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On the steady flow of compressible viscous fluid and its stability with respect to initial disturbance

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1 Introduction

The motion of a compressible viscous isotropic Newtonian fluid is formulated by the following initial value problem of the Navier-Stokes equation for viscous compressible fluid:

$$\begin{cases} \rho_t + \nabla\cdot(\rho v) = G(x), \\ v_t + (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v) - \dfrac{\nabla(P(\rho))}{\rho} + F(x), \\ (\rho, v)(0,x) = (\rho_0, v_0)(x), \end{cases} \tag{1.1}$$

where $t \geq 0$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; $\rho = \rho(t,x)\,(>0)$ and $v = (v_1(t,x), v_2(t,x), v_3(t,x))$ denote the density and velocity respectively, which are unknown; $P(\cdot)$ $(P' > 0)$ denotes the pressure; μ and μ' are the viscosity coefficients which satisfy the condition: $\mu > 0$ and $\mu' + 2\mu/3 \geq 0$; $F(x) = (F_1(x), F_2(x), F_3(x))$ is a given external force and $G(x)$ is a given mass source. The stationary problem corresponding to the initial value problem (1.1) is

$$\begin{cases} \nabla\cdot(\rho v) = G(x), \\ (v\cdot\nabla)v = \dfrac{\mu}{\rho}\Delta v + \dfrac{\mu+\mu'}{\rho}\nabla(\nabla\cdot v) - \dfrac{\nabla(P(\rho))}{\rho} + F(x), \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; $\rho = \rho(x)\,(>0)$ and $v = (v_1(x), v_2(x), v_3(x))$ are unknown functions; $F(x)$, $G(x)$ and the other symbols are the same as in (1.1). In this note, we consider the case where the external force F is given by following form:

$$F = \nabla\cdot F_1 + F_2, \tag{1.3}$$

where $F_1 = (F_{1,ij}(x))_{1\leq i,j\leq 3}$ and $F_2 = (F_{2,i}(x))_{1\leq i\leq 3}$.

Before stating our results, we introduce some function spaces. Let L_p denote the usual L_p space, $\mathscr{S}'$ the set of all tempered distributions both on $\mathbb{R}^3$. We put

$$H^k = \{\, u \in L_{1,loc} \mid \|u\|_k < \infty \,\} = \{\, u \in \mathscr{S}' \mid \|\mathscr{F}^{-1}[(1+|\xi|^2)^{k/2}\hat{u}]\| < \infty \,\},$$

$$\hat{H}^k = \{\, u \in L_{1,loc} \mid \nabla u \in H^{k-1} \,\}, \quad \|u\| = \|u\|_{L_2}, \quad \|u\|_k = \textstyle\sum_{\nu=0}^{k} \|\nabla^\nu u\|$$

and furthermore for short we use the notation:

$$\mathscr{H}^{k,\ell} = \{\, (\sigma, v) \mid \sigma \in H^k,\ v \in H^\ell \,\}, \quad \hat{\mathscr{H}}^{k,\ell} = \{\, (\sigma, v) \mid \sigma \in \hat{H}^k,\ v \in \hat{H}^\ell \,\},$$

$$\mathscr{H}^{j,k,\ell} = \{\, (\sigma, v, w) \mid \sigma \in H^j,\ v \in H^k,\ w \in H^\ell \,\},$$

$$\|(\sigma, v)\|_{k,\ell} = \|\sigma\|_k + \|v\|_\ell, \quad \|(\sigma, v, w)\|_{j,k,\ell} = \|\sigma\|_j + \|v\|_k + \|w\|_\ell.$$

This note is based on a joint work with Prof. Y. Shibata, Department of Mathematical Sciences, Waseda

Definition 1.1

$$I_\epsilon^k = \{ \sigma \in H^k \mid \|\sigma\|_{I^k} < \epsilon \}, \quad J_\epsilon^k = \{ u \in \hat{H}^k \mid \|v\|_{J^k} < \epsilon \},$$

where

$$\|\sigma\|_{I^k} = \|\sigma\|_{L_6} + \left\| \frac{\sigma}{|x|} \right\| + \textstyle\sum_{\nu=1}^k \|(1+|x|)^\nu \nabla^\nu \sigma\| + \|(1+|x|)^2 \sigma\|_{L_\infty},$$

$$\|v\|_{J^k} = \|v\|_{L_6} + \left\| \frac{v}{|x|} \right\| + \textstyle\sum_{\nu=1}^k \|(1+|x|)^{\nu-1} \nabla^\nu v\| + \sum_{\nu=0}^1 \|(1+|x|)^{\nu+1} \nabla^\nu v\|_{L_\infty}.$$

Moreover we put

$$\begin{aligned} \mathscr{I}_\epsilon^{k,\ell} &= \{ (\sigma, v) \mid \sigma \in I_\epsilon^k, \ v \in J_\epsilon^\ell \}, \\ \dot{\mathscr{I}}_\epsilon^{k,\ell} &= \{ (\sigma, v) \in \mathscr{I}_\epsilon^{k,\ell} \mid \nabla \cdot v = \nabla \cdot V_1 + V_2 \quad \textit{for some} \ \ V_1, V_2 \\ &\qquad \textit{such that} \ \ \|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1} V_2\|_{L_1} \le \epsilon \}, \\ \|(\sigma, v)\|_{\mathscr{I}^{k,\ell}} &= \|\sigma\|_{I^k} + \|v\|_{J^\ell}. \end{aligned}$$

The first theorem is about the existence of stationary solution for (1.2) and its weighted-L_2, L_∞ estimates.

Theorem 1.1 *Let $\bar{\rho}$ be any positive constant. Then, there exist small constants $c_0 > 0$ and $\epsilon > 0$ depending on $\bar{\rho}$ such that if (F, G) satisfies the estimate:*

$$\begin{aligned} &\textstyle\sum_{\nu=0}^3 \|(1+|x|)^{\nu+1} \nabla^\nu F\| + \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1} \\ &\quad + \|(1+|x|) G\| + \textstyle\sum_{\nu=1}^4 \|(1+|x|)^\nu \nabla^\nu G\| \\ &\quad + \textstyle\sum_{\nu=0}^1 \|(1+|x|)^{\nu+2} \nabla^\nu G\|_{L_\infty} + \|(1+|x|)^{-1} G\|_{L_1} \le c_0 \epsilon, \end{aligned}$$

then (1.2) admits a solution of the form: $(\rho, v) = (\bar{\rho} + \sigma, v)$ where $(\sigma, v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$. Furthermore the solution is unique in the following sense:

> *There exists an ϵ_1 with $0 < \epsilon_1 \le \epsilon$ such that if $(\bar{\rho} + \sigma_1, v_1)$ and $(\bar{\rho} + \sigma_2, v_2)$ satisfy (1.2) with the same (F, G), and $\|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}}, \|(\sigma_1, v_1)\|_{\mathscr{I}^{3,4}} \le \epsilon_1$, then $(\sigma_1, v_1) = (\sigma_2, v_2)$.*

Next we consider the stability of the stationary solution of (1.2) with respect to initial disturbance. Let (ρ^*, v^*) be a solution of (1.2) obtained in Theorem 1.1. The stability of (ρ^*, v^*) means the solvability of the non-stationary problem (1.1). Let us introduce the class of functions which solutions of (1.1) belong to.

Definition 1.2

$$\begin{aligned} \mathscr{C}(0, T; \mathscr{H}^{k,\ell}) = \{ (\sigma, v) \mid \ & \sigma(t, x) \in C^0(0, T; H^k) \cap C^1(0, T; H^{k-1}), \\ & w(t, x) \in C^0(0, T; H^\ell) \cap C^1(0, T; H^{\ell-2}) \}. \end{aligned}$$

Then, we have the following theorem.

Theorem 1.2 *There exist $C > 0$ and $\delta > 0$ such that if $\|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3} \le \delta$ then (1.1) admits a unique solution: $(\rho, v) = (\rho^* + \sigma, v^* + w)$ globally in time, where $(\sigma, w) \in \mathscr{C}(0, \infty; \mathscr{H}^{3,3})$, $\nabla \sigma$, $w_t \in L_2(0, \infty; H^2)$, $\nabla w \in L_2(0, \infty; H^3)$. Moreover the (σ, w) satisfies the estimate:*

$$\|(\sigma, w)(t)\|_{3,3}^2 + \int_0^t \|(\nabla \sigma, \nabla w, w_t)(s)\|_{2,3,2}^2 \, ds \le C \|(\rho_0 - \rho^*, v_0 - v^*)\|_{3,3}^2 \tag{1.4}$$

for any $t \ge 0$.

Remark 1.1 When Theorem 1.2 holds, we shall say that the stationary solution (ρ^*, v^*) of (1.2) is stable in the H^3-framework with respect to small initial disturbance.

Matsumura and Nishida [4] first proved the stability of constant state $(\bar{\rho}, 0)$ in H^3-framework with respect to initial disturbance, namely they proved Theorem 1.2 in the case where $(\rho^*, v^*) = (\bar{\rho}, 0)$. When the external force is given by the potential: $F = -\nabla\Phi$, $F_2 = G = 0$ in (1.2) and (1.3) where Φ is a scalar function, the stationary solution $(\rho^*, v^*)(x)$ of (1.2) in a neighborhood of $(\bar{\rho}, 0)$ in $\mathscr{H}^{2,2}$ has the form:

$$\int_{\bar{\rho}}^{\rho^*(x)} \frac{P'(\eta)}{\eta}\, d\eta + \Phi(x) = 0, \quad v^*(x) = 0.$$

In this case, Matsumura and Nishida [5] proved the stability of $(\rho^*(x), 0)$ in the H^3-framework with respect to initial disturbance in an exterior domain.

The purpose of this note is to consider the case where the external force is given by the general formula (1.3) and also mass source G appears. In this case, the stationary solution $(\rho^*, v^*)(x)$ is non-trivial in general, especially $v^* \not\equiv 0$. We are interested only in strong solutions. Then, when F is small enough in a certain norm and $G = 0$, Novotny and Padula [6] proved a unique existence theorem of solutions to (1.2) in an exterior domain. In their proof, they decomposed the equations into the Stokes equation, transport equation and Laplace equation. Since we consider the problem in $\mathbb{R}^3$, that is, the boundary condition is not imposed, we can solve (1.2) without any such decomposition technique. In fact, in §2, we establish the corresponding linear theory to (1.2) in the L_2-framework by the usual Banach closed range theorem, after obtaining some weighted-L_2 estimates for solutions.

The stability of the stationary solutions $(\rho^*, v^*)(x)$ of (1.2) in H^3-framework has not been studied yet. As we stated in Remark 1, Theorem 2 tells us the stability of stationary solutions $(\rho^*, v^*)(x)$ in H^3-framework. The main step of our proof of Theorem 2 is to obtain a priori estimate for solutions of (1.1) as usual. In §3, we shall obtain a priori estimates by choosing several multipliers and using the integration by parts. Compared with the case where $v^* = 0$, we have to give more consideration to choice of multipliers.

Recently, Kawashita [3] and Danchin [1,2] consider the optimal class of initial data regarding the regularity. We think that our result will be improved in this direction.

2 Stationary Problem

We study the stationary problem (1.2). Take any constant $\bar{\rho} > 0$. Substituting $\rho = \bar{\rho} + \sigma$ into (1.2) and putting $\gamma = P'(\bar{\rho})$, (1.2) is reduced to the equation:

$$\begin{cases} \nabla\cdot v + \Big(\dfrac{v}{\bar{\rho}+\sigma}\cdot\nabla\Big)\sigma = \dfrac{G}{\bar{\rho}+\sigma}, \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(\bar{\rho}+\sigma)(v\cdot\nabla)v \\ \qquad - \big[P'(\bar{\rho}+\sigma) - P'(\bar{\rho})\big]\nabla\sigma + (\bar{\rho}+\sigma)F. \end{cases} \tag{2.1}$$

Our goal in this part is to prove Theorem 1.1 by application of weighted-L_2 method to the linearized problem for (2.1).

2.1 Weighted-L_2 theory for linearized problem

In this section, let k be an integer fixed to $k = 3$ or $k = 4$. We shall consider the linearized equation of (2.1):

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)\sigma = g, & (2.2) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(b\cdot\nabla)c + f, & (2.3) \end{cases}$$

where $a = (a_1(x), a_2(x), a_3(x))$, $b = (b_1(x), b_2(x), b_3(x))$, $c = (c_1(x), c_2(x), c_3(x))$ and $(f,g) \in \mathscr{H}^{k-1,k}$ are given. Throughout this section, we assume that

$$a \in \hat{H}^4, \quad \|(1+|x|)a\|_{L_\infty} + \textstyle\sum_{\nu=1}^4 \|(1+|x|)^{\nu-1}\nabla^\nu a\| \le \delta, \quad b, c \in J_\delta^{k+1}, \tag{2.4}$$

$$\textstyle\sum_{\nu=0}^{k-1} \|(1+|x|)^{\nu+1}\nabla^\nu f\| + \|(1+|x|)g\| + \sum_{\nu=1}^{k} \|(1+|x|)^\nu \nabla^\nu g\| < \infty. \tag{2.5}$$

Solution to approximate problem. First, we solve the approximate problem:

$$\begin{cases} \nabla\cdot v + (a\cdot\nabla)\sigma - \epsilon\Delta\sigma + \epsilon\sigma = g, & (2.6) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma + \epsilon v = -(b\cdot\nabla)c + f \equiv h & (2.7) \end{cases}$$

in $\mathscr{H}^{2,2}$. In the next lemma, we shall prove some fundamental a priori estimate needed later.

Lemma 2.1 *There exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in* (2.4) *satisfies $\delta \le \delta_0$ then we have the following estimates:*

(i) *If $0 < \epsilon \le 1$ and $(\sigma, v) \in \mathscr{H}^{2,2}$ is a solution to* (2.6)–(2.7), *then*

$$\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 + \epsilon\{\|v\|^2 + \|\sigma\|^2 + \|\nabla^2\sigma\|^2\} \le C\epsilon^{-1}\|(h,g)\|^2. \tag{2.8}$$

(ii) *If $0 \le \epsilon \le 1$ and $(\sigma, v) \in \mathscr{H}^{2,2}$ is a solution to* (2.6)–(2.7), *then*

$$\|(\nabla\sigma, \nabla^2 v)\| \le C\{\|v\| + \|(h, \nabla g)\|\}. \tag{2.9}$$

Here, $C > 0$ is a constant depending only on μ, μ' and γ.

Proof. (i) Multiplying (2.6) and (2.7) by σ and v respectively; using integration by parts, we have

$$(h, v) = \mu\|\nabla v\|^2 + (\mu+\mu')\|\nabla\cdot v\|^2 + \gamma(\nabla\sigma, v) + \epsilon\|v\|^2,$$
$$(g, \sigma) = -(v, \nabla\sigma) + (a\cdot\nabla\sigma, \sigma) + \epsilon\|\nabla\sigma\|^2 + \epsilon\|\sigma\|^2.$$

Canceling the term of $(\nabla\sigma, v)$ in the above two relations, we obtain

$$\mu\|\nabla v\|^2 + \epsilon\gamma\|\sigma\|^2 + \epsilon\|v\|^2 \le \gamma|(a\cdot\nabla\sigma, \sigma)| + |(h, v)| + \gamma|(g, \sigma)|. \tag{2.10}$$

Differentiating (2.6) and (2.7), and employing the same argument, we have

$$\mu\|\nabla^2 v\|^2 + \epsilon\gamma\|\nabla^2\sigma\|^2 \le \gamma|(\nabla(a\cdot\nabla\sigma), \nabla\sigma)| + |(\nabla h, \nabla v)| + \gamma|(\nabla g, \nabla\sigma)|. \tag{2.11}$$

Adding (2.10) and (2.11), we have

$$\begin{aligned} &\mu\|\nabla v\|_1^2 + \epsilon\{\|v\|^2 + \gamma\|\sigma\|^2 + \gamma\|\nabla^2\sigma\|^2\} \\ &\le \textstyle\sum_{\nu=0}^1 \big[\gamma|(\nabla^\nu(a\cdot\nabla\sigma), \nabla^\nu\sigma)| + |(\nabla^\nu h, \nabla^\nu v)| + \gamma|(\nabla^\nu g, \nabla^\nu\sigma)|\big]. \end{aligned} \tag{2.12}$$

$$\|\nabla\sigma\|^2 \le C_{\gamma,\mu,\mu'}\{\|\nabla^2 v\|^2 + \epsilon\|v\|^2 + \|h\|^2\} \tag{2.13}$$

as follows from (2.7), it follows from (2.12) that

$$\begin{aligned} &\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 + \epsilon\{\|v\|^2 + \|\sigma\|^2 + \|\nabla^2\sigma\|^2\} \\ &\le C_1 \textstyle\sum_{\nu=0}^1 |(\nabla^\nu(a\cdot\nabla\sigma), \nabla^\nu\sigma)| \\ &\quad + C_2\big[\|h\|^2 + \textstyle\sum_{\nu=0}^1\{|(\nabla^\nu h, \nabla^\nu v)| + |(\nabla^\nu g, \nabla^\nu\sigma)|\}\big] \equiv I_1 + I_2, \end{aligned} \tag{2.14}$$

where the constants $C_j > 0$ ($j = 1, 2$) depend only on μ, μ' and γ. Now, integration by parts and the Hardy inequality imply that

$$\begin{aligned} I_1 &\le C_1\Big[\Big|\Big(|x|a\cdot\nabla\sigma, \frac{\sigma}{|x|}\Big)\Big| + \sum_{|\alpha|=1}\Big\{|(\partial_x^\alpha a\cdot\nabla\sigma, \partial_x^\alpha\sigma)| + \frac{1}{2}|((\nabla\cdot a)\partial_x^\alpha a, \partial_x^\alpha\sigma)|\Big\}\Big] \\ &\le C_3\{\|(1+|x|)a\|_{L_\infty} + \|\nabla a\|_{L_\infty}\}\|\nabla\sigma\|^2 \le C_3\delta\|\nabla\sigma\|^2, \end{aligned} \tag{2.15}$$

whereas

$$\begin{aligned} I_2 &\le C_2\|h\|^2 + \frac{1}{2}\{\epsilon\|v\|^2 + \|\nabla^2 v\|^2 + \epsilon\|\sigma\|^2 + \epsilon\|\nabla^2\sigma\|^2\} \\ &\quad + \frac{C_2^2}{2}\{\epsilon^{-1}\|h\|^2 + \|h\|^2 + \epsilon^{-1}\|g\|^2 + \epsilon^{-1}\|g\|^2\}. \end{aligned} \tag{2.16}$$

Combining (2.14)–(2.16), we have (2.8) if $\delta \le 1/4C_3$.

(ii) Using the Friedrichs mollifier, we may assume that $(\sigma, v) \in \mathscr{H}^{\infty,\infty}$. Employing the same argument as in the beginning of proof for (i), we have (2.11) and (2.13). Adding (2.11) and (2.13), we have

$$\begin{aligned} \|(\nabla\sigma, \nabla^2 v)\|^2 &\le C_1\big[\|(v,h)\|^2 + |(\nabla(a\cdot\nabla\sigma), \nabla\sigma)| + \{|(\nabla h, \nabla v)| + |(\nabla g, \nabla\sigma)|\}\big] \\ &\equiv C_1\{\|(v,h)\|^2 + I_1 + I_2\}, \end{aligned} \tag{2.17}$$

where the constant $C_1 > 0$ depends only on μ, μ' and γ. By the same calculation as in (2.15)

$$I_1 \le C_2\delta\|\nabla\sigma\|^2 \quad (C_2 \text{ depends only on } \mu, \mu' \text{ and } \gamma), \tag{2.18}$$

whereas integration by parts implies that

$$I_2 \le \frac{1}{2C_1}\{\|\nabla^2 v\|^2 + \|\nabla\sigma\|^2\} + \frac{C_1}{2}\{\|h\|^2 + \|\nabla g\|^2\}. \tag{2.19}$$

Combining (2.17)–(2.19), we have (2.9) if $\delta \le 1/4C_2$. ∎

Now, we employ the closed range theorem to prove the existence of solution. We introduce the operator A defined on $D(A) \subset L_2$ into L_2 by $A(\sigma, v) = (A_1(\sigma, v), A_2(\sigma, v))$, where $D(A) = \mathscr{H}^{2,2}$ and

$$\begin{aligned} A_1(\sigma, v) &= \nabla\cdot v + (a\cdot\nabla)\sigma - \epsilon\Delta\sigma + \epsilon\sigma, \\ A_2(\sigma, v) &= -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma + \epsilon v. \end{aligned}$$

Lemma 2.1 (i) implies that for each $0 < \epsilon \le 1$ the range of A is closed. Since the dual operator of A has the essentially the same form as A itself, we can show the a priori estimate for the dual of A, and therefore we have the follwing proposition.

Proposition 2.1 *There exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in (2.4) satisfies $\delta \le \delta_0$ then for $0 < \epsilon \le 1$, (2.6)–(2.7) has a solution $(\sigma, v) \in \mathscr{H}^{2,2}$, which satisfies*

$$\|(\sigma, v)\|_{2,2} \le C(\epsilon)\|(h, g)\|, \tag{2.20}$$

where the constant $C(\epsilon)$ depends on $\mu, \mu', \gamma, \epsilon$ and $C(\epsilon) \to \infty$ as $\epsilon \downarrow 0$.

Furthermore, by the regularity theorem of the properly elliptic operator, we have

Corollary 2.1 *Let $(\sigma, v) \in \mathscr{H}^{2,2}$ be solution to (2.6)–(2.7) obtained in Proposition 2.1. Then $(\sigma, v) \in \mathscr{H}^{k+1,k+1}$ and*

$$\|(\sigma, v)\|_{k+1,k+1} \le C(\epsilon)\|(h, g)\|_{k-1,k-1}, \tag{2.21}$$

where the constant $C(\epsilon) > 0$ depends on $\mu, \mu', \gamma, \epsilon$ and $C(\epsilon) \to \infty$ as $\epsilon \downarrow 0$.

Solution to linearized problem (2.2)–(2.3) *and its L_2 estimate.* Next, we shall discuss the estimate for solution to (2.6)–(2.7) independent of $0 < \epsilon \le 1$.

Lemma 2.2 *Let $0 < \epsilon \le 1$ and $(\sigma, v) \in \mathscr{H}^{k+1,k+1}$ be solution to (2.6)–(2.7) which satisfies (2.21). Then, there exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in (2.4) satisfies $\delta \le \delta_0$, then we have the estimate:*

$$\|(\nabla\sigma, \nabla v)\|_{k-1,k} \le C\{\|(1 + |x|)(h, g)\| + \|(\nabla h, \nabla g)\|_{k-2,k-1}\}, \tag{2.22}$$

where the constant C depends only on μ, μ' and γ.

Proof. By aid of the Friedrichs mollifier, we may assume that $(\sigma, v) \in \mathscr{H}^{\infty,\infty}$. The same argument as in the proof of Lemma 2.1 (i) implies that

$$\|\nabla v\|_1^2 + \|\nabla\sigma\|^2 \le C\big[\|h\|^2 + \textstyle\sum_{\nu=0}^{1}\{|(\nabla^\nu h, \nabla^\nu v)| + |(\nabla^\nu g, \nabla^\nu \sigma)|\}\big].$$

For the right hand side, using the Hardy inequality, we have

$$\begin{aligned}\textstyle\sum_{\nu=0}^{1}\{|(\nabla^\nu h, \nabla^\nu v)| + |(\nabla^\nu g, \nabla^\nu\sigma)|\} &\le \frac{1}{2C}\{\|\nabla v\|_1^2 + \|\nabla\sigma\|^2\}\\ &+C'\{\|(1 + |x|)h\|^2 + \|(1 + |x|)g\|^2 + \|\nabla g\|^2\}.\end{aligned}$$

So we obtain

$$\|(\nabla\sigma, \nabla v)\|_{0,1} \le C\{\|(1 + |x|)(h, g)\| + \|\nabla g\|\}, \tag{2.23}$$

where the constant $C > 0$ depends only on μ, μ' and γ. Moreover, for any multi-index α with $1 \le |\alpha| \le k - 1$, applying ∂_x^α to (2.6)–(2.7) and employing Lemma 2.1 (ii) for the resultant equations, we have

$$\|(\nabla^{|\alpha|+1}\sigma, \nabla^{|\alpha|+2}v)\| \le C\{\|\nabla^{|\alpha|}v\| + \|\nabla\sigma\|_{|\alpha|-1} + \|(\nabla^{|\alpha|}h, \nabla^{|\alpha|+1}g)\|\}, \tag{2.24}$$

if $\delta > 0$ is small enough. Combining (2.23) and (2.24), we obtain (2.22). ∎

From Proposition 2.1, Corollary 2.1 and Lemma 2.2, it follows that for each $0 < \epsilon \le 1$, (2.6)–(2.7) admits a solution $(\sigma^\epsilon, v^\epsilon) \in \mathscr{H}^{k+1,k+1}$ such that $\|(\sigma^\epsilon, v^\epsilon)\|_{L_6} + \|(\sigma^\epsilon, v^\epsilon)/|x|\| +$

$\|(\nabla\sigma^\epsilon, \nabla v^\epsilon)\|_k \le CK$, where $K = \|(1+|x|)(h,g)\| + \|(\nabla h, \nabla g)\|_{k-2,k-1}$. Choosing an appropriate subsequence, there exists $(\sigma, v) \in L_6$, $(\theta, w) \in L_2$, $(\theta^i, w^i) \in \mathscr{H}^{k-1,k}$ such that

$$(\sigma^\epsilon, v^\epsilon) \rightharpoonup (\sigma, v) \text{ weakly in } L_6, \quad \frac{(\sigma^\epsilon, v^\epsilon)}{|x|} \rightharpoonup (\theta, w) \text{ weakly in } L_2,$$
$$\Big(\frac{\partial\sigma^\epsilon}{\partial x_i}, \frac{\partial v^\epsilon}{\partial x_i}\Big) \rightharpoonup (\theta^i, w^i) \text{ weakly in } \mathscr{H}^{k-1,k}$$

as $\epsilon \downarrow 0$. Thus, we have

Proposition 2.2 *There exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in (2.4) satisfies $\delta \le \delta_0$ then for $0 < \lambda \le 1$, (2.2)–(2.3) admits a solution $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ which satisfies the estimate:*

$$\|(\sigma, v)\|_{L_6} + \Big\|\frac{(\sigma, v)}{|x|}\Big\| + \|(\nabla\sigma, \nabla v)\|_{k-1,k} \le C\big\{\|(1+|x|)(h,g)\| + \|(\nabla h, \nabla g)\|_{k-2,k-1}\big\}, \quad (2.25)$$

where the constant $C > 0$ depends only on μ, μ' and γ.

Weighted-L_2 estimate for solution to the linearized equation (2.2)–(2.3). At last, we shall give weighted-L_2 estimate for the solution to (2.2)–(2.3).

Lemma 2.3 *Let $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ be solution to (2.2)–(2.3) which satisfies (2.27). Then, there exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in (2.4) satisfies $\delta \le \delta_0$ then for any integer with $1 \le \ell \le k$, we have the estimate:*

$$\begin{aligned}\textstyle\sum_{\nu=1}^{\ell}\|(1+|x|)^\nu(\nabla^\nu\sigma, \nabla^{\nu+1}v)\| \le C\big[&\|b\|_{J^{k+1}}\|c\|_{J^{k+1}} \\ &+ \|\nabla v\| + \textstyle\sum_{\nu=1}^{\ell}\|(1+|x|)^\nu(\nabla^{\nu-1}f, \nabla^\nu g)\|\big],\end{aligned} \quad (2.26)$$

where C is a constant depending only on μ, μ' and γ.

Proof. Let $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ be a solution to (2.2)–(2.3) satisfying (2.25). We shall prove the lemma by induction on ℓ. Let ℓ be any integer with $1 \le \ell \le k$ and if $\ell \ge 2$, we assume that

$$\begin{aligned}\textstyle\sum_{\nu=1}^{\ell-1}\|(1+|x|)^\nu(\nabla^\nu\sigma, \nabla^{\nu+1}v)\| \le C\big[&\|b\|_{J^{k+1}}\|c\|_{J^{k+1}} \\ &+ \|\nabla v\| + \textstyle\sum_{\nu=1}^{\ell-1}\|(1+|x|)^\nu(\nabla^{\nu-1}f, \nabla^\nu g)\|\big],\end{aligned} \quad (2.27)$$

Using the Fiedrichs mollifier and a cut-off function, we may assume that $(\sigma, v) \in C_0^\infty(\mathbb{R}^3)$. We apply ∂_x^α $(1 \le |\alpha| \le 4)$ to (2.2) and (2.3); multiply the resultant equation by $(1+|x|)^{2|\alpha|}\partial_x^\alpha\sigma$ and $(1+|x|)^{2|\alpha|}\partial_x^\alpha v$ respectively. Summing up the resultant equations and canceling the term of $\big(\nabla\partial_x^\alpha\sigma, (1+|x|)^{2\ell}\partial_x^\alpha v\big)$, we obtain

$$\begin{aligned}\|(1+|x|)^\ell\nabla^{\ell+1}v\|^2 \le C\big[&\big(|\nabla^{\ell+1}v|, (1+|x|)^{2\ell-1}|\nabla^\ell v|\big) \\ &+ \big(|\nabla^\ell v|, (1+|x|)^{2\ell-1}|\nabla^\ell\sigma|\big) + \big|\big(\nabla^\ell(a\cdot\nabla\sigma), (1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big| \\ &+ \big|\big(\nabla^\ell f, (1+|x|)^{2\ell}\nabla^\ell v\big)\big| + \big|\big(\nabla^\ell g, (1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big| \\ &+ \big|\big(\nabla^\ell\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell v\big)\big|\big],\end{aligned} \quad (2.28)$$

where the constant C depends only on μ, μ' and γ. Since

$$\begin{aligned}\|(1+|x|)^\ell\nabla^\ell\sigma\|^2 \le C_{\gamma,\mu,\mu'}\big[&\|(1+|x|)^\ell\nabla^{\ell+1}v\|^2 \\ &+ \|(1+|x|)^\ell\nabla^{\ell-1}f\|^2 + \big|\big(\nabla^{\ell-1}\{(b\cdot\nabla)c\}, (1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big|\big].\end{aligned}$$

as follows from (2.3), combining this with (2.28), we have

$$\begin{aligned}
&\|(1+|x|)^\ell(\nabla^{\ell+1}v,\nabla^\ell\sigma)\|^2 \le C_1\big|\big(\nabla^\ell(a\cdot\nabla\sigma),(1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big| \\
&\quad + C_2\big[\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|(1+|x|)^\ell\nabla^{\ell-1}f\|^2 \\
&\qquad\qquad + \big|\big(\nabla^\ell f,(1+|x|)^{2\ell}\nabla^\ell v\big)\big| + \big|\big(\nabla^\ell g,(1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big|\big] \\
&\quad + C_3\big|\big(\nabla^\ell\{(b\cdot\nabla)c\},(1+|x|)^{2\ell}\nabla^\ell v\big)\big| \\
&\quad + C_4\big|\big(\nabla^{\ell-1}\{(b\cdot\nabla)c\},(1+|x|)^{2\ell}\nabla^\ell\sigma\big)\big| \equiv I_1+I_2+I_3+I_4,
\end{aligned} \tag{2.29}$$

where the constants C_j $(j=1,2,3)$ depend only on μ,μ' and γ.

Now, we estimate the right hand side of (2.29) respectively. Integration by parts and the Sobolev inequality imply that

$$\begin{aligned}
&I_1 \le C\epsilon\textstyle\sum_{\nu=1}^{\ell}\|(1+|x|)^\nu\nabla^\nu\sigma\|^2 \quad \text{in the same way as (2.15)}, \\
&I_2 \le \frac{1}{5}\|(1+|x|)^\ell(\nabla^\ell\sigma,\nabla^{\ell+1}v)\|^2 \\
&\qquad + C\{\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|(1+|x|)^\ell(\nabla^{\ell-1}f,\nabla^\ell g)\|^2\}.
\end{aligned} \tag{2.30}$$

Moreover, noting that for multi-index α, β with $|\alpha|$, $|\beta|\le k+1$

$$\big\|(1+|x|)^{|\alpha|+|\beta|+1}|\partial^\alpha b||\partial^\beta c|\big\| \le C\|b\|_{J^{k+1}}\|c\|_{J^{k+1}} \quad \text{if } |\alpha|\le 1 \text{ or } |\beta|\le 1, \tag{2.31}$$

we can show that

$$\begin{aligned}
&I_3+I_4 \le \frac{1}{5}\|(1+|x|)^\ell(\nabla^\ell\sigma,\nabla^{\ell+1}v)\|^2 \\
&\quad + C\begin{cases} \|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}} & \ell=1,2,3, \\ \|(1+|x|)^3(\nabla^3\sigma,\nabla^4 v)\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}} & \ell=4. \end{cases}
\end{aligned} \tag{2.32}$$

Indeed, I_3 is estimated as follows: If $\ell=1$ or 2, since $(1+|x|)^{\ell+1}\nabla^\ell\{(b\cdot\nabla)c\}\in L_2$ as follows from (2.31), we have

$$I_3 \le C\{\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\}.$$

If $\ell=3$ or $\ell=4$, reforming I_3 into the following two parts:

$$\begin{aligned}
I_3 &= C_3\sum_{|\alpha|=\ell}\Big(\sum_{\substack{\beta\le\alpha\\|\beta|=\ell-2}}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^\beta c + \sum_{\substack{\beta\le\alpha\\|\beta|=1}}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^\beta c,\ (1+|x|)^{2\ell}\partial_x^\alpha v\Big) \\
&+ C_3\sum_{|\alpha|=\ell}\Big(\Big\{\sum_{\substack{\beta\le\alpha\\|\beta|=0}} + \sum_{\substack{\beta\le\alpha\\|\beta|=\ell-1}} + \sum_{\substack{\beta\le\alpha\\|\beta|=\ell}}\Big\}\binom{\alpha}{\beta}(\partial_x^{\alpha-\beta}b\cdot\nabla)\partial_x^\beta c,\ (1+|x|)^{2\ell}\partial_x^\alpha v\Big) \equiv I_{31}+I_{32},
\end{aligned} \tag{2.33}$$

Using integration by parts for I_{31}, we have

$$\begin{aligned}
I_{31} &\le C\|(1+|x|)^2\nabla b\|_{L_\infty}\big[\|(1+|x|)^{\ell-2}\nabla^{\ell-1}c\|\|(1+|x|)^\ell\nabla^{\ell+1}v\| \\
&\qquad\qquad + \textstyle\sum_{\nu=\ell-2}^{\ell-1}\|(1+|x|)^\nu\nabla^{\nu+1}c\|\|(1+|x|)^{\ell-1}\nabla^\ell v\|\big] \\
&\quad + (\text{the same term except for the exchange of } b \text{ and } c) \\
&\le \frac{1}{5}\|(1+|x|)^\ell\nabla^{\ell+1}v\|^2 + C\{\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\},
\end{aligned}$$

and for I_{32} we can use (2.31) directly as in the case $\ell = 1$ or 2,

$$I_{32} \le C\{\|(1+|x|)^{\ell-1}\nabla^\ell v\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\},$$

where the constant C depends only on μ, μ' and γ. Further, as for I_4: If $\ell = 1, 2$ or 3, since $(1+|x|)^\ell \nabla^{\ell-1}\{(b\cdot\nabla)c\} \in L_2$ as follows from (2.31), we have

$$I_4 \le \frac{1}{5}\|(1+|x|)^\ell \nabla^\ell \sigma\|^2 + C\|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}.$$

If $\ell = 4$, integration by parts implies that

$$I_4 \le C_4 \sum_{|\alpha|=3}\Big[\big|(\nabla\cdot\partial_x^\alpha\{(b\cdot\nabla)c\}, (1+|x|)^8\partial_x^\alpha\sigma)\big| + \Big|\Big(\partial_x^\alpha\{(b\cdot\nabla)c\}, 8(1+|x|)^7\frac{x}{|x|}\partial_x^\alpha\sigma\Big)\Big|\Big].$$

Then, decomposing each term as in (2.33) (the first term same as I_3 with $\ell = 4$ and the second term same as I_3 with $\ell = 3$) and using integration by parts, we have

$$I_4 \le \frac{1}{5}\|(1+|x|)^4\nabla^4\sigma\|^2 + C\{\|(1+|x|)^3\nabla^3\sigma\|^2 + \|b\|^2_{J^{k+1}}\|c\|^2_{J^{k+1}}\},$$

where the constant C depends only on μ, μ' and γ.

Combining (2.29), (2.30), (2.32) and (2.27) if $\ell \ge 2$, we obtain (2.26). This completes the proof of Lemma 2.3. ∎

Now combining Proposition 2.2 and Lemma 2.3, we have the following theorem.

Theorem 2.1 *There exists $\delta_0 = \delta_0(\gamma, \mu, \mu') > 0$ such that if δ in (2.4) satisfies $\delta \le \delta_0$, then (2.2)–(2.3) admits a solution $(\sigma, v) \in \hat{\mathscr{H}}^{k,k+1}$ which satisfies the estimate:*

$$\begin{aligned}\|(\sigma,v)\|_{L_6} &+ \Big\|\frac{(\sigma,v)}{|x|}\Big\| + \textstyle\sum_{\nu=1}^k\|(1+|x|)^\nu\nabla^\nu\sigma\| + \sum_{\nu=1}^{k+1}\|(1+|x|)^{\nu-1}\nabla^\nu v\| \\ &\le C\big[\|b\|^2_{J^{k+1}} + \textstyle\sum_{\nu=0}^{k-1}\|(1+|x|)^{\nu+1}\nabla^\nu f\| + \|(1+|x|)g\| + \sum_{\nu=1}^k\|(1+|x|)^\nu\nabla^\nu g\|\big],\end{aligned}$$

where the constant $C > 0$ is depending only on μ, μ' and γ. Furthermore the uniqueness is held in $\hat{\mathscr{H}}^{1,2} \cap L_6$.

Proof. The existence and the estimate follows from Proposition 2.2 and Lemma 2.3 directly. The uniqueness follows from an argument similar to Lemma 2.1 (ii). ∎

2.2 A Proof of Theorem 1.1

In this section, we shall construct a solution to (2.1), by use of the contraction mapping principle in $\mathscr{I}_\epsilon^{4,5}$. We employ the following system of equations:

$$\begin{cases} \nabla\cdot v + \Big(\dfrac{\tilde v}{\bar\rho + \tilde\sigma}\cdot\nabla\Big)\sigma = \dfrac{G}{\bar\rho + \tilde\sigma}, & (2.34) \\ -\mu\Delta v - (\mu+\mu')\nabla(\nabla\cdot v) + \gamma\nabla\sigma = -(\bar\rho+\tilde\sigma)(\tilde v\cdot\nabla)\tilde v \\ \qquad -\big[P'(\bar\rho+\tilde\sigma) - P'(\bar\rho)\big]\nabla\tilde\sigma + (\bar\rho+\tilde\sigma)F, & (2.35)\end{cases}$$

where $(\tilde\sigma, \tilde v)(x) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ is given.

Introduction of the solution map T for (2.34)–(2.35). First and foremost, we put

$$a = \tilde{v}/(\bar{\rho}+\tilde{\sigma}), \quad b = c = \bar{\rho}^{\frac{1}{2}}\tilde{v}, \quad g = G/(\bar{\rho}+\tilde{\sigma}),$$
$$f = -\tilde{\sigma}(\tilde{v}\cdot\nabla)\tilde{v} - \left[P'(\bar{\rho}+\tilde{\sigma}) - P'(\bar{\rho})\right]\nabla\tilde{\sigma} + (\bar{\rho}+\tilde{\sigma})F. \tag{2.36}$$

If we assume that

$$K_0 \equiv \|(1+|x|)G\| + \textstyle\sum_{\nu=0}^{3}\|(1+|x|)^{\nu+1}\nabla^{\nu}F\| + \sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^{\nu}G\| < \infty, \tag{2.37}$$

then we can check (2.4)–(2.5) easily and additionally we have

$$\|(1+|x|)g\| + \textstyle\sum_{\nu=0}^{3}\|(1+|x|)^{\nu+1}\nabla^{\nu}f\| + \sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^{\nu}g\| \le C\{\epsilon^2 + K_0\} \tag{2.38}$$

for some constant $C = C(\bar{\rho},\mu,\mu')$. Applying Theorem 2.1 with $k=4$ for (2.34)–(2.35), we have the following lemma.

Lemma 2.4 *Let $(F,G) \in \mathscr{H}^{3,4}$ satisfies* (2.37). *Then, there exists ϵ_0 such that if $\epsilon \le \epsilon_0$ then* (2.34)–(2.35) *with $(\tilde{\sigma},\tilde{v}) \in \mathscr{I}_\epsilon^{4,5}$ has a solution $(\sigma, v) \in \hat{\mathscr{H}}^{4,5}$ which satisfies the estimate:*

$$\begin{aligned}\|(\sigma,v)\|_{L_6} + \left\|\frac{(\sigma,v)}{|x|}\right\| &+ \textstyle\sum_{\nu=1}^{5}\|(1+|x|)^{\nu-1}\nabla^{\nu}v\| \\ &+ \textstyle\sum_{\nu=1}^{4}\|(1+|x|)^{\nu}\nabla^{\nu}\sigma\| \le C\{\epsilon^2 + K_0\},\end{aligned} \tag{2.39}$$

where the constant $C>0$ depends only on μ,μ' and $\bar{\rho}$.

Hence, we can consider the solution map $T : (\tilde{\sigma},\tilde{v}) \longmapsto (\sigma, v)\ ;\ \mathscr{I}_\epsilon^{4,5} \longrightarrow \hat{\mathscr{H}}^{4,5}$ for (2.34)–(2.35).

Next, we have to show that $(\tilde{\sigma},\tilde{v}) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ leads to $(\sigma,v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$. The following lemma plays an important role when we estimate the solution by L_∞-norm.

Lemma 2.5 *Let $E(x)$ be a scalar function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+1}} \quad (|\alpha| = 0,1,2).$$

(i) *If $\phi(x)$ is a smooth scalar function of the form: $\phi = \nabla\cdot\phi_1 + \phi_2$ satisfying*

$$L_1(\phi) \equiv \|(1+|x|)^3\phi\|_{L_\infty} + \|(1+|x|)^2\phi_1\|_{L_\infty} + \|\phi_2\|_{L_1} < \infty,$$

then we have for any multi-index α with $|\alpha| = 0,1$

$$|\partial_x^\alpha(E*\phi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|+1}} L_1(\phi).$$

(ii) *If $\phi(x)$ is a smooth scalar function of the form: $\phi = \phi_1\phi_2$ satisfying*

$$L_2(\phi) \equiv \|(1+|x|)^2\phi\|_{L_\infty} + \|(1+|x|)^3(\nabla\phi_1)\phi_2\|_{L_\infty} + \|(1+|x|)^3\phi_1(\nabla\phi_2)\|_1 < \infty,$$

then we have for any multi-index α with $|\alpha| = 1,2$

$$|\partial_x^\alpha(E*\phi)(x)| \le \frac{C_\alpha}{|x|^{|\alpha|}} L_2(\phi).$$

Here, C_α denotes a constant depending only on α.

Now, with aid of the Helmholtz decomposition and the Fourier transform, we shall estimate L_∞-norm of the solution to (2.34)–(2.35).

Lemma 2.6 *Let (F, G) satisfy following estimate (for K_0 defined by (2.37)):*

$$K \equiv K_0 + \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1} + \textstyle\sum_{\nu=0}^{1} \|(1+|x|)^{\nu+2}\nabla^\nu G\|_{L_\infty} < \infty.$$

Then, if $(\sigma, v) \in \hat{\mathscr{H}}^{4,5}$ is a solution to (2.34)–(2.35) with $(\tilde\sigma, \tilde v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ and satisfies (2.39) then (σ, v) satisfies the estimate:

$$\|(1+|x|)^2\sigma\|_{L_\infty} + \textstyle\sum_{\nu=0}^{1} \|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty} \le C\{\epsilon^2 + K\}, \tag{2.40}$$

where the constant $C > 0$ depends only on μ, μ' and $\bar\rho$.

Proof. In view of the Helmholtz decomposition, v is written of the form: $v = w + \nabla p$ ($w \in \dot{L}_6$, $\nabla p \in M_6$). Here and hereafter

$$M_6 = \{\nabla p \mid p \in L_{6,loc},\ \nabla p \in L_6\},\quad \dot{L}_6 = \overline{\{w \in C_0^\infty \mid \nabla\cdot w = 0\}}^{L_6},$$

where $\overline{\cdot}^{L_6}$ means the compepletion of $\cdot$ with respect to the L_6-norm. Substituting this formula into (2.34)–(2.35), we have

$$\begin{cases} \Delta p + \Big(\dfrac{\tilde v}{\bar\rho + \tilde\sigma}\cdot\nabla\Big)\sigma = \dfrac{G}{\bar\rho+\tilde\sigma}, & (2.41)\\ -\mu\Delta w + \nabla\Phi = -\bar\rho(\tilde v\cdot\nabla)\tilde v + f \equiv h, & (2.42)\\ \Phi = \gamma\sigma - (2\mu+\mu')\Delta p. & (2.43)\end{cases}$$

Thus we have the representation for Φ and w:

$$\Phi = \sum_{i=1}^{3} \frac{\partial E_0}{\partial x_i} * h_i,\quad w_j(x) = \sum_{i=1}^{3} E_{ij} * h_i(x), \tag{2.44}$$

where $E_0(x) = -(4\pi)^{-1}/|x|$ and $E_{ij}(x) = (8\pi\mu)^{-1}\big(\delta_{ij}/|x| - x_i x_j/|x|^3\big)$.

We shall apply Lemma 2.5 (i) to estimate Φ and w. Therefore, in order to estimate (2.44) we need to take a look at h. By $(\tilde\sigma, \tilde v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$, there exist $\tilde V_1 = (\tilde V_{1,i})_{1\le i\le 3}$ and $\tilde V_2$ such that

$$\nabla\cdot\tilde v = \nabla\cdot\tilde V_1 + \tilde V_2,\quad \|(1+|x|)^3\tilde V_1\|_{L_\infty} + \|(1+|x|)^{-1}\tilde V_2\|_{L_1} \le \epsilon \tag{2.45}$$

and so we can calculate

$$\begin{aligned} h_i &= \Big[\bar\rho\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\{-\tilde v_i\tilde v_j + \tilde v_i\tilde V_{1,j}\} + \nabla\cdot\{(\bar\rho+\tilde\sigma)F_{1,i\cdot}\}\Big]\\ &\quad + \Big\{-\bar\rho(\tilde V_1\cdot\nabla)\tilde v_i + \bar\rho\tilde V_2\tilde v_i - \tilde\sigma(\tilde v\cdot\nabla)\tilde v_i - Q(\sigma)\sigma\frac{\partial\tilde\sigma}{\partial x_i} - \nabla\sigma\cdot F_{1,i\cdot} + (\bar\rho+\tilde\sigma)F_{2,i}\Big\}\\ &\equiv \nabla\cdot h_1^i + h_2^i, \end{aligned}$$

where $Q(\sigma) = \int_0^1 P''(\bar\rho + \theta\sigma)\,d\theta$. By $(\tilde\sigma, \tilde v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ and (2.45), using the Sobolev inequality, we have

$$\|(1+|x|)^3 h_i\|_{L_\infty} + \|(1+|x|)^2 h_1^i\|_{L_\infty} + \|h_2^i\|_{L_1} \le C\{\epsilon^2 + K_1\},$$

where K_1 is defined by $K_1 \equiv \|(1+|x|)^3 F\|_{L_\infty} + \|(1+|x|)^2 F_1\|_{L_\infty} + \|F_2\|_{L_1}$ and $C > 0$ is a constant depending only on $\bar{\rho}$. Thus, applying Lemma 2.5 (i) to (2.44), we have

$$|x|^2|\Phi(x)| + \textstyle\sum_{\nu=0}^{1}|x|^{\nu+1}|\nabla^\nu w(x)| \le CK_1. \tag{2.46}$$

As for p, we have from (2.41)

$$p = E_0 * \Big(-\sum_{i=1}^{3}\frac{\tilde{v}_i}{\bar{\rho}+\tilde{\sigma}}\frac{\partial\sigma}{\partial x_i} + \frac{G}{\bar{\rho}+\tilde{\sigma}}\Big) \equiv -E_0 * \sum_{i=1}^{3} q_1^i q_2^i + E_0 * r. \tag{2.47}$$

Since $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\epsilon^{4,5}$, it follows from (2.39) and the Sobolev inequality that

$$\|(1+|x|)^2 q_1^i q_2^i\|_{L_\infty} + \|(1+|x|)^3(\nabla q_1^i) q_2^i\|_{L_\infty} + \|(1+|x|)^3 q_1^i(\nabla q_2^i)\|_1 \le C\{\epsilon^2 + K_0\},$$
$$\textstyle\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+2}\nabla^\nu r\|_{L_\infty} \le C\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+2}\nabla^\nu G\|_{L_\infty} \equiv K_2,$$

where the constant $C > 0$ depends only on $\bar{\rho}$. Applying Lemma 2.5 (ii) to each term of (2.47) respectively, we also have

$$\textstyle\sum_{\nu=1}^{2}|x|^\nu|\nabla^\nu p(x)| \le C\{\epsilon^2 + K_0 + K_2\}. \tag{2.48}$$

Now, we are ready to estimate v and σ. First, we consider the case where $|x| \ge 1$. Returning to $v = w + \nabla p$ and combining (2.46) and (2.48), we obtain

$$\textstyle\sum_{\nu=0}^{1}(1+|x|)^{\nu+1}|\nabla^\nu v(x)| \le C\{\epsilon^2 + K_0 + K_1 + K_2\}. \tag{2.49}$$

Besides by (2.43) we have $\sigma = \gamma^{-1}\{(2\mu+\mu')\Delta p + \Phi\}$. Combining (2.46) and (2.48), we get

$$(1+|x|)^2|\sigma(x)| \le C\{\epsilon^2 + K_0 + K_1 + K_2\}. \tag{2.50}$$

Next, we consider the case where $|x| < 1$. The Sobolev inequality and the Hardy inequality imply that

$$(1+|x|)^2|\sigma(x)| + \textstyle\sum_{\nu=0}^{1}(1+|x|)^{\nu+1}|\nabla^\nu v(x)| \le C\|(\nabla\sigma, \nabla v)\|_{1,2} \le C\{\epsilon^2 + K_0\}. \tag{2.51}$$

Consequently by (2.49), (2.50) and (2.51), we have

$$\|(1+|x|)^2\nabla^2\sigma\|_{L_\infty} + \textstyle\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+1}\nabla^\nu v\|_{L_\infty} \le C\big[\epsilon^2 + \sum_{j=0}^{2}K_j\big] \le C\{\epsilon^2 + K\}.$$

This completes the proof of Lemma 2.6. ∎

We combine Lemmas 2.4 and 2.6 to prove that the solution $(\sigma, v) \in \dot{\mathscr{I}}_\epsilon^{4,5}$.

Proposition 2.3 *There exist $c_0 > 0$ and $\epsilon > 0$ such that if $(F, G) \in \mathscr{H}^{3,4}$ satisfies*

$$K + \|(1+|x|)^{-1}G\|_{L_1} \le c_0\epsilon \quad (K \text{ is defined in Lemma 2.6}), \tag{2.52}$$

then (2.34)–(2.35) with $(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ admits a solution $(\sigma, v) = T(\tilde{\sigma}, \tilde{v}) \in \dot{\mathscr{I}}_\epsilon^{4,5}$.

Proof. By Lemmas 2.4, 2.6 and (2.52), it follows that (2.34)–(2.35) has a solution $(\sigma, v) \in \hat{\mathscr{H}}^{4,5}$, which satisfies

$$\|(\sigma, v)\|_{\mathscr{I}^{4,5}} \le C\{\epsilon^2 + K\} \le C\{\epsilon^2 + c_0\epsilon\},$$

where the constant $C > 0$ depends only on μ, μ' and $\bar{\rho}$. Thus if we take $c_0 \leq 1/2C$ and $\epsilon > 0$ sufficiently small, it follows that $(\sigma, v) \in \mathscr{I}_\epsilon^{4,5}$. At last, we define V_1 and V_2 by

$$V_1 = -\frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}\sigma, \quad V_2 = \Big(\nabla\cdot\frac{\tilde{v}}{\bar{\rho}+\tilde{\sigma}}\Big) + \frac{G}{\bar{\rho}+\tilde{\sigma}}.$$

Then immediately from (2.34)

$$\nabla\cdot v = \nabla\cdot V_1 + V_2.$$

Moreover, by $(\tilde{\sigma}, \tilde{v}) \in \mathscr{I}_\epsilon^{4,5}$ and (2.40), using Sovolev inequality, we have

$$\|(1+|x|)^3 V_1\|_{L_\infty} + \|(1+|x|)^{-1}V_2\|_{L_1} \leq C\{\epsilon^2 + K + \|(1+|x|)^{-1}G\|_{L_1}\},$$

further by (2.52)

$$\leq C\{\epsilon^2 + c_0\epsilon\} \leq C\epsilon^2 + \epsilon/2 \leq \epsilon,$$

if $c_0 \leq 1/2C$ and $\epsilon > 0$ is sufficiently small. This completes the proof of Proposition 2.3. ∎

Contraction of the solution map T. Finally, we shall show that the solution map T for (2.34)–(2.35) is contract. We suppose that $(\tilde{\sigma}^j, \tilde{v}^j) \in \mathscr{I}_\epsilon^{4,5}$ and $(\sigma^j, v^j) = T(\tilde{\sigma}^j, \tilde{v}^j)$ for $j = 1, 2$. Then it immediately follows from (2.34)–(2.35) that

$$\begin{cases} \nabla\cdot(v^1 - v^2) - \Big(\frac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1}\cdot\nabla\Big)(\sigma^1 - \sigma^2) = g, \\ -\mu\Delta(v^1 - v^2) - (\mu+\mu')\nabla\{\nabla\cdot(v^1 - v^2)\} + \gamma\nabla(\sigma^1 - \sigma^2) \\ \qquad = -\bar{\rho}(\tilde{v}^1\cdot\nabla)\tilde{v}^1 + \bar{\rho}(\tilde{v}^2\cdot\nabla)\tilde{v}^2 + f, \end{cases} \tag{2.53}$$

where $(f, g) \in \mathscr{H}^{3,3}$ is

$$\begin{aligned} g &= -\Big(\frac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1} - \frac{\tilde{v}^2}{\bar{\rho}+\tilde{\sigma}^2}\Big)\cdot\nabla\sigma^2 + \Big(\frac{G}{\bar{\rho}+\tilde{\sigma}^1} - \frac{G}{\bar{\rho}+\tilde{\sigma}^2}\Big), \\ f &= -\tilde{\sigma}^1(\tilde{v}^1\cdot\nabla)\tilde{v}^1 + \tilde{\sigma}^2(\tilde{v}^2\cdot\nabla)\tilde{v}^2 - \big[P'(\bar{\rho}+\tilde{\sigma}^1) - P'(\bar{\rho})\big]\nabla\tilde{\sigma}^1 \\ &\quad - \big[P'(\bar{\rho}+\tilde{\sigma}^2) - P'(\bar{\rho})\big]\nabla\tilde{\sigma}^2 + (\tilde{\sigma}^1 - \tilde{\sigma}^2)F. \end{aligned}$$

Since

$$\begin{aligned} &\textstyle\sum_{\nu=0}^2\|(1+|x|)^{\nu+1}\nabla^\nu f\| + \|(1+|x|)g\| + \sum_{\nu=1}^3\|(1+|x|)^\nu\nabla^\nu g\| \\ &\qquad \leq C\{\epsilon + K_0\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3,4}}. \end{aligned}$$

as follows from the Sobolev inequality for K_0 defined in (2.37), by application of Theorem 2.1 with $k = 3$ to (2.53), we obtain

$$\begin{aligned} &\|(\sigma^1 - \sigma^2, v^1 - v^2)\|_{L_6} + \Big\|\frac{(\sigma^1 - \sigma^2, v^1 - v^2)}{|x|}\Big\| \\ &\quad + \textstyle\sum_{\nu=1}^3\|(1+|x|)^\nu\nabla^\nu(\sigma^1 - \sigma^2)\| + \sum_{\nu=1}^4\|(1+|x|)^{\nu-1}\nabla^\nu(v^1 - v^2)\| \\ &\leq C\{\epsilon + K_0\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3,4}}. \end{aligned} \tag{2.54}$$

Next, we decompose (2.53) as in the proof of Lemma 2.6: Putting $v^1 - v^2 = w + \nabla p$ ($w \in \dot{L}_6$, $\nabla p \in M_6$), we have

$$\begin{cases} \Delta p + \Big(\dfrac{\tilde{v}^1}{\bar{\rho}+\tilde{\sigma}^1}\cdot\nabla\Big)(\sigma^1 - \sigma^2) = g, \\ -\mu\Delta w + \nabla\Phi = -\bar{\rho}(\tilde{v}^1\cdot\nabla)\tilde{v}^1 + \bar{\rho}(\tilde{v}^2\cdot\nabla)\tilde{v}^2 + f \equiv h, \\ \qquad \Phi = \gamma(\sigma^1 - \sigma^2) - (2\mu + \mu')\Delta p. \end{cases}$$

The same argument as in the proof of Lemma 2.6 implies that

$$\begin{aligned} &\|(1+|x|)^2(\sigma^1 - \sigma^2)\|_{L_\infty} + \textstyle\sum_{\nu=0}^{1}\|(1+|x|)^{\nu+1}\nabla^\nu(v^1 - v^2)\|_{L_\infty} \\ &\quad \le C\{\epsilon + K\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3.4}} \\ &\quad\quad + C\epsilon\big[\|(1+|x|)^3(\tilde{V}_1^1 - \tilde{V}_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(\tilde{V}_2^1 - \tilde{V}_2^2)\|_{L_1}\big], \end{aligned} \tag{2.55}$$

where $\tilde{V}_1^j$, $\tilde{V}_2^j$ ($j = 1,2$) are functions satisfying

$$\nabla\cdot\tilde{v}^j = \nabla\cdot\tilde{V}_1^j + \tilde{V}_2^j, \quad \|(1+|x|)^3\tilde{V}_1^j\|_{L_\infty} + \|(1+|x|)^{-1}\tilde{V}_2^j\|_{L_1} \le \epsilon. \tag{2.56}$$

Moreover, if we define V_1^j, V_2^j ($j = 1,2$) as

$$V_1^j = -\frac{\tilde{v}^j}{\bar{\rho}+\tilde{\sigma}^j}\sigma^j, \quad V_2^j = \Big(\nabla\cdot\frac{\tilde{v}^j}{\bar{\rho}+\tilde{\sigma}^j}\Big) + \frac{G}{\bar{\rho}+\tilde{\sigma}^j}, \tag{2.57}$$

then

$$\begin{aligned} &\|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\quad \le C\{\epsilon + \|(1+|x|)G\|_{L_\infty}\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3.4}}. \end{aligned} \tag{2.58}$$

Combining (2.54), (2.55) and (2.58), we obtain

$$\begin{aligned} &\|(\sigma^1 - \sigma^2, v^1 - v^2)\|_{\mathscr{I}^{3,4}} + \|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\quad \le C\{\epsilon + K\}\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3.4}} \\ &\quad\quad + C\epsilon\big[\|(1+|x|)^3(\tilde{V}_1^1 - \tilde{V}_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(\tilde{V}_2^1 - \tilde{V}_2^2)\|_{L_1}\big]. \end{aligned}$$

Therefore, we have the following proposition.

Proposition 2.4 *There exist $c_0 > 0$ and $\epsilon > 0$ such that if $(F, G) \in \mathscr{H}^{3,4}$ satisfies $K \le c_0\epsilon$ (K is defined in Lemma 2.6), then for $(\tilde{\sigma}^j, \tilde{v}^j) \in \dot{\mathscr{I}}_\epsilon^{4,5}$ and $(\sigma^j, v^j) = T(\tilde{\sigma}^j, \tilde{v}^j)$ ($j = 1,2$)*

$$\begin{aligned} &\|(\sigma^1 - \sigma^2, v^1 - v^2)\|_{\mathscr{I}^{3,4}} + \|(1+|x|)^3(V_1^1 - V_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(V_2^1 - V_2^2)\|_{L_1} \\ &\le \frac{1}{2}\big[\|(\tilde{\sigma}^1 - \tilde{\sigma}^2, \tilde{v}^1 - \tilde{v}^2)\|_{\mathscr{I}^{3.4}} + \|(1+|x|)^3(\tilde{V}_1^1 - \tilde{V}_1^2)\|_{L_\infty} + \|(1+|x|)^{-1}(\tilde{V}_2^1 - \tilde{V}_2^2)\|_{L_1}\big], \end{aligned}$$

where $(\tilde{V}_1^j, \tilde{V}_2^j)$ ($j = 1,2$) are functions satisfying (2.56) *and (V_1^j, V_2^j) ($j = 1,2$) are defined by* (2.57).

Hence, by Propositions 2.3 and 2.4, the contraction mapping principle implies the existence and uniqueness of solution to (1.2) which we have stated in Theorem 1.1.

3 Non-stationary Problem

In this section, we consider stability of the stationary solution with respect to the initial disturbance (ρ_0, v_0). Let $\bar{\rho}$ be a positive constant and let (F, G) be small in the sense of Theorem 1.1. We denote the corresponding stationary solution obtained in Theorem 1.1 by (ρ^*, v^*). Putting

$$\rho(t,x) = \rho^*(x) + \sigma(t,x), \quad v(t,x) = v^*(x) + w(t,x)$$

into (1.1), we have the system of equations for (σ, w):

$$\begin{cases} \sigma_t(t) + \nabla\cdot\{(\rho^* + \sigma(t))w(t)\} = -\nabla\cdot(v^*\sigma(t)), & (3.1)\\ w_t(t) - \dfrac{1}{\rho^*}\left[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\right] + A(t)\nabla\sigma(t) = f(t), & (3.2)\\ (\sigma, w)(0,x) = (\rho_0 - \rho^*, v_0 - v^*)(x), & (3.3)\end{cases}$$

where $A(t) = P'(\rho^* + \sigma(t))/(\rho^* + \sigma(t))$ and

$$f(t) = -(v^*\cdot\nabla)w(t) - (w(t)\cdot\nabla)(v^* + w(t)) - \frac{1}{\rho^*}\{P'(\rho^* + \sigma(t)) - P'(\rho^*)\}\nabla\rho^*$$
$$-\frac{\sigma(t)}{\rho^*(\rho^* + \sigma(t))}\left[\mu\Delta(v^* + w(t)) + (\mu+\mu')\nabla\{\nabla\cdot(v^* + w(t))\} - P'(\rho^* + \sigma(t))\nabla\rho^*\right].$$

Our goal in this section is to give a proof of Theorem 1.2. The proof consists of the following two steps. One is local existence:

Proposition 3.1 *If $(\sigma, w)(0) \in \mathscr{H}^{3,3}$, then there exists $t_0 > 0$ such that the initial value problem (3.1)–(3.2) with initial data $(\sigma, w)(0)$ admits a unique solution $(\sigma, w)(t) \in \mathscr{C}(0, t_0; \mathscr{H}^{3,3})$. Moreover, $(\sigma, w)(t)$ satisfies*

$$\|(\sigma, w)(t)\|_{3,3}^2 \le 2\|(\sigma, w)(0)\|_{3,3}^2$$

for any $t \in [0, t_0]$.

And the other is a priori estimate:

Proposition 3.2 *Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2). Then there exists $\epsilon_0 > 0$ such that if $\epsilon \le \epsilon_0$ and $\sup_{0\le t\le t_1} \|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \epsilon$, then*

$$\|(\sigma, w)(t)\|_{3,3}^2 + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2\, ds \le C\|(\sigma, w)(0)\|_{3,3}^2 \qquad (3.4)$$

for any $t \in [0, t_1]$, where $C > 0$ is a constant depending only on μ and μ'.

Concerning the local existence, we can apply the Matsumura-Nishida [4] method directly. So we shall devote the following sections to the proof of Proposition 3.2.

Some estimates for $f(t)$ and its derivatives.

Lemma 3.1 *Let α be a multi-index with $0 \le |\alpha| \le 3$ and let us write $\partial_x^\alpha f(t)$ of the form:*

$$\partial_x^\alpha f(t) = -\frac{\sigma(t)}{\rho^*(\rho^* + \sigma(t))}\left[\mu\Delta\partial_x^\alpha w(t) + (\mu+\mu')\nabla(\nabla\cdot\partial_x^\alpha w(t))\right] + F_\alpha(t).$$

Then, there exists $\epsilon > 0$ *such that if* $\|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \epsilon$ *then* $F_\alpha(t)$ *satisfies*

$$|F_\alpha(t)| \le C \begin{cases} |\nabla v^*||w(t)| + \big(|v^*| + |w(t)|\big)|\nabla w(t)| + \big(|\nabla\rho^*| + |\nabla^2 v^*|\big)|\sigma(t)| \\ \qquad\qquad\qquad\qquad \text{if } \alpha = 0, \\ |\nabla^{|\alpha|+1} v^*||w(t)| + \sum_{\nu=1}^{|\alpha|+1} |\nabla^\nu w(t)| + \sum_{\nu=1}^{|\alpha|+1}\big(|\nabla^\nu \rho^*| + |\nabla^{\nu+1} v^*|\big)|\sigma(t)| \\ \qquad + \sum_{\nu=1}^{|\alpha|} |\nabla^\nu \sigma(t)| + R_{|\alpha|}(t) \quad \text{if } |\alpha| = 1, 2, 3. \end{cases} \tag{3.5}$$

Here, $C > 0$ *is a constant depending only on* μ, μ'; $R_1(t) = 0$ *and* $R_k(t)$ $(k = 2, 3)$ *satisfies the following estimates:*

$$\|R_2(t)\| \le C\epsilon\|\nabla^3 w(t)\|, \quad \|R_k(t)\|_{L_{\frac{3}{2}}} \le C\epsilon\|(\nabla^2\sigma, \nabla^2 w)(t)\|_{k-2,k-2} \ (k = 2, 3). \tag{3.6}$$

Proof. By combination of the Leibniz rule and the Sobolev embedding: $H^2 \hookrightarrow L_\infty$, we can easily check (3.5) with

$$R_k(t) = \begin{cases} 0 \quad \text{if } k = 1, \qquad |\nabla^2 w(t)||\nabla^2\sigma(t)| \quad \text{if } k = 2, \\ |\nabla^2 w(t)||\nabla^3 \sigma(t)| + \big(|\nabla^2 w(t)| + |\nabla^3 w(t)|\big)|\nabla^2\sigma(t)| + \\ \big(|\nabla^3\rho^*| + |\nabla^4 v^*|\big)|\nabla\sigma(t)| + \big(|\nabla^3\rho^*| + |\nabla^2 w(t)|\big)|\nabla^2 w(t)| \end{cases} \quad \text{if } k = 3.$$

Then, using the Gagliard-Nirenberg inequality and the Sobolev inequality, we obtain (3.6). ∎

Estimates for $\nabla w(t)$ *and its derivatives up to* $\nabla^4 w(t)$.

Lemma 3.2 *Let* $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ *be a solution to* (3.1)–(3.2). *Then, there exist* ϵ_0, $\lambda_0 > 0$ *and* $\alpha_k > 0$ *such that if* $\epsilon \le \epsilon_0$ *and* $\|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \bar{\rho}, v^*)\|_{\mathscr{I}^{4,5}} \le \epsilon$ *then*

$$\frac{d}{dt}\big[\|\sigma(t)\|^2 + \big(B(t)w(t), w(t)\big)\big] + \alpha_0\|\nabla w(t)\|^2 \le C\epsilon\|\nabla\sigma(t)\|^2, \tag{3.7}$$

$$\begin{aligned} \frac{d}{dt}\big[\|\nabla^k\sigma(t)\|^2 + \big(B(t)\nabla^k w(t), \nabla^k w(t)\big)\big] + \alpha_k\|\nabla^{k+1} w(t)\|^2 \\ \le C(\epsilon + \lambda)\|(\nabla\sigma, w_t)(t)\|_{k-1,k-1}^2 + C\lambda^{-1}\|\nabla w(t)\|_{k-1}^2 \end{aligned} \tag{3.8}$$

for $1 \le k \le 3$ *and any* λ *with* $0 < \lambda < \lambda_0$, *where* $C > 0$ *is a constant depending only on* μ, μ' *and* $B(t) = (\rho^* + \sigma(t))^2/P'(\rho^* + \sigma(t))$.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_0; \mathscr{H}^{\infty,\infty})$. For any multi-index α with $0 \le |\alpha| \le 3$, applying ∂_x^α to (3.1) and (3.2); multiplying the resultant equation by $\partial_x^\alpha\sigma(t)$ and $(\rho + \sigma(t))A(t)^{-1}\partial_x^\alpha w(t)$ respectively, we have

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\sigma(t)\|^2 - \big((\rho^* + \sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t)\big) = \big(-\partial_x^\alpha(v^*\sigma(t)) + I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\big),$$

$$\begin{aligned} \big(B(t)\partial_x^\alpha w_t(t), \partial_x^\alpha w(t)\big) - \Big(\frac{B(t)}{\rho^*}\partial_x^\alpha\{\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla\cdot w(t))\}, \partial_x^\alpha w(t)\Big) \\ + \big((\rho^* + \sigma(t))\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)\big) = \big(\partial_x^\alpha f(t) + J_\alpha(t), B(t)\partial_x^\alpha w(t)\big) \end{aligned}$$

by integration with respect to x, where $I_\alpha(t)$ and $J_\alpha(t)$ are defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\big(\partial_x^{\alpha-\beta}(\rho^* + \sigma(t))\big)\,\partial_x^\beta w(t),$$

$$J_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\Big[\Big(\partial_x^{\alpha-\beta}\frac{1}{\rho^*}\Big)\partial_x^\beta\{\mu\Delta w(t) + (\mu + \mu')\nabla(\nabla\cdot w(t))\} + \big(\partial_x^{\alpha-\beta}A(t)\big)\nabla\partial_x^\beta w(t)\Big].$$

Canceling the term of $((\rho+\sigma(t))\partial_x^\alpha w(t), \nabla\partial_x^\alpha\sigma(t))$ by the above two formulas and writing the first term of second formula as follows:

$$(B(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t)) = \frac{1}{2}\frac{d}{dt}(B(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)) - \frac{1}{2}(B_t(t)\partial_x^\alpha(t), \partial_x^\alpha w(t)),$$

and using integration by parts for the second term of second formula, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\|\partial_x^\alpha\sigma(t)\|^2 + (B(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t))\} + B_0\mu\|\nabla\partial_x^\alpha w(t)\|^2 \\
&\le |(\partial_x^\alpha(v^*\sigma(t)), \nabla\partial_x^\alpha\sigma(t))| + |(\partial_x^\alpha f(t), B(t)\partial_x^\alpha w(t))| \\
&\quad + \Big[|(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t))| + |(J_\alpha(t), B(t)\partial_x^\alpha w(t))|\Big] + \frac{1}{2}|(B_t(t)\partial_x^\alpha w(t), \partial_x^\alpha w(t))| \\
&\quad + \Big[\mu\Big|\Big(\big(\nabla\frac{B(t)}{\rho^*}\big)\nabla\partial_x^\alpha w(t), \partial_x^\alpha w(t)\Big)\Big| + (\mu+\mu')\Big|\Big(\big(\nabla\frac{B(t)}{\rho^*}\big)\nabla\partial_x^\alpha w(t), \partial_x^\alpha w(t)\Big)\Big|\Big] \\
&\equiv K_1 + K_2 + K_3 + K_4 + K_5,
\end{aligned} \tag{3.9}$$

where $B_0 = \min_{\rho_0/2\le s\le 2\rho_0} s^2/P'(s)$.

Now we estimate the right hand side of (3.9), using the Sobolev inequality and the Gagliard-Nirenberg inequality. If $\alpha = 0$, employing the Hardy inequality, we have

$$K_1 \le \|(1+|x|)v^*\|_{L_\infty}\Big\|\frac{\sigma(t)}{|x|}\Big\|\|\nabla\sigma(t)\| \le C\epsilon\|\nabla\sigma(t)\|^2. \tag{3.10}$$

If $1 \le |\alpha| \le 3$, by integration by parts, we can show that

$$K_1 \le C\epsilon\|\nabla\sigma(t)\|^2_{|\alpha|-1}. \tag{3.11}$$

To use Lemma 3.1, we divide K_2 into the following two parts:

$$\begin{aligned}
K_2 &\le (F_\alpha(t), |\partial_x^\alpha w(t)|) + \Big|\Big(\frac{\sigma(t)}{\rho^*(\rho^*+\sigma(t))}\big[\mu\Delta\partial_x^\alpha w(t) + (\mu+\mu')\nabla(\nabla\cdot\partial_x^\alpha w(t))\big], \partial_x^\alpha w(t)\Big)\Big| \\
&\equiv K_{21} + K_{22}.
\end{aligned}$$

Concerning K_{22}, using integration by parts, we have

$$K_{22} \le C\epsilon\|\nabla\partial_x^\alpha w(t)\|^2. \tag{3.12}$$

To estimate K_{21}, we use (3.5). If $\alpha = 0$,

$$\begin{aligned}
K_{21} &\le C\Big\{\|(1+|x|)^2\nabla v^*\|_{L_\infty}\Big\|\frac{w(t)}{|x|}\Big\|^2 + \|(1+|x|)v^*\|_{L_\infty}\|\nabla w(t)\|\Big\|\frac{w(t)}{|x|}\Big\| \\
&\quad + \|w(t)\|_{L_3}\|\nabla w(t)\|\|w(t)\|_{L_6} + \|(1+|x|)(\nabla\rho^*, \nabla^2 v^*)\|_{L_3}\Big\|\frac{\sigma(t)}{|x|}\Big\|\|w(t)\|_{L_6}\Big\} \\
&\le C\epsilon\|(\nabla\sigma, \nabla w)(t)\|^2,
\end{aligned} \tag{3.13}$$

and if $1 \le |\alpha| \le 3$,

$$\begin{aligned}
K_{21} &\le C\{\|\nabla^{|\alpha|+1}v^*\|\|w(t)\|_{L_6}\|\nabla^{|\alpha|}w(t)\|_{L_3} + \textstyle\sum_{\nu=1}^{|\alpha|+1}\|\nabla^\nu w(t)\|\|\nabla^{|\alpha|}w(t)\| \\
&\quad + \textstyle\sum_{\nu=1}^{|\alpha|+1}\|(\nabla^\nu\rho^*, \nabla^{\nu+1}v^*)\|\|\sigma(t)\|_{L_6}\|\nabla^{|\alpha|}w(t)\|_{L_3} \\
&\qquad + \textstyle\sum_{\nu=1}^{|\alpha|}\|\nabla^\nu\sigma(t)\|\|\nabla^{|\alpha|}w(t)\| + \|R_{|\alpha|}(t)\|_{L_{\frac{3}{2}}}\|\nabla^{|\alpha|}w(t)\|_{L_3}\} \\
&\le C(\epsilon+\lambda)\|(\nabla\sigma(t), \nabla w(t))\|^2_{|\alpha|-1,|\alpha|} + C\lambda^{-1}\|\nabla^{|\alpha|}w(t)\|^2.
\end{aligned} \tag{3.14}$$

For $1 \leq |\alpha| \leq 2$, we can easily check that

$$K_3 \leq C\epsilon \|(\nabla\sigma(t), \nabla w(t))\|^2_{|\alpha|-1,|\alpha|} \tag{3.15}$$

It also turns out to be true for $|\alpha| = 3$, using the following inequality: $\|w(t)/(1+|x|)\|_{L_\infty} \leq C\|\nabla w(t)\|_1$, which follows from combination of the Sobolev inequality and the Hardy inequality. In order to estimate K_4, we use (3.1). Then

$$\begin{aligned}
K_4 &= \left|\left(\tilde{B}(t)\sigma_t(t)\,\partial_x^\alpha w(t), \partial_x^\alpha w(t)\right)\right| \\
&= \left|\left(\nabla\cdot\{(\rho^*+\sigma(t))w(t) + v^*\sigma(t)\}, \tilde{B}(t)\,\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\right)\right| \\
&\leq C\left|\left(w(t) + v^*\sigma(t), \nabla\{\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\} + \{\nabla\tilde{B}(t)\}\,\partial_x^\alpha w(t)\cdot\partial_x^\alpha w(t)\right)\right| \\
&\leq C\{(\|w(t)\|_{L_3} + \|v^*\|_{L_6}\|\sigma(t)\|_{L_6})\|\nabla\partial_x^\alpha w(t)\|\|\partial_x^\alpha w(t)\|_{L_6} \\
&\qquad + \|(w,\sigma)(t)\|_{L_6}\|(\nabla\rho^*, \nabla\sigma(t))\|\|\partial_x^\alpha w(t)\|^2_{L_6}\} \\
&\leq C\epsilon\|\nabla\partial_x^\alpha w(t)\|^2,
\end{aligned} \tag{3.16}$$

where $\tilde{B}(t)$ is defined by $\tilde{B}(t) = (\rho^*+\sigma(t))\left[2 - P''(\rho^*+\sigma(t))(\rho^*+\sigma(t))/P'(\rho^*+\sigma(t))\right]/P'(\rho^*+\sigma(t))$. We also have

$$K_5 \leq C\|(\nabla\rho^*, \nabla\sigma(t))\|_{L_3}\|\nabla\partial_x^\alpha w(t)\|\|\partial_x^\alpha w(t)\|_{L_6} \leq C\epsilon\|\nabla\partial_x^\alpha w(t)\|^2. \tag{3.17}$$

Combining (3.9)–(3.17), we obtain (3.7) and (3.8), if we choose ϵ, $\lambda > 0$ small enough. ∎

Estimates for $w_t(t)$ and its derivatives up to $\nabla^2 w_t(t)$.

Lemma 3.3 *Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1; \mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2). Then, there exist $\epsilon_0 > 0$ and $\beta_k > 0$ such that if $\epsilon \leq \epsilon_0$ and $\|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \rho_0, v^*)\|_{\mathscr{I}^{4,5}} \leq \epsilon$ then we have*

$$\frac{d}{dt}(w(t), \nabla\sigma(t)) + \beta_1\|w_t(t)\|^2 \leq C\epsilon\|\nabla\sigma(t)\|^2 + C\|\nabla w(t)\|_1^2, \tag{3.18}$$

$$\frac{d}{dt}(\nabla^{k-1}w(t), \nabla^k\sigma(t)) + \beta_k\|\nabla^{k-1}w_t(t)\|^2 \leq C\|(\nabla\sigma, \nabla w, \nabla^{k-2}w_t)(t)\|^2_{k-2,k,0} \tag{3.19}$$

for $2 \leq k \leq 3$, where $C > 0$ is a constant depending only on μ and μ'.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_0; \mathscr{H}^{\infty,\infty})$. For any multi-index α with $0 \leq |\alpha| \leq 2$, applying $\partial_x^\alpha A(t)^{-1}$ to (3.2) and multiplying the resultant equation by $\partial_x^\alpha w_t(t)$, we have

$$\begin{aligned}
&(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w_t(t)) + \left(\frac{1}{A(t)}\partial_x^\alpha w_t(t), \partial_x^\alpha w_t(t)\right) \\
&= \left(\partial_x^\alpha\left\{\frac{1}{\rho^* A(t)}\left[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\right] + \frac{1}{A(t)}f(t) - I_\alpha(t)\right\}, \partial_x^\alpha w_t(t)\right),
\end{aligned}$$

where $I_\alpha(t)$ is defined by

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\left(\partial_x^{\alpha-\beta}\frac{1}{A(t)}\right)\partial_x^\beta w_t(t).$$

The first term can be written in following form:

$$(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w_t(t)) = \frac{d}{dt}(\nabla\partial_x^\alpha\sigma(t), \partial_x^\alpha w(t)) + (\partial_x^\alpha\sigma_t(t), \nabla\cdot\partial_x^\alpha w(t)).$$

So putting $A_1 = \max_{\rho_0/2\le s\le 2\rho_0} P'(s)/s$, we have

$$
\begin{aligned}
&\frac{d}{dt}(\nabla\partial_x^\alpha\sigma(t),\partial_x^\alpha w(t)) + \frac{1}{A_1}\|\partial_x^\alpha w_t(t)\|^2 \\
&\le \left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^* A(t)}[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))]\right\},\partial_x^\alpha w_t(t)\right)\right| + |(I_\alpha(t),\partial_x^\alpha w_t(t))| \\
&\quad + \left|\left(\partial_x^\alpha\left\{\frac{1}{A(t)}f(t)\right\},\partial_x^\alpha w_t(t)\right)\right| + |(\partial_x^\alpha\sigma_t(t),\nabla\cdot\partial_x^\alpha w(t))| \equiv K_1+K_2+K_3+K_4
\end{aligned}
\tag{3.20}
$$

Now we estimate the right hand side of (3.20), using the Sobolev inequality and the Gagliard-Nirenberg inequality. First, we can easily check that

$$
K_1 \le \lambda\|\nabla^{|\alpha|}w_t(t)\|^2 + C\lambda^{-1}\|\nabla^2 w(t)\|_{|\alpha|},\quad K_2 \le C\epsilon\|\nabla w_t(t)\|_{|\alpha|-1}^2, \tag{3.21}
$$

furthermore if $\alpha = 0$, (by using Lemma 3.1)

$$
K_3 \le C\epsilon\|(\nabla\sigma,\nabla w, w_t)(t)\|_{0,1,0}^2 \tag{3.22}
$$

If $1 \le |\alpha| \le 2$, we divide K_3 as

$$
K_3 \le \sum_{\beta<\alpha}|(\{\partial_x^{\alpha-\beta}A(t)^{-1}\}\partial_x^\beta f(t),\partial_x^\alpha w_t(t))| + |(A(t)^{-1}\partial_x^\alpha f(t),\partial_x^\alpha w_t(t))| \equiv K_{31}+K_{32}.
$$

Then, using Lemma 3.1, we have

$$
\begin{aligned}
K_{31} &\le C\epsilon\|(\nabla\sigma,\nabla w,\nabla^{|\alpha|}w_t)(t)\|_{|\alpha|-1,|\alpha|+1,0}^2, \\
K_{32} &\le C\Big\{\|\nabla v^*\|_{L_3}\|w(t)\|_{L_6} + \textstyle\sum_{\nu=1}^{|\alpha|+1}\|(\nabla^\nu\rho^*,\nabla^{\nu+1}v^*)\|_{L_3}\|\sigma(t)\|_{L_6} \\
&\qquad + \|(\nabla\sigma(t),\nabla w(t))\|_{|\alpha|-1,|\alpha|+1} + \|R_{|\alpha|}(t)\|\Big\}\|\nabla^{|\alpha|}w_t(t)\| \\
&\le \lambda\|\nabla^{|\alpha|}w_t(t)\|^2 + C\lambda^{-1}\|(\nabla\sigma(t),\nabla w(t))\|_{|\alpha|-1,|\alpha|+1}^2.
\end{aligned}
\tag{3.23}
$$

At last, in order to estimate K_4, we substitute (3.1) into σ_t as in (3.16): Indeed, if $\alpha = 0$,

$$
\begin{aligned}
K_4 &\le |(\nabla\cdot\{(\rho^*+\sigma(t))w(t)\},\nabla\cdot w(t))| + |(v^*\sigma(t),\nabla(\nabla\cdot w(t)))| \\
&\le C\Big\{\|(\nabla\rho^*,\nabla\sigma(t))\|_{L_3}\|w(t)\|_{L_6}\|\nabla w(t)\| \\
&\qquad + \|\nabla w(t)\|^2 + \|(1+|x|)v^*\|_{L_\infty}\left\|\frac{\sigma(t)}{|x|}\right\|\|\nabla^2 w(t)\|\Big\} \\
&\le C\epsilon\|(\nabla\sigma(t),\nabla^2 w(t))\|^2 + C\|\nabla w(t)\|^2,
\end{aligned}
\tag{3.24}
$$

and if $1 \le |\alpha| \le 2$,

$$
\begin{aligned}
K_4 &\le |(\partial_x^\alpha\{(\rho^*+\sigma(t))w(t) + v^*\sigma(t)\},\nabla(\nabla\cdot\partial_x^\alpha w(t)))| \\
&\le C\sum_{\beta<\alpha}\Big\{\|(\partial_x^{\alpha-\beta}\rho^*,\partial_x^{\alpha-\beta}\sigma(t))\|_{L_3}\|\partial_x^\beta w(t)\|_{L_6} + \|\partial_x^\alpha w(t)\| \\
&\qquad + \|\partial_x^{\alpha-\beta}v^*\|_{L_3}\|\partial_x^\beta\sigma(t)\|_{L_6} + \|\partial_x^\alpha\sigma(t)\|\Big\}\|\nabla^2\partial_x^\alpha w(t)\| \\
&\le C\|(\nabla\sigma(t),\nabla w(t))\|_{|\alpha|-1,|\alpha|+1}^2.
\end{aligned}
\tag{3.25}
$$

Combining (3.20)–(3.25), we obtain (3.18) and (3.19), if we take ϵ, $\lambda > 0$ small enough. ∎

Estimates for $\nabla\sigma(t)$ and its derivatives up to $\nabla^3\sigma(t)$.

Lemma 3.4 *Let $(\sigma, w)(t) \in \mathscr{C}(0,t_1\,;\,\mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2). Then, there exist $\epsilon_0 > 0$ and $\beta_k > 0$ such that if $\epsilon \le \epsilon_0$ and $\|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \rho_0, v^*)\|_{\mathscr{I}^{4,5}} \le \epsilon$ then we have*

$$\|\nabla\sigma(t)\|^2 \le \|(\nabla w, w_t)(t)\|_{1,0}^2, \tag{3.26}$$

$$\|\nabla^k\sigma(t)\|^2 \le C\|(\nabla\sigma, \nabla w, \nabla^{k-1}w_t)(t)\|_{k-2,k,0}^2 \tag{3.27}$$

for $2 \le k \le 3$, where $C > 0$ is a constant depending only on μ and μ'.

Proof. Using the Friedrichs mollifier, we may assume that $(\sigma, w)(t) \in \mathscr{C}(0, t_0\,;\,\mathscr{H}^{\infty,\infty})$. For any multi-index α with $0 \le |\alpha| \le 2$, applying ∂_x^α to (3.2) and multiplying the resultant equation by $\nabla\partial_x^\alpha\sigma(t)$, we have

$$\begin{aligned} A_0\|\nabla\partial_x^\alpha\sigma(t)\|^2 &\le \left|\left(\partial_x^\alpha w_t(t), \nabla\partial_x^\alpha\sigma(t)\right)\right| \\ &+ \left|\left(\partial_x^\alpha\left\{\frac{1}{\rho^*}\left[\mu\Delta w(t) + (\mu+\mu')\nabla(\nabla\cdot w(t))\right]\right\}, \nabla\partial_x^\alpha\sigma(t)\right)\right| + \left|\left(I_\alpha(t), \nabla\partial_x^\alpha\sigma(t)\right)\right| \\ &+ \left|\left(\partial_x^\alpha f(t), \nabla\partial_x^\alpha\sigma(t)\right)\right| \equiv K_1 + K_2 + K_3 + K_4, \end{aligned} \tag{3.28}$$

where $A_0 = \min_{\rho_0/2 \le s \le 2\rho_0} P'(s)/s$ and

$$I_\alpha(t) = \sum_{\beta<\alpha}\binom{\alpha}{\beta}\left(\partial_x^{\alpha-\beta}A(t)\right)\nabla\partial_x^\beta\sigma(t).$$

It immediately follows from the Sobolev inequality that

$$\begin{aligned} &K_1 \le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|\nabla^{|\alpha|}w_t(t)\|^2, \\ &K_2 \le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|\nabla^2 w(t)\|_{|\alpha|}^2, \quad K_3 \le C\epsilon\|\nabla\sigma(t)\|_{|\alpha|}^2. \end{aligned} \tag{3.29}$$

We employ Lemma 3.1 to estimate K_4. Using the Sobolev inequality and the Gagliard-Nirenberg inequality, we have

$$\begin{aligned} K_4 &\le C\Big\{\|\nabla v^*\|_{L_3}\|w(t)\|_{L_6} + \textstyle\sum_{\nu=1}^{|\alpha|+1}\|(\nabla^\nu\rho^*, \nabla^{\nu+1}v^*)\|_{L_3}\|\sigma(t)\|_{L_6} \\ &\quad + \|(\nabla\sigma(t), \nabla w(t))\|_{|\alpha|-1,|\alpha|+1} + \|R_{|\alpha|}(t)\|\Big\}\|\nabla^{|\alpha|+1}\sigma(t)\| \\ &\le \lambda\|\nabla^{|\alpha|+1}\sigma(t)\|^2 + C\lambda^{-1}\|(\nabla\sigma(t), \nabla w(t))\|_{|\alpha|-1,|\alpha|+1}^2 \end{aligned} \tag{3.30}$$

for $1 \le |\alpha| \le 2$. This calculation is also true for $\alpha = 0$, if we regard $R_0(t)$ and $\|\nabla\sigma(t)\|_{-1}$ as zero. Combining (3.28)–(3.30), we obtain (3.26) and (3.27) if we take ϵ, $\lambda > 0$ small enough. ∎

A Proof of Proposition 3.2. Let $(\sigma, w)(t) \in \mathscr{C}(0, t_1\,;\,\mathscr{H}^{3,3})$ be a solution to (3.1)–(3.2) locally in time. Furthermore, we suppose that $\sup_{0\le t\le t_1}\|(\sigma, w)(t)\|_{3,3}$, $\|(\rho^* - \rho_0, v^*)\|_{\mathscr{I}^{4,5}} \le \epsilon$, where $\epsilon > 0$ is small enough such that at least we can use Lemmas 3.2 through 3.4. We use the notation:

$$[\sigma, w]_B(t) \equiv \|\sigma(t)\|^2 + \left(B(t)w(t), w(t)\right), \quad B(t) = \frac{(\rho^* + \sigma(t))^2}{P'(\rho^* + \sigma(t))}.$$

Summing up (3.7), (3.8) with $k = 1$, (3.18) and (3.26) (after multiplying (3.7), (3.8), (3.18) with small numbers respectively), we have

$$\frac{d}{dt}\Big\{\textstyle\sum_{\nu=0}^{1}\alpha_\nu[\nabla^\nu\sigma, \nabla^\nu w]_B + \beta_1\left(w, \nabla\sigma\right)\Big\} + \|(\nabla\sigma, \nabla w, w_t)\|_{0,1,0}^2 \le 0, \tag{3.31}$$

if we take ϵ, $\lambda > 0$ sufficiently small. Here and hereafter, α_k, $\beta_k > 0$ are constants depending only on μ and μ'. Similarly, summing up (3.8), (3.19), (3.27) with $k = 2$ and (2.31), we have

$$\frac{d}{dt}\{ \textstyle\sum_{\nu=0}^{2}\alpha_\nu[\nabla^\nu\sigma, \nabla^\nu w]_B + \sum_{\nu=1}^{2}\beta_\nu(\nabla^{\nu-1}w, \nabla^\nu\sigma)\} + \|(\nabla\sigma, \nabla w, w_t)\|_{1,2,1}^2 \leq 0. \tag{3.32}$$

Also, by (3.8), (3.19), (3.27) with $k = 3$ and (2.32), we obtain

$$\frac{d}{dt}\{ \textstyle\sum_{\nu=0}^{3}\alpha_\nu[\nabla^\nu\sigma, \nabla^\nu w]_B + \sum_{\nu=1}^{3}\beta_\nu(\nabla^{\nu-1}w, \nabla^\nu\sigma)\} + \|(\nabla\sigma, \nabla w, w_t)\|_{2,3,2}^2 \leq 0, \tag{3.33}$$

for any $t \in [0, t_1]$. Then, integration of (3.33) on $[0, t]$ implies that

$$N_B[\sigma, w](t) + \int_0^t \|(\nabla\sigma, \nabla w, w_t)(s)\|_{2,3,2}^2\, ds \leq N_B[\sigma, w](0), \tag{3.34}$$

where $N_B[\sigma, w](s)$ is defined by

$$N_B[\sigma, w](s) \equiv \textstyle\sum_{\nu=0}^{3}\alpha_\nu[\nabla^\nu\sigma, \nabla^\nu w]_B(s) + \sum_{\nu=1}^{3} \beta_\nu(\nabla^{\nu-1}w(s), \nabla^\nu\sigma(s))$$

for each $s \geq 0$.

Let us denote $B_0 = \min_{\rho_0/2\leq s\leq 2\rho_0}\{s^2/P'(s), 1\}$ and $B_1 = \max_{\rho_0/2\leq s\leq 2\rho_0}\{s^2/P'(s), 1\}$. Since we may assume without loss of generality that $\alpha_k \leq \alpha_{k-1}$ and $\beta_k \leq \alpha_k \min\{B_0, 1\}/4$ for $k = 1, 2, 3$, it follows from a simple calculation that

$$\frac{\alpha_3}{4}B_0\,\|(\sigma, w)(s)\|_{3,3}^2 \leq N_B[\sigma, w](s) \leq 2B_1\,\|(\sigma, w)(s)\|_{3,3}^2 \tag{3.35}$$

for each $s \in [0, t_1]$. Combining (3.34) and (3.35), we obtain (3.4), which completes the proof of Proposition 3.2. ∎

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