nabla_d = (E_d^*)^{-1}\nabla, \\
\boldsymbol{h}_d = \dfrac{\partial y}{\partial t}\circ e_d, \quad \boldsymbol{n}_d = \dfrac{\nabla_d\eta}{|\nabla_d\eta|}, \\
\mathbf{D}_d(\boldsymbol{v}) = \mathbf{D}(\boldsymbol{v})\circ e_d, \quad \Pi_d g = \Pi g\circ e_d,
\end{cases}
$$

and $E_d = (a_{ij})$ is the Jacobian matrix of the mapping from y to x, a^{ij} is the $ij-$component of $(E_d^*)^{-1}$ and E_d^* is the transposed matrix of E_d.

Extentions $\hat\theta_\sigma^{(1)} \in C^{4+\alpha,\frac{4+\alpha}{2}}(Q_T^{(1)})$, $\hat\theta_\sigma^{(2)} \in C^{4+\alpha,\frac{4+\alpha}{2}}(Q_T^{(2)})$, $\hat d_\sigma \in C^{5+\alpha,\frac{5+\alpha}{2}}(\mathcal{M}\times[0,T])$, $\hat{\boldsymbol{v}}_\sigma \in C^{3+\alpha,\frac{3+\alpha}{2}}(Q_T^{(1)})$, $\nabla\hat p_\sigma \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_T^{(1)})$ of the initial data can be constructed to satisfy the conditions:

$$
\begin{cases}
\hat\theta_\sigma^{(1)}(y,0) = \theta_{\sigma,0}^{(1)}(y), \quad \dfrac{\partial\hat\theta_\sigma^{(1)}}{\partial t}(y,0) = \theta_\sigma^{(1)[1]}(y), \\
\hat\theta_\sigma^{(2)}(y,0) = \theta_{\sigma,0}^{(2)}(y), \quad \dfrac{\partial\hat\theta_\sigma^{(2)}}{\partial t}(y,0) = \theta_\sigma^{(2)[1]}(y), \\
\hat d_\sigma(\omega,0) = 0, \quad \dfrac{\partial\hat d_\sigma}{\partial t}(\omega,0) = d_\sigma^{[1]}(\omega), \quad \dfrac{\partial^2\hat d_\sigma}{\partial t^2}(\omega,0) = d_\sigma^{[2]}(\omega), \\
\hat{\boldsymbol{v}}_\sigma(y,0) = \boldsymbol{v}_{\sigma,0}(y), \quad \dfrac{\partial\hat{\boldsymbol{v}}_\sigma}{\partial t}(y,0) = \boldsymbol{v}_\sigma^{[1]}(y), \\
\nabla\hat p_\sigma(y,0) = -\boldsymbol{v}_\sigma^{[1]}(y) - (\boldsymbol{v}_{\sigma,0}(y)\cdot\nabla)\boldsymbol{v}_{\sigma,0}(y) + \nu\nabla^2\boldsymbol{v}_{\sigma,0}(y) + \boldsymbol{f}(\theta_{\sigma,0}(y))
\end{cases}
$$

and the ineauality:

$$
(2.3)\quad \begin{aligned}
&\sum_{i=1,2}|\hat\theta_\sigma^{(i)}|_{Q_T^{(i)}}^{(4+\alpha,\frac{4+\alpha}{2})} + |\hat d_\sigma|_{\Gamma_T}^{(5+\alpha,\frac{5+\alpha}{2})} + |\nabla\hat p_\sigma|_{Q_T^{(1)}}^{(1+\alpha,\frac{1+\alpha}{2})} + |\hat{\boldsymbol{v}}_\sigma|_{Q_T^{(1)}}^{(3+\alpha,\frac{3+\alpha}{2})} \\
&\qquad \le c\left(\sum_{i=1,2}|\theta_{\sigma,0}^{(i)}|_{\bar\Omega^{(i)}}^{(4+\alpha)} + |\boldsymbol{v}_{\sigma,0}|_{\bar\Omega^{(1)}}^{(3+\alpha)}\right),
\end{aligned}
$$

with a constant c being bounded as $T \longrightarrow 0$. Here the functions $\theta_\sigma^{(1)[1]}, \theta_\sigma^{(2)[2]}, d_\sigma^{[1]}, d_\sigma^{[2]}$, $\boldsymbol{v}_\sigma^{[1]}$ are defined from the compatibility conditions between the equations and the data in (2.1) and (2.1) (see, [7]). Furthermore, the inequality

$$\text{(2.4)}\quad \sum_{i=1,2} |\hat{\theta}^{(i)}_\sigma - \hat{\theta}^{(i)}|^{(4+\alpha,\frac{4+\alpha}{2})}_{Q^{(i)}_T} + |\hat{d}_\sigma - \hat{d}|^{(5+\alpha,\frac{5+\alpha}{2})}_{\Gamma_T}$$
$$+|\nabla\hat{p}_\sigma - \nabla\hat{p}|^{(1+\alpha,\frac{1+\alpha}{2})}_{Q^{(1)}_T} + |\hat{v}_\sigma - \hat{v}|^{(3+\alpha,\frac{3+\alpha}{2})}_{Q^{(1)}_T}$$
$$\leq c\left(\sum_{i=1,2} |\theta^{(i)}_{\sigma,0} - \theta^{(i)}_0|^{(4+\alpha)}_{\bar{\Omega}^{(i)}} + |v_{\sigma,0} - v_0|^{(3+\alpha)}_{\bar{\Omega}^{(1)}}\right),$$

obviously holds, where $(\hat{v}, \nabla\hat{p}, \hat{\theta}^{(1)}, \hat{\theta}^{(2)})$ is a extension corresponding to problem (P). Then by introducing the new unknown functions $w^{(i)} \equiv \theta^{(i)} - \hat{\theta}^{(i)}_\sigma - \chi(\nabla\eta\cdot\nabla\hat{\theta}^{(i)}_\sigma)\delta$, $(i = 1,2)$, $\delta \equiv d - \hat{d}_\sigma$, $u \equiv v - \hat{v}_\sigma$ and $\nabla q \equiv \nabla p - \nabla\hat{p}_\sigma$, problem (2.1)-(2.2) can be written in the equivalent form

$$\text{(2.5)}\quad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \nabla q = \mathcal{F}_1(u, \nabla q, w^{(1)}, \delta) \quad \text{in} \quad Q^{(1)}_T, \\ \nabla\cdot u = \mathcal{F}_2(u,\delta) \quad \text{in} \quad Q^{(1)}_T, \\ u|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(1)}, \\ u = -\hat{v}_\sigma \quad \text{on} \quad \Sigma_T, \\ \begin{cases} u\cdot n = \mathcal{F}_3(u,\delta) \\ 2\nu\Pi\mathbf{D}(u)n = \mathcal{F}_4(u,\delta) \quad \text{on} \quad \Gamma_T, \end{cases} \end{cases}$$

$$\text{(2.6)}\quad \begin{cases} \dfrac{\partial w^{(1)}}{\partial t} - \dfrac{1}{\rho C^{(1)}_p}\nabla\cdot\left(\kappa^{(1)}(\hat{\theta}^{(1)}_\sigma)\nabla w^{(1)}\right) - \dfrac{4\nu}{C^{(1)}_p}\mathbf{D}(u):\mathbf{D}(\hat{v}_\sigma) \\ \qquad = \mathcal{G}_1(u, w^{(1)}, \delta) \quad \text{in} \quad Q^{(1)}_T, \\ \dfrac{\partial w^{(2)}}{\partial t} - \dfrac{1}{\rho_e C^{(2)}_p}\nabla\cdot\left(\kappa^{(2)}(\hat{\theta}^{(2)}_\sigma)\nabla w^{(2)}\right) = \mathcal{G}_2(w^{(2)},\delta) \quad \text{in} \quad Q^{(2)}_T, \\ w^{(1)}|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(1)}, \\ w^{(2)}|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(2)}, \\ w^{(1)} = \theta_2 - \hat{\theta}^{(1)}_\sigma \quad \text{on} \quad \Sigma_T, \\ \begin{cases} \dfrac{\partial\delta}{\partial t} + \dfrac{1}{l\rho_e}\kappa^{(1)}(\hat{\theta}^{(1)}_\sigma)(\nabla\eta\cdot\nabla w^{(1)}) \\ \qquad -\dfrac{1}{l\rho_e}\kappa^{(2)}(\hat{\theta}^{(2)}_\sigma)(\nabla\eta\cdot\nabla w^{(2)}) = \mathcal{G}_3(w^{(1)}, w^{(2)}, \delta), \\ w^{(1)} + \dfrac{\partial\hat{\theta}^{(1)}_\sigma}{\partial n}\delta - \dfrac{\theta_1\sigma}{l}\dfrac{1}{|\nabla\eta|}\displaystyle\sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\dfrac{\partial^2\delta}{\partial\omega_i\partial\omega_j} = \mathcal{G}_4(\delta), \\ w^{(2)} + \dfrac{\partial\hat{\theta}^{(2)}_\sigma}{\partial n}\delta - \dfrac{\theta_1\sigma}{l}\dfrac{1}{|\nabla\eta|}\displaystyle\sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\dfrac{\partial^2\delta}{\partial\omega_i\partial\omega_j} \\ \qquad = \mathcal{G}_5(\delta) \quad \text{on} \quad \Gamma_T, \end{cases} \end{cases}$$

where by $\mathcal{F}_i$, $i = 1,\cdots,4$, in (2.5) and $\mathcal{G}_i$, $i = 1,\cdots,5$, in (2.6) we mean nonlinear terms derived by the above linearization. The explicit representations of $\mathcal{F}_i$, $i = 1,\cdots,4$, and $\mathcal{G}_i$, $i = 1,\cdots,3$, have the same form given in [7], hence we omit them here. The representations of $\mathcal{G}_4$ and $\mathcal{G}_5$ will be given in section 4.

3. The linear problems. In this section we are concerned with the lin
lem

$$
(3.1)\begin{cases}
\dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + \nabla q = \boldsymbol{F}_1, \qquad \nabla\cdot \boldsymbol{u} = F_2 \quad \text{in} \quad Q_T^{(1)},\\
\boldsymbol{u}|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(1)},\\
\boldsymbol{u} = \boldsymbol{H}_1 \quad \text{on} \quad \Sigma_T,\\
\boldsymbol{u}\cdot\boldsymbol{n} = F_3, \qquad 2\nu\Pi\mathbf{D}(\boldsymbol{u}) = \boldsymbol{F}_4 \quad \text{on} \quad \Gamma_T,\\
\dfrac{\partial w^{(1)}}{\partial t} - \dfrac{1}{\rho C_p^{(1)}}\nabla\cdot\left(\kappa^{(1)}(\hat{\theta}_\sigma^{(1)})\nabla w^{(1)}\right)\\
\qquad -\dfrac{4\nu}{C_p^{(1)}}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\hat{\boldsymbol{v}}_\sigma) = G_1 \quad \text{in} \quad Q_T^{(1)},\\
\dfrac{\partial w^{(2)}}{\partial t} - \dfrac{1}{\rho_e C_p^{(2)}}\nabla\cdot\left(\kappa^{(2)}(\hat{\theta}_\sigma^{(2)})\nabla w^{(2)}\right) = G_2 \quad \text{in} \quad Q_T^{(2)},\\
w^{(1)}|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(1)},\\
w^{(2)}|_{t=0} = 0 \quad \text{on} \quad \bar{\Omega}^{(2)},\\
w^{(1)} = H_2 \quad \text{on} \quad \Sigma_T,\\
\begin{cases}
\dfrac{\partial \delta}{\partial t} + \dfrac{1}{l\rho_e}\kappa^{(1)}(\hat{\theta}_\sigma^{(1)})(\nabla\eta\cdot\nabla w^{(1)})\\
\qquad -\dfrac{1}{l\rho_e}\kappa^{(2)}(\hat{\theta}^{(2)})(\nabla\eta\cdot\nabla w^{(2)}) = G_3,\\
w^{(1)} + \dfrac{\partial\hat{\theta}_\sigma^{(1)}}{\partial \boldsymbol{n}}\delta - \dfrac{\theta_{1\sigma}}{l}\displaystyle\sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\dfrac{\partial^2\delta}{\partial\omega_i\partial\omega_j} = G_4,\\
w^{(2)} + \dfrac{\partial\hat{\theta}_\sigma^{(2)}}{\partial \boldsymbol{n}}\delta - \dfrac{\theta_{2\sigma}}{l}\displaystyle\sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\dfrac{\partial^2\delta}{\partial\omega_i\partial\omega_j} = G_5 \qquad \text{on} \quad \Gamma_T
\end{cases}
\end{cases}
$$

We treat the above problrem separately, that is,

$$
(3.2)\qquad\begin{cases}
\dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + \nabla q = \boldsymbol{F}_1, \qquad \nabla\cdot \boldsymbol{u} = F_2 \qquad \text{in} \quad Q_T^{(1)},\\
\boldsymbol{u}|_{t=0} = 0 \qquad \text{on} \quad \bar{\Omega}^{(1)},\\
\boldsymbol{u} = \boldsymbol{H}_1 \qquad \text{on} \quad \Sigma_T,\\
\boldsymbol{u}\cdot\boldsymbol{n} = F_3, \qquad \Pi\mathbf{D}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F}_4 \qquad \text{on} \quad \Gamma_T
\end{cases}
$$

$$
(3.3)\begin{cases}
\dfrac{\partial w^{(1)}}{\partial t}-\dfrac{1}{\rho C_p^{(1)}}\nabla\cdot\left(\kappa^{(1)}(\hat{\theta}_\sigma^{(1)})\nabla w^{(1)}\right) \\
\qquad\qquad = G_1+\dfrac{4\nu}{C_p^{(1)}}\mathbf{D}(\boldsymbol{u}):\mathbf{D}(\hat{\boldsymbol{v}}_\sigma) \qquad \text{in} \quad Q_T^{(1)}, \\
\dfrac{\partial w^{(2)}}{\partial t}-\dfrac{1}{\rho_e C_p^{(2)}}\nabla\cdot\left(\kappa^{(2)}(\hat{\theta}_\sigma^{(2)})\nabla w^{(2)}\right)=G_2 \qquad \text{in} \quad Q_T^{(2)}, \\
w^{(1)}|_{t=0}=0 \qquad \text{on} \quad \bar{\Omega}^{(1)}, \\
w^{(2)}|_{t=0}=0 \qquad \text{on} \quad \bar{\Omega}^{(2)}, \\
w^{(1)}=H_2 \qquad \text{on} \quad \Sigma_T, \\
\begin{cases}
\dfrac{\partial\delta}{\partial t}+\dfrac{1}{l\rho_e}\kappa^{(1)}(\hat{\theta}_\sigma^{(1)})(\nabla\eta\cdot\nabla w^{(1)}) \\
\qquad -\dfrac{1}{l\rho_e}\kappa^{(2)}(\hat{\theta}_\sigma^{(2)})(\nabla\eta\cdot\nabla w^{(2)})=G_3, \\
w^{(1)}+\dfrac{\partial\hat{\theta}_\sigma^{(1)}}{\partial n}\delta-\dfrac{\theta_1\sigma}{l}\displaystyle\sum_{i,j=1,2}a_{ij}(\omega,0,0,0)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j}=G_4, \\
w^{(2)}+\dfrac{\partial\hat{\theta}_\sigma^{(2)}}{\partial n}\delta-\dfrac{\theta_1\sigma}{l}\displaystyle\sum_{i,j=1,2}a_{ij}(\omega,0,0,0)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j}=G_5 \qquad \text{on} \quad \Gamma_T.
\end{cases}
\end{cases}
$$

For problem (3.2) we have already obtained the following theorem (see [6]).

THEOREM 3.1. *Let us assume that*

$$
\boldsymbol{F}_1\in C_0^{\alpha,\frac{\alpha}{2}}(Q_T^{(1)}), \quad F_2\in C_0^{1+\alpha,\frac{1+\alpha}{2}}(Q_T^{(1)}), \quad \boldsymbol{H}_1\in C_0^{2+\alpha,\frac{2+\alpha}{2}}(\Sigma_T),
$$

$$
F_3\in C_0^{2+\alpha,\frac{2+\alpha}{2}}(\Gamma_T), \quad \boldsymbol{F}_4\in C_0^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma_T).
$$

and there exist a vector field $\boldsymbol{r}\in C_0^{\alpha,\frac{\alpha}{2}}(Q_T^{(1)})$ *and a tensor* $\boldsymbol{R}$ *satisfying*

$$
\frac{\partial F_2}{\partial t}-\nabla\cdot\boldsymbol{F}_1=\nabla\cdot\boldsymbol{r}, \quad \boldsymbol{r}=\nabla\cdot\boldsymbol{R}, \quad \langle\langle\boldsymbol{R}\rangle\rangle_{Q_T^{(1)}}^{(1+\alpha,\gamma)}<\infty,
$$

in the sense of distribution and

$$
\int_\Omega F_2dx=-\int_\Gamma F_3d\Gamma-\int_\Sigma \boldsymbol{G}_1\cdot\boldsymbol{n}d\Sigma.
$$

Then probrem (3.2) has a unique solution $\boldsymbol{u}\in C_0^{2+\alpha,\frac{2+\alpha}{2}}(Q_T^{(1)}), \nabla q\in C_0^{\alpha,\frac{\alpha}{2}}(Q_T^{(1)})$ *which satisfies*

$$
|\boldsymbol{u}|_{Q_T^{(1)}}^{(2+\alpha,\frac{2+\alpha}{2})}+|\nabla q|_{Q_T^{(1)}}^{(\alpha,\frac{\alpha}{2})}\le C\Big(|\boldsymbol{F}_1|_{Q_T^{(1)}}^{(\alpha,\frac{\alpha}{2})}+|F_2|_{Q_T^{(1)}}^{(1+\alpha,\frac{1+\alpha}{2})}+|\boldsymbol{H}_1|_{\Sigma_T}^{(2+\alpha,\frac{2+\alpha}{2})}
$$
$$
+|F_3|_{\Gamma_T}^{(2+\alpha,\frac{2+\alpha}{2})}+|\boldsymbol{F}_4|_{\Gamma_T}^{(1+\alpha,\frac{1+\alpha}{2})}+|\boldsymbol{r}|_{Q_T^{(1)}}^{(\alpha,\frac{\alpha}{2})}+\langle\langle\boldsymbol{R}\rangle\rangle_{Q_T^{(1)}}^{(1+\alpha,\gamma)}\Big)
$$

where a constant C *does not depend on* F_j, $j=1,\cdots,4$, $\boldsymbol{H}_1$, *and remains bounded as*

For problem (3.3), we first consider the model problem in the half space:

$$(3.4)\quad \begin{cases} \dfrac{\partial w_1}{\partial t} - a_1 \Delta w_1 = g_1 & \text{in} \quad D^3_\infty, \\ \dfrac{\partial w_2}{\partial t} - a_2 \Delta w_2 = g_2 & \text{in} \quad \tilde{D}^3_\infty, \\ w_1|_{t=0} = 0 \quad \text{on} \quad R^3_+, \\ w_2|_{t=0} = 0 \quad \text{on} \quad R^3_-, \\ \dfrac{\partial \delta}{\partial t} + d_1 \dfrac{\partial w_1}{\partial z_3} - d_2 \dfrac{\partial w_2}{\partial z_3} = g_3 & \text{on} \quad R^2_\infty, \\ w_1 + b_1 \delta - c\sigma \Delta_{z'} \delta = g_4 & \text{on} \quad R^2_\infty, \\ w_2 + b_2 \delta - c\sigma \Delta_{z'} \delta = g_5 & \text{on} \quad R^2_\infty. \end{cases}$$

Here $D^3_\infty \equiv \{(z_1, z_2, z_3, t) \in \mathbf{R}^4 | z_3 > 0, t > 0\}$, $\tilde{D}^3_\infty \equiv \{(z_1, z_2, z_3, t) \in \mathbf{R}^4 | z_3 < 0, t > 0\}$, $R^2_\infty \equiv \{(z_1, z_2, t) \in \mathbf{R}^3 | t > 0\}$, $R^3_+ \equiv \{(z_1, z_2, z_3) \in \mathbf{R}^3 | z_3 > 0\}$, $R^3_- \equiv \{(z_1, z_2, z_3) \in \mathbf{R}^3 | z_3 < 0\}$, and $a_1, a_2, b_1, b_2, c, d_1, d_2$ are positive constants. We begin with the derivation of a estimate of a solution to the following homogeneous problem:

$$(3.5)\quad \begin{cases} \dfrac{\partial w_1'}{\partial t} - a_1 \Delta w_1' = 0 & \text{in} \quad D^3_\infty, \\ \dfrac{\partial w_2'}{\partial t} - a_2 \Delta w_2' = 0 & \text{in} \quad \tilde{D}^3_\infty, \\ w_1'|_{t=0} = 0 \quad \text{on} \quad R^3_+, \\ w_2'|_{t=0} = 0 \quad \text{on} \quad R^3_-, \\ \dfrac{\partial \delta}{\partial t} + d_1 \dfrac{\partial w_1'}{\partial z_3} - d_2 \dfrac{\partial w_2'}{\partial z_3} = g_3 & \text{on} \quad R^2_\infty, \\ w_1' + b_1 \delta - c\sigma \Delta_{z'} \delta = 0 & \text{on} \quad R^2_\infty, \\ w_2' + b_2 \delta - c\sigma \Delta_{z'} \delta = 0 & \text{on} \quad R^2_\infty. \end{cases}$$

Making use of the Fourier transformation with respect to $z' = (z_1, z_2)$ and the Laplace transformation with respect to t:

$$FL[f](\xi', z_3, s) \equiv \tilde{f}(\xi', z_3, s) \equiv \int_0^\infty e^{-st} dt \int_{\mathbf{R}^2} e^{-iz' \cdot \xi'} f(z', z_3, t) dz',$$

we have a representation of a solution of the transformed problem of (3.5) as follows:

$$(3.6)\quad \tilde{w_1'} = -(b_1 + c\sigma |\xi'|^2) \tilde{\delta} \exp \left[- \left(\frac{s + a_1 |\xi'|^2}{a_1} \right)^{1/2} z_3 \right],$$

$$(3.7)\quad \tilde{w_2'} = -(b_2 + c\sigma |\xi'|^2) \tilde{\delta} \exp \left[\left(\frac{s + a_2 |\xi'|^2}{a_2} \right)^{1/2} z_3 \right]$$

and

$$(3.8)\quad \tilde{\delta} = \frac{\tilde{g_3}}{s + \sum\limits_{i=1,2} d_i (b_i + c\sigma |\xi'|^2) \left(\dfrac{s + a_i |\xi'|^2}{a_i} \right)^{1/2}}.$$

The following theorem in [3] makes it possible to estimate these transformed functions in Hölder norms.

THEOREM 3.2. *Suppose that a function $f(x,t)$ belongs to $C_0^{\alpha,\frac{\alpha}{2}}(\mathbf{R}^2_\infty)$ for some $\alpha > 0$ and a symbol $\tilde{K}(\xi,s)$ $(s = a + i\xi_0, a \geq 0)$, satisfies the condition:*

$$\Gamma_h^{\nu_j}(\tilde{K}) \equiv \int_0^\infty \frac{dz_0}{z_0^{3/2}} \int_0^\infty \frac{dz_1}{z_1^{3/2}} \int_0^\infty \frac{dz_2}{z_2^{3/2}} \times \|\Delta_0(z_0)\Delta_1(z_1)\Delta_2(z_2)[\eta_j^{\nu_j}\tilde{\Phi}(\eta,\eta_0)\tilde{K}(\eta_h,s_h)]\|_{L_2(\mathbf{R}^3_{\eta,\eta_0})} \leq Ch^l$$

for sufficiently large ν_j, $j = 0,1,2$, where $\eta_h = \eta/h, s_h = a + i\xi_0/h^2$, (if $j = 0$), 1 (if $j= 1,2$) and C is a positive constant independent of h.
Then the convolution $u = K * f$ satisfies the inequality

$$\langle u_a\rangle_{R^2_\infty}^{(l+\alpha,\frac{l+\alpha}{2})} \leq C\langle f_a\rangle_{R^2_\infty}^{(\alpha,\frac{\alpha}{2})},$$

where the notation f_a means the function defined as $f_a \equiv F^{-1}[([FL]f)]$, and C is a positive constant independent of f_a. Here $\Delta_i(z_i)$ are finite difference of step size z_i in the variable η_i and $\Phi(x,t) = \phi(x_1)\phi(x_2)\phi(t)$, where $\phi(x) = \sum_{k=1}^{N} \frac{(-1)^{k+1}N!}{k!(N-k)!}\frac{1}{k}\omega\left(\frac{x}{k}\right)$, N is a sufficiently large positive integer and ω is a function belongs to $C^\infty(\mathbf{R})$ satisfying $\mathrm{supp}\omega \subset [0,1], \omega \geq 0$ *and* $\int \omega(x)dx = 1$.

Introduce the notaion

$$\begin{cases} R(\xi',s) \equiv s + \sum_{i=1,2} d_i(b_i + c\sigma|\xi'|^2)\left(\dfrac{s + a_i|\xi'|^2}{a_i}\right)^{1/2}, \\ P(\xi',s) \equiv s + \sum_{i=1,2} d_i b_i \left(\dfrac{s + a_i|\xi'|^2}{a_i}\right)^{1/2}, \quad r_i(\xi',s) \equiv \left(\dfrac{s + a_i|\xi'|^2}{a_i}\right)^{1/2} (i = 1,2), \end{cases}$$

then considering $\arg r_1$, $\arg r_2 \in (-\pi/4, \pi/4)$, we have

$$|R(\xi',s)| \geq C|P(\xi',s)|,$$

where C is an arbitraly positive constant satisfying $0 < C < (2 - 2^{1/2})^{1/2}/2$. This inequality plays essential role to derive uniform estimates of $R(\xi',s)$ with respect to $\sigma \geq 0$. Indeed, by the caluculation given in [7] with this inequality, it is easily seen that $R(\xi',s)$ satisfies the following lemma.

LEMMA 3.3. *The symbol $R(\xi',s)$ satisfies*

$$\Gamma_h^{\nu_j}(R(\xi',s)) \leq Ch, \quad j = 0,1,2,$$

where C is a positive constant independent of h and $\sigma \geq 0$, and $\nu_j's$ are sufficiently large positive constants.

Then firstly we have the estimate

$$|\delta|_{\mathbf{R}^2_T}^{(3+\alpha,\frac{3+\alpha}{2})} \leq C|g_3|_{\mathbf{R}^2_T}^{(2+\alpha,\frac{2+\alpha}{2})}.$$

Furthermore the following lemma obviously holds because of the homogeneity of the symbol r_i.

LEMMA 3.4. *The symbols r_i (i=1,2)* satisfy

$$\Gamma_h^{\nu_j}(r_i) \leq Ch, \quad j = 0,1,2,$$

where C is a positive constant independent of h, and $\nu_j' s$ are sufficiently large positive constants.

Then the relation

$$\left(s + c\sigma|\xi'|^2 \sum_{j=1,2} d_j r_j\right)\tilde{\delta} = -\sum_{j=1,2} d_j b_j r_j \tilde{\delta} + \tilde{g}_3$$

implies the estimate

$$\left|\frac{\partial \delta}{\partial t}\right|^{(2+\alpha,\frac{2+\alpha}{2})}_{\mathbf{R}^2_T} + \sigma|\Delta_{z'}\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\mathbf{R}^2_T}$$
$$\leq C\left(|\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\mathbf{R}^2_T} + |g_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{\mathbf{R}^2_T}\right) \leq C|g_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{\mathbf{R}^2_T}.$$

Hence we have

$$|\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,\mathbf{R}^2_T} \leq C|g_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{\mathbf{R}^2_T},$$

where $C's$ are positive constants which is independent of σ. Since w_i', $i = 1, 2$, in (3.5) can be considered as solutions of the Dirichlet problem of heat equations, we have also the estimate:

$$|w_1'|^{(3+\alpha,\frac{3+\alpha}{2})}_{D^3_T} + |w_2'|^{(3+\alpha,\frac{3+\alpha}{2})}_{\tilde{D}^3_T}$$
$$\leq C\left((b_1+b_2)|\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\mathbf{R}^2_T} + c\sigma|\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,\mathbf{R}^2_T}\right) \leq C|g_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{R^2_T}.$$

The solution of the non-homogeneous problem (3.4) is given by adding the above w_i' to a solution of the problem

$$\begin{cases} \dfrac{\partial w_1''}{\partial t} - a_1\Delta w_1'' = g_1 & \text{in } D^3_\infty, \\ \dfrac{\partial w_2''}{\partial t} - a_2\Delta w_2'' = g_2 & \text{in } \tilde{D}^3_\infty, \\ w_1''|_{t=0} = 0 \quad \text{on } R^3_+, \\ w_2''|_{t=0} = 0 \quad \text{on } R^3_-, \\ w_1'' = g_4 \quad \text{on } R^2_\infty, \\ w_2'' = g_5 \quad \text{on } R^2_\infty. \end{cases}$$

Hence we have the following theorem:

THEOREM 3.5. *Suppose that*

$$g_1 \in C_0^{1+\alpha,\frac{1+\alpha}{2}}(D^3_T), \quad g_2 \in C_0^{1+\alpha,\frac{1+\alpha}{2}}(\tilde{D}^3_T),$$

$$g_3 \in C_0^{2+\alpha,\frac{2+\alpha}{2}}(R^2_T), \quad g_4 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(R^2_T), \quad g_5 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(R^2_T),$$

then problem (3.4) *has a unique solution*

$$w_1 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(D^3_T), \quad w_2 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(\tilde{D}^3_T), \quad \delta \in C_{\sigma,0}^{3+\alpha,\frac{3+\alpha}{2}}(R^2_T)$$

satisfying

$$|w_1|^{(3+\alpha,\frac{3+\alpha}{2})}_{D^3_T} + |w_2|^{(3+\alpha,\frac{3+\alpha}{2})}_{\tilde{D}^3_T} + |\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,R^2_T}$$
$$\le C\Big(|g_1|^{(1+\alpha,\frac{1+\alpha}{2})}_{D^3_T} + |g_2|^{(1+\alpha,\frac{1+\alpha}{2})}_{D^3_T}$$
$$+|g_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{R^2_T} + |g_4|^{(3+\alpha,\frac{3+\alpha}{2})}_{R^2_T} + |g_5|^{(3+\alpha,\frac{3+\alpha}{2})}_{R^2_T}\Big),$$

where C is a constant independent of σ, g_i, $i = 1,\cdots,5$, and remains bounded as $T \longrightarrow 0$.

On the basis of this theorm, we can solve problem (3.3) by the method of regularizer (see [8], [7], [5]).

THEOREM 3.6. *Let us assume that*

$$G_1 \in C_0^{1+\alpha,\frac{1+\alpha}{2}}(Q_T^{(1)}),\quad G_2 \in C_0^{1+\alpha,\frac{1+\alpha}{2}}(Q_T^{(2)}),\quad H_2 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(\Sigma_T),$$

$$G_3 \in C_0^{2+\alpha,\frac{2+\alpha}{2}}(\Gamma_T),\quad G_4 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(\Gamma_T),\quad G_5 \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(\Gamma_T).$$

Then probrem (3.3) has a unique solution $w^{(i)} \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(Q_T^{(i)})$, $i = 1,2$, $\delta \in C_{\sigma,0}^{3+\alpha,\frac{3+\alpha}{2}}(\Gamma_T)$ which satisfies

$$\sum_{i=1,2} |w^{(i)}|^{(3+\alpha,\frac{3+\alpha}{2})}_{Q_T^{(i)}} + |\delta|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,\Gamma_T} \le C\Big(|G_1|^{(1+\alpha,\frac{1+\alpha}{2})}_{Q_T^{(1)}} + |G_2|^{(1+\alpha,\frac{1+\alpha}{2})}_{Q_T^{(1)}}$$
$$+|H_2|^{(3+\alpha,\frac{3+\alpha}{2})}_{\Sigma_T} + |G_3|^{(2+\alpha,\frac{2+\alpha}{2})}_{\Gamma_T} + |G_4|^{(3+\alpha,\frac{3+\alpha}{2})}_{\Gamma_T} + |G_5|^{(3+\alpha,\frac{3+\alpha}{2})}_{\Gamma_T}\Big)$$

where a constant C does not depend on σ, G_j, $j = 1,\cdots,5$, H_2, and remains bounded as $T \longrightarrow 0$.

4. Proof of Theorem 1.1.

LEMMA 4.1. *Let $0 < \alpha < 1$. The following inequalities hold for any $\boldsymbol{u}_1, \boldsymbol{u}_2 \in C_0^{2+\alpha,\frac{2+\alpha}{2}}(Q_T^{(1)}), \nabla q_1, \nabla q_2 \in C_0^{\alpha,\frac{\alpha}{2}}(Q_T^{(1)})$,
$\delta_1, \delta_2 \in C_{\sigma,0}^{3+\alpha,\frac{3+\alpha}{2}}(\Gamma_T), w_1^{(1)}, w_2^{(1)} \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(Q_T^{(1)}), w_1^{(2)}, w_2^{(2)} \in C_0^{3+\alpha,\frac{3+\alpha}{2}}(Q_T^{(2)})$.*

$$\left|\mathcal{F}_1(\boldsymbol{u}_1,\nabla q_1, w_1^{(1)},\delta_1) - \mathcal{F}_1(\boldsymbol{u}_2,\nabla q_2, w_2^{(1)},\delta_2)\right|^{(\alpha,\frac{\alpha}{2})}_{Q_T^{(1)}}$$
$$\le c(T)\Big(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^{(2+\alpha,\frac{2+\alpha}{2})}_{Q_T^{(1)}} + |\nabla q_1 - \nabla q_2|^{(\alpha,\frac{\alpha}{2})}_{Q_T^{(1)}}$$
$$+|\delta_1 - \delta_2|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,\Gamma_T} + |w_1^{(1)} - w_2^{(1)}|^{(3+\alpha,\frac{3+\alpha}{2})}_{Q_T^{(1)}}\Big),$$

$$|\mathcal{F}_2(\boldsymbol{u}_1,\delta_1) - \mathcal{F}_2(\boldsymbol{u}_2,\delta_2)|^{(1+\alpha,\frac{1+\alpha}{2})}_{Q_T^{(1)}}$$
$$\le c(T)\Big(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^{(2+\alpha,\frac{2+\alpha}{2})}_{Q_T^{(1)}} + |\delta_1 - \delta_2|^{(3+\alpha,\frac{3+\alpha}{2})}_{\sigma,\Gamma_T}\Big),$$

$$
\begin{aligned}
&|\mathcal{F}_3(\boldsymbol{u}_1,\delta_1)-\mathcal{F}_3(\boldsymbol{u}_2,\delta_2)|_{\Gamma_T}^{(2+\alpha,\frac{2+\alpha}{2})}\\
&\le\left(c(T)+\left|1-\frac{\rho_e}{\rho}\right|\right)\\
&\qquad\times\left(|\boldsymbol{u}_1-\boldsymbol{u}_2|_{Q_T^{(1)}}^{(2+\alpha,\frac{2+\alpha}{2})}+|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathcal{F}_4(\boldsymbol{u}_1,\delta_1)-\mathcal{F}_4(\boldsymbol{u}_2,\delta_2)|_{\Gamma_T}^{(1+\alpha,\frac{1+\alpha}{2})}\\
&\qquad\le c(T)\left(|\boldsymbol{u}_1-\boldsymbol{u}_2|_{Q_T^{(1)}}^{(2+\alpha,\frac{2+\alpha}{2})}+|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathcal{G}_1(\boldsymbol{u}_1,w_1^{(1)},\delta_1)-\mathcal{G}_1(\boldsymbol{u}_2,w_2^{(1)},\delta_2)\right|_{Q_T^{(1)}}^{(1+\alpha,\frac{1+\alpha}{2})}\\
&\le c(T)\left(|\boldsymbol{u}_1-\boldsymbol{u}_2|_{Q_T^{(1)}}^{(2+\alpha,\frac{2+\alpha}{2})}\right.\\
&\qquad\left.+|w_1^{(1)}-w_2^{(1)}|_{Q_T^{(1)}}^{(3+\alpha,\frac{3+\alpha}{2})}+|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathcal{G}_2(w_1^{(2)},\delta_1)-\mathcal{G}_2(w_2^{(2)},\delta_2)\right|_{Q_T^{(2)}}^{(1+\alpha,\frac{1+\alpha}{2})}\\
&\qquad\le c(T)\left(|w_1^{(2)}-w_2^{(2)}|_{Q_T^{(2)}}^{(3+\alpha,\frac{3+\alpha}{2})}+|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\mathcal{G}_3(w_1^{(1)},w_1^{(2)},\delta_1)-\mathcal{G}_3(w_2^{(1)},w_2^{(2)},\delta_2)\right|_{\Gamma_T}^{(2+\alpha,\frac{2+\alpha}{2})}\\
&\le\left(c(T)+\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)\nabla\theta_1\left(1-\frac{\sigma}{l}H_0\right)\cdot\boldsymbol{\tau}\right|_{\Gamma}^{(0)}\right)\\
&\quad\times\left(|w_1^{(1)}-w_2^{(1)}|_{Q_T^{(1)}}^{(3+\alpha,\frac{3+\alpha}{2})}\right.\\
&\qquad\left.+|w_1^{(2)}-w_2^{(2)}|_{Q_T^{(2)}}^{(3+\alpha,\frac{3+\alpha}{2})}+|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\right),
\end{aligned}
$$

$$
|\mathcal{G}_4(\delta_1)-\mathcal{G}_4(\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\le c(\epsilon,T)|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})},
$$

$$
|\mathcal{G}_5(\delta_1)-\mathcal{G}_5(\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}\le c(\epsilon,T)|\delta_1-\delta_2|_{\sigma,\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})},
$$

where $c(T)$ is a positive constant depending on $T,\boldsymbol{u}_1,\boldsymbol{u}_2,q_1,q_2,w_1^{(1)},w_2^{(1)},w_1^{(2)},w_2^{(2)}$, δ_1,δ_2 which converges to 0 as $T\longrightarrow 0$ and $c(\epsilon,T)$ is a positive constant depending not *only on T and the above functions but also on ϵ which will appear below, and be taken arbitrally small for suitably chosen ϵ and T.*

Proof. Terms $\mathcal{F}_i$ ($i = 1, \cdots, 7$) and $\mathcal{G}_i$ ($i = 1, \cdots, 3$) can be treated by ju same way as in [7], hence here we give the estimates only for $\mathcal{G}_i$ ($i = 4, 5$).

Set

$$\begin{aligned}\mathcal{G}_4(\delta) = & -\hat{\theta}_\sigma^{(1)} + \theta_1 - \frac{\theta_1\sigma}{l}\frac{1}{|\nabla\eta|}\sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j} \\ & + \frac{\theta_1\sigma}{l}\frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|}\sum_{i,j=1,2} a_{ij}\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right)\frac{\partial^2(\hat{d}_\sigma + \delta)}{\partial\omega_i\partial\omega_j} \\ & + \frac{\theta_1\sigma}{l}\frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|} b\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right) \\ \equiv & -\hat{\theta}_\sigma^{(1)} + \theta_1 - \frac{\theta_1\sigma}{l}(\mathcal{A}(\delta) + \mathcal{B}(\delta)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}(\delta) \equiv & -\frac{1}{|\nabla\eta|}\sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j} \\ & + \frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|}\sum_{i,j=1,2} a_{ij}\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right)\frac{\partial^2(\hat{d}_\sigma + \delta)}{\partial\omega_i\partial\omega_j}\end{aligned}$$

and

$$\mathcal{B}(\delta) \equiv -\frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|} b\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right).$$

Firstly we rewrite $\mathcal{A}(\delta)$ as follows:

$$\begin{aligned}\mathcal{A}(\delta) = & \left(\frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|}\right)\sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j} \\ & + \frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|}\sum_{i,j}\Big(a_{ij}(\omega, 0, 0, 0) \\ & \qquad - a_{ij}\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right)\Big)\frac{\partial^2\delta}{\partial\omega_i\partial\omega_j} \\ & + \frac{1}{|\nabla_{\hat{d}_\sigma+\delta}\eta|}\sum_{i,j=1,2} a_{ij}\left(\omega, \hat{d}_\sigma + \delta, \nabla_{\hat{d}_\sigma+\delta}(\hat{d}_\sigma + \delta)\right)\frac{\partial^2\hat{d}_\sigma}{\partial\omega_i\partial\omega_j}.\end{aligned}$$

For example, the first term is evaluated as follows:

$$\begin{aligned}\Bigg| & \left(\frac{1}{|\nabla_{\hat{d}_\sigma+\delta_1}\eta|} - \frac{1}{|\nabla_{\hat{d}_\sigma+\delta_2}\eta|}\right)\sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0)\frac{\partial^2\delta_1}{\partial\omega_i\partial\omega_j} \\ & + \left(\frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{\hat{d}_\sigma+\delta_2}\eta|}\right)\sum_{i,j=1,2} a_{ij}(\omega, 0, 0, 0)\frac{\partial^2(\delta_1 - \delta_2)}{\partial\omega_i\partial\omega_j}\Bigg|^{(3+\alpha, \frac{3}{}}_{\Gamma_T}\end{aligned}$$

$$\leq \left| \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_1}\eta|} - \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_2}\eta|} \right|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} \left| \sum_{i,j=1,2} a_{ij=1,2}(\omega,0,0,0)\frac{\partial^2\delta_1}{\partial\omega_i\partial\omega_j} \right|_{\Gamma_T}^{(0)}$$

$$+ \left| \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_1}\eta|} - \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_2}\eta|} \right|_{\Gamma_T}^{(0)} \left| \sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\frac{\partial^2\delta_1}{\partial\omega_i\partial\omega_j} \right|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}$$

$$+ \left| \frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_2}\eta|} \right|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} \left| \sum_{i,j} a_{ij}(\omega,0,0,0)\frac{\partial^2(\delta_1-\delta_2)}{\partial\omega_i\partial\omega_j} \right|_{\Gamma_T}^{(0)}$$

$$+ \left| \frac{1}{|\nabla\eta|} - \frac{1}{|\nabla_{\hat{d_\sigma}+\delta_2}\eta|} \right|_{\Gamma_T}^{(0)} \left| \sum_{i,j=1,2} a_{ij}(\omega,0,0,0)\frac{\partial^2(\delta_1-\delta_2)}{\partial\omega_i\partial\omega_j} \right|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}$$

$$\leq c(T)|\Delta_\Gamma(\delta_1-\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})},$$

where by Δ_Γ we denote the operator $\sum_{i,j} a_{ij}(\omega,0,0,0)\frac{\partial^2}{\partial\omega_i\partial\omega_j}$. Taking into account the definition of a_{ij}, the second and the last terms are also estimated by $|\Delta_\Gamma(\delta_1 - \delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})}$. Furthermore, considering the following estimate

$$\begin{aligned} &\Big| b\left(\omega, \hat{d_\sigma}+\delta_1, \nabla_{\hat{d_\sigma}+\delta_1}(\hat{d_\sigma}+\delta_1)\right) \\ &\qquad -b\left(\omega, \hat{d_\sigma}+\delta_2, \nabla_{\hat{d_\sigma}+\delta_2}(\hat{d_\sigma}+\delta_2)\right)\Big|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} \\ &\leq C_1|\nabla(\delta_1-\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} \\ &\leq C_1\left(\epsilon|\Delta_\Gamma(\delta_1-\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} + C(\epsilon)|\nabla(\delta_1-\delta_2)|_{\Gamma_T}^{(0)}\right), \end{aligned}$$

$\mathcal{B}(\delta_1) - \mathcal{B}(\delta_2)$ is obviously estimated in the form

$$|\mathcal{B}(\delta_1)-\mathcal{B}(\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})} \leq C_2\left(\epsilon + c(T)C(\epsilon)\right)|\Delta_\Gamma(\delta_1-\delta_2)|_{\Gamma_T}^{(3+\alpha,\frac{3+\alpha}{2})},$$

where C_1 and C_2 are positive constants independent of σ. □

Now, on the basis of this lemma, we construct a solution of problem (2.5)-(2.6) in the function space:

$$X_{\sigma,T}^{k+\alpha} \equiv \left\{ \begin{array}{|l} (\boldsymbol{u}, \nabla q, w^{(1)}, w^{(2)}, \delta) \in \mathcal{X}_{\sigma,T}^{k+\alpha} \\ \|(w^{(1)}, w^{(2)}, \delta, \boldsymbol{u}, \nabla q)\|_{X_{\sigma,T}^{k+\alpha}} \equiv |\boldsymbol{u}|_{Q_T^{(1)}}^{(k+2+\alpha,\frac{k+2+\alpha}{2})} \\ +|\nabla q|_{Q_T^{(1)}}^{(k+\alpha,\frac{k+\alpha}{2})} + |w^{(1)}|_{Q_T^{(1)}}^{(k+3+\alpha,\frac{k+3+\alpha}{2})} \\ +|w^{(2)}|_{Q_T^{(2)}}^{(k+3+\alpha,\frac{k+3+\alpha}{2})} + |\delta|_{\sigma,\Gamma_T}^{(k+3+\alpha,\frac{k+3+\alpha}{2})} \\ \leq KM \end{array} \right\},$$

where

$$\begin{aligned} M \equiv\; &|\mathcal{F}_1(0,0,0,0)|_{Q_T^{(1)}}^{(k+\alpha,\frac{k+\alpha}{2})} + |\mathcal{F}_2(0,0)|_{Q_T^{(1)}}^{(k+1+\alpha,\frac{k+1+\alpha}{2})} \\ &+|\mathcal{F}_3(0,0)|_{\Gamma_T}^{(k+2+\alpha,\frac{k+2+\alpha}{2})} + |\mathcal{F}_4(0,0)|_{\Gamma_T}^{(k+1+\alpha,\frac{1+\alpha}{2})} \end{aligned}$$

$$+|\mathcal{G}_1(0,0,0)|_{Q_T^{(1)}}^{(k+1+\alpha,\frac{1+\alpha}{2})}+|\mathcal{G}_2(0,0)|_{Q_T^{(2)}}^{(k+1+\alpha,\frac{k+1+\alpha}{2})}$$
$$+|\mathcal{G}_3(0,0,0)|_{\Gamma_T}^{(k+2+\alpha,\frac{k+2+\alpha}{2})}+|\mathcal{G}_4(0)|_{\Gamma_T}^{(k+3+\alpha,\frac{k+3+\alpha}{2})}$$
$$+|\mathcal{G}_5(0)|_{\Gamma_T}^{(k+3+\alpha,\frac{k+3+\alpha}{2})},$$

and K and T are positive constants to be determined later. Choose $(u,\nabla q,w^{(1)},w^{(2)},\delta)\in X^{\alpha}_{\sigma,T}$ arbitrarily, and $(\hat{u},\nabla\hat{q},\hat{w}^{(1)},\hat{w}^{(2)},\hat{\delta})$ be a solution of problem (3.1) with $(\boldsymbol{F}_1,F_2,0,\boldsymbol{H}_1,F_3,\boldsymbol{F}_4,G_1,G_2,0,0,H_2,G_3,G_4,G_5)$ replaced by $(\mathcal{F}_1,\mathcal{F}_2,0,-\hat{\boldsymbol{v}}_\sigma,\mathcal{F}_3,\mathcal{F}_4,\mathcal{G}_1,\mathcal{G}_2,0,0,\theta_2-\hat{\theta}^{(1)}_\sigma,\mathcal{G}_3,\mathcal{G}_4,\mathcal{G}_5)$. Let P be a mapping corresponds $(u,\nabla q,w^{(1)},w^{(2)},\delta)$ to $(\hat{u},\nabla\hat{q},\hat{w}^{(1)},\hat{w}^{(2)},\hat{\delta})$. Then theorem 3.1 and theorem 3.6 garantee that P maps $X^{\alpha}_{\sigma,T}$ into itself. Actually, it is shown as follows.

$$\begin{aligned}
&||P(u,\nabla q,w^{(1)},w^{(2)},\delta)||_{X^{\alpha}_{\sigma,T}}\equiv||(\hat{u},\nabla\hat{q},\hat{w}^{(1)},\hat{w}^{(2)},\hat{\delta})||_{X^{\alpha}_{\sigma,T}}\\
&\le||P(u,\nabla q,w^{(1)},w^{(2)},\delta)-P(0,0,0,0,0)||_{X^{\alpha}_{\sigma,T}}\\
&\qquad+||P(0,0,0,0,0)||_{X^{\alpha}_{\sigma,T}}\\
&\le C_3\left[\epsilon+c(T)(C(\epsilon)+1)+\left|1-\frac{\rho_e}{\rho}\right|\right.\\
&\qquad+\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)(\nabla\theta_1)\cdot\boldsymbol{\tau}\right|_{\Gamma}^{(0)}\\
&\qquad\left.+\frac{\sigma}{l}\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)\nabla(\theta_1H_0)\cdot\boldsymbol{\tau}\right|_{\Gamma}^{(0)}\right]KM\\
&\qquad+M\\
&\equiv(L(\epsilon,T)K+1)\,M,
\end{aligned}$$

where H_0 is the twice mean curvature of Γ and C_3 is a positive constant independent of σ. Considerting the smallness assumptions for $|\rho-\rho_e|$ and $|\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})|$ given in theorem 1.1, we can take σ satisfying

$$\begin{aligned}
(4.1)\qquad &C_3\frac{\sigma}{l}\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)\nabla(\theta_1H_0)\cdot\boldsymbol{\tau}\right|_{\Gamma}^{(0)}\\
&\le\frac{1}{2}-C_3\left(\left|1-\frac{\rho_e}{\rho}\right|+\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)(\nabla\theta_1)\cdot\boldsymbol{\tau}\right|_{\Gamma}^{(0)}\right).
\end{aligned}$$

Hence, for some ϵ_0 satisfying $C_3\epsilon_0<1/2$, there exsist $T_0>0$ independent of $\sigma\in(0,\sigma*)$ such as $L(\epsilon_0,T_0)<1$. Here by σ^* we denote the upperbound of σ satisfying (4.1). Then taking $K>0$ larger than $1/(1-L(\epsilon_0,T_0))$, we have $||P(u_\sigma,\nabla q_\sigma,w_\sigma^{(1)},w_\sigma^{(2)},\delta_\sigma)||_{X^{\alpha}_{\sigma,T_0}}\le KM$.

Contractiveness of mapping P also follows from $L(\epsilon_0,T_0)<1$. Hence the contractive mapping theorem yields a unique solution of the problem.

Moreover, the convergence of the solution of problem (P_σ) can be proved as follows. Let $(u_{\sigma_1},\nabla q_{\sigma_1},w_{\sigma_1}^{(1)},w_{\sigma_1}^{(2)},\delta_{\sigma_1})$ in $X^{2+\alpha}_{\sigma_1,T_0}$ and $(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})$ in $X^{2+\alpha}_{\sigma_2,T_0}$ be solutions of problems (P_{σ_1}) and (P_{σ_2}), respectively. Then we have

$$\begin{aligned}
&||(u_{\sigma_1},\nabla q_{\sigma_1},w_{\sigma_1}^{(1)},w_{\sigma_1}^{(2)},\delta_{\sigma_1})-(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})||_{X^{\alpha}_{T_0}}\\
&\le C_4||(\boldsymbol{v}_{\sigma_1,0},\theta^{(1)}_{\sigma_1,0},\theta^{(2)}_{\sigma_1,0})-(\boldsymbol{v}_{\sigma_2,0},\theta^{(1)}_{\sigma_2,0},\theta^{(2)}_{\sigma_2,0})||_{\mathcal{H}^{\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&+C_5\left(c(T_0)+\left|1-\frac{\rho_e}{\rho}\right|+\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)(\nabla\theta_1)\cdot\tau\right|_\Gamma^{(0)}\right)\\
&\quad\times\|(u_{\sigma_1},\nabla q_{\sigma_1},w_{\sigma_1}^{(1)},w_{\sigma_1}^{(2)},\delta_{\sigma_1})-(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})\|_{\mathcal{X}_{T_0}^\alpha}\\
&\quad+\frac{|\sigma_1-\sigma_2|}{l}\left(\left|\left(\kappa^{(1)}(\theta_0^{(1)})-\kappa^{(2)}(\theta_0^{(2)})\right)\nabla(\theta_1 H_0)\cdot\tau\right|_\Gamma^{(0)}+C_6\right)\\
&\quad\times\left(\|(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})\|_{\mathcal{X}_{\sigma,T_0}^{2+\alpha}}+\|(v_{\sigma_2,0},\theta_{\sigma_2,0}^{(1)},\theta_{\sigma_2,0}^{(2)})\|_{\mathcal{H}^{1+\alpha}}\right)\\
&\equiv M(T_0)\|(u_{\sigma_1},\nabla q_{\sigma_1},w_{\sigma_1}^{(1)},w_{\sigma_1}^{(2)},\delta_{\sigma_1})-(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})\|_{\mathcal{X}_{T_0}^\alpha}\\
&\quad+|\sigma_1-\sigma_2|K'M,
\end{aligned}
$$

where C_i, i=4,5,6, are positive constants independent of σ, and K is a positive constnant satisfying $K' > K$. Noting that $M(T_0) < L(\epsilon_0, T_0) < 1$, we have

$$
\begin{aligned}
&\|u_{\sigma_1},\nabla q_{\sigma_1},w_{\sigma_1}^{(1)},w_{\sigma_1}^{(2)},\delta_{\sigma_1})-(u_{\sigma_2},\nabla q_{\sigma_2},w_{\sigma_2}^{(1)},w_{\sigma_2}^{(2)},\delta_{\sigma_2})\|_{\mathcal{X}_{T_0}^\alpha}\\
&\le\frac{|\sigma_1-\sigma_2|K'M}{1-M(T_0)}\longrightarrow 0\quad(\sigma_1,\sigma_2\longrightarrow 0).
\end{aligned}
$$

Thus $\{(u_\sigma,\nabla q_\sigma,w_\sigma^{(1)},w_\sigma^{(2)},\delta_\sigma)\}$ is a Cauchy sequence in $\mathcal{X}_{T_0}^\alpha$ as $\sigma\longrightarrow 0$. Hence the proof of theorem 1.1 is completed.

REFERENCES

[1] B.V. Bazaliĭ and S.P. Degtyarev, *The classical Stefan problem as the limit case of the Stefan problem with a kinetic condition at the free boundary*, in Free Boundary Problems in Continuum Mechanics, S. N. Antontsev, K. H. Hoffmann, and A. M. Khludnev, eds., Birkhauser Verlag, Besel, 1992, pp.83-90.

[2] X. Chen and F. Reitich, *Local existence and uniqueness of the Stefan problem with kinetic undercooling*, J. Math. Anal. Appl., 164(1992), pp.350-362.

[3] K.K. Golovkin and V.A. Solonnikov, *On estimates of convolution operators*, Zap. Nauch. Sem. Leningrad. Otdel. Mat. Inst. Steklov(LOMI) 7(1968), pp.6-86.(Russian)

[4] E. Hanzawa, *Classical solutions of the Stefan problem*, Tohoku Math. J., 33(1981), pp. 297-335.

[5] Y. Kusaka *Classical solvability and uniquness of the Stefan problem with surface tension in a viscous incompressible fluid flow*, (to be submitted).

[6] Y. Kusaka and A. Tani, *On the classical solvability of the Stefan problem in a viscous incompressible fluid flow*, SIAM J. Math. Anal. 30(1999), pp. 584-602.

[7] Y. Kusaka and A. Tani, *Classical solvability and the two-phase Stefan problem in a viscous incompressible fluid flow*, (to appear).

[8] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs, Vol.23, Amer. Math. Soc., Providence, 1968.

[9] I.Sh. Moghilevskii and V.A. solonnikov, *Solvability of a non-coercive initial-boundary value problem for the nonstatiionary Stokes system in Hölder spaces (the case of half-space)*, Zeitschr. für Anal. und ihre Anwend. 8 (4) (1989), pp. 329-347.