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THE CANONICAL MAPPING i_n AND k-SUBSPACES OF FREE TOPOLOGICAL GROUPS ON METRIZABLE SPACES

静岡大学教育学部　山田　耕三　(Kohzo Yamada)
Faculty of Education, Shizuoka University

1 Definitions

Let $F(X)$ and $A(X)$ be respectively the *free topological group* and the *free abelian topological group* on a Tychonoff space X in the sense of Markov [6]. As an abstract group, $F(X)$ is free on X and it carries the finest group topology that induces the original topology of X; every continuous map from X to an arbitrary topological group lifts in a unique fashion to a continuous homomorphism from $F(X)$. Similarly, as an abstract group, $A(X)$ is the free abelian group on X, having the finest group topology that induces the original topology of X, so that every continuous map from X to an arbitrary abelian topological group extends to a unique continuous homomorphism from $A(X)$.

For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose length is less than or equal to n. It is known that X itself and each $F_n(X)$ are closed in $F(X)$. The subspace $A_n(X)$ is defined similarly and each $A_n(X)$ is closed in $A(X)$. Let e be the identity of $F(X)$ and 0 be that of $A(X)$. For each $n \in \mathbb{N}$ and an element $(x_1, x_2, \dots, x_n)$ of $(X \oplus X^{-1} \oplus \{e\})^n$ we call $x_1x_2\cdots x_n$ a *form*. In the abelian case, $x_1+x_2+\cdots+x_n$ is also called a *form* for $(x_1, x_2, \dots, x_n) \in (X\oplus -X\oplus\{0\})^n$. We remark that a form may contain some reduced letter. Then the reduced form of $x_1x_2\cdots x_n$ is a word of $F(X)$ and that of $x_1 + x_2 + \cdots + x_n$ is a word of $A(X)$. For each $n \in \mathbb{N}$ we denote the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ onto $F_n(X)$ by i_n and we also use the same symbol i_n in the abelian case, that is, i_n means the natural mapping from $(X \oplus -X \oplus \{0\})^n$ onto $A_n(X)$. Clearly the mapping i_n is continuous for each $n \in \mathbb{N}$.

All topological spaces are assumed to be Tychonoff. By $\mathbb{N}$ we denote the set of all positive natural numbers. Our terminology and notations follow [3]. We refer to [5] for elementary properties of topological groups and to [1] and [4] for the main properties of free topological groups.

2 The mappings i_n and $F_n(X)$

The following problems have been studied by several mathematicians and were described in [9].

Problem 1 *Characterize spaces X for which the mapping i_n is quotient (closed, z-closed, R-quotient, etc.), $n \in \mathbb{N}$.*

Problem 2 *Find general conditions on X implying tat $F(X)$ (or $F_n(X)$ for each $n \in \mathbb{N}$) is a k-space.*

Problem 1 was completely solved for $n = 2$ by Pestov [7]. He proved that the mapping i_2 is quotient iff X is strongly collectionwise normal, i.e., if every neighborhood of the diagonal in X^2 contains a uniform neighborhood of the diagonal. Furthermore, the author [12] proved that i_2 is quotient iff i_2 is closed. The author also proved in the same paper that for a metrizable space X the mapping i_n is closed for each $n \in \mathbb{N}$ iff X is compact or discrete. They are also true for abelian case.

On the other hand, about Problem 2, Arhangel'skiĭ, Okunev and Pestov [2] gave a characterization of a metrizable space X such that $F(X)$ ($A(X)$) is a k-space, respectively.

Theorem 2.1 ([2]) *For a metrizable space X the following are equivalent:*

(1) *$F(X)$ is a k-space,*

(2) *$F(X)$ is a k_ω-space or discrete,*

(3) *X is locally compact separable or discrete.*

Theorem 2.2 ([2]) *For a metrizable space X the following are equivalent:*

(1) *$A(X)$ is a k-space,*

(2) *$A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space,*

(3) *X is locally compact and the set of all nonisolated points is separable.*

Furthermore, about Problem 1, the author [11] obtained a characterization of a metrizable space such that every i_n is quotient for abelian case. He proved that for a metrizable space X the mapping i_n for abelian case is quotient for each $n \in \mathbb{N}$ if and only if either X is locally compact and the set dX of all nonisolated points in X is separable, or dX is compact. As the author mentioned in [11, Proposition 4.1], for a Dieudonné complete, and hence metrizable space X i_n is quotient iff $F_n(X)$ ($A_n(X)$) is a k-space for each $n \in \mathbb{N}$. That is, the author obtained, in [11], the following results which are answers to Problem 2 for the free abelian topological group on a metrizable space.

Theorem 2.3 *For a metrizable space X the following are equivalent:*

(1) *$A_n(X)$ is a k-space for each $n \in \mathbb{N}$,*

(2) *$A_4(X)$ is a k-space,*

(3) *i_n is quotient for each $n \in \mathbb{N}$,*

(4) *i_4 is quotient,*

(5) *either X is locally compact and the set dX of all nonisolated points in X is separable, or dX is compact.*

Theorem 2.4 *For a metrizable space X the following are equivalent:*

(1) *$A_3(X)$ is a k-space,*

(2) *i_3 is quotient,*

(3) *X is locally compact or the set of all nonisolated points is compact.*

The aim of this note is to solve the above problems for the non-abelian free topological group on a metrizable space. To do that, we need a neighborhood base of e defined by Uspanskiĭ [10].

Let $P(X)$ be the set of all continuous pseudometrics on a space X. Put

$$F_0(X) = \{h = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_{2n}^{\varepsilon_{2n}} \in F(X) : \sum_{i=1}^{2n} \varepsilon_i = 0, x_i \in X \text{ for } i = 1, 2, \dots, n, n \in \mathbb{N}\}$$

Then $F_0(X)$ is a clopen normal subgroup of $F(X)$. It is well-known that every $h \in F_0(X)$ can be represented as

$$h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} g_2 x_2^{\varepsilon_2} y_2^{-\varepsilon_2} g_2^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_n} g_n^{-1}$$

for some $n \in \mathbb{N}$, where $x_i, y_i \in X$, $\varepsilon_i = \pm 1$ and $g_i \in F(X)$ for $i = 1, 2, \ldots, n$. Take an arbitrary $r = \{\rho_g : g \in F(X)\} \in P(X)^{F(X)}$. Let

$$p_r(h) = \inf\{\sum_{i=1}^{n} \rho_{g_i}(x_i, y_i) : h = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_n} g_n^{-1}, n \in \mathbb{N}\}$$

for each $h \in F_0(X)$. Then Uspenskiĭ [10] proved that:

(1) p_r is a continuous seminorm on $F_0(X)$ and

(2) $\{\{h \in F_0(X) : p_r(h) < \delta\} : r \in P(X)^{F(X)}, \delta > 0\}$ is a neighborhood base of e in $F(X)$. (Note that $p_r(e) = 0$ for each $r \in P(X)^{F(X)}$.)

Applying the above neighborhood, we can prove the following.

Theorem 2.5 *For a metrizable space X if $F_n(X)$ is a k-space for each $n \in \mathbb{N}$, then X is locally compact separable or discrete.*

Corollary 2.6 *For a metrizable space X if the mapping i_n is quotient for each $n \in \mathbb{N}$, then X is locally compact separable or discrete.*

Pestov and the author [8] showed that for a metrizable space X $F(X)$ is a k-space iff $F(X)$ has the inductive limit topology, i.e. a subset U of $F(X)$ is open if each $U \cap F_n(X)$ is open in $F_n(X)$. Consequently, from Theorem 2.1, Theorem 2.5, Theorem 2.6 and the above result, we can obtain the following.

Theorem 2.7 *For a metrizable space X, the following are equivalent:*

(1) *$F(X)$ is a k-space,*

(2) *$F_n(X)$ is a k-space for each $n \in \mathbb{N}$,*

(3) *$F(X)$ has the inductive limit topology,*

(4) *i_n is quotient for each $n \in \mathbb{N}$,*

(5) *X is locally compact separable or discrete.*

As compered with the abelian case, the above result is interesting. For, by Theorem 2.2 and Theorem 2.3, there is a metrizable space X, for example the hedgehog space such that each spininess is a sequence which converges to the center point, such that each $A_n(X)$ is a k-space, and hence i_n for abelian case is quotient, but $A(X)$ is not a k-space. On the other hand, for non-abelian case, Theorem 2.7 shows that there is not such a metrizable space.

3 A simple description of the topology of $F(X)$

As is well known, for a Tychonoff space X every compact subset of $F(X)$ is contained in some $F_n(X)$, $n \in \mathbb{N}$. Hence, $F(X)$ is a k-space if and only if the two conditions hold: first, $F(X)$ has the inductive limit topology and second, $F_n(X)$ is a k-space $n \in \mathbb{N}$. If a space X is Diedonné complete, then the above second condition can be replaced by the quotientness of i_n. We consider a simple description of the topology of $F(X)$, as follows;

$$\text{a set } U \subseteq F(X) \text{ is open in } F(X) \text{ if and only if } \quad i_n^{-1}(U \cap F_n(X)) \text{ is open in } (X \oplus X^{-1} \oplus \{e\})^n \text{ for each } n \in \mathbb{N}.$$

Clearly, if $F(X)$ has the inductive limit topology and i_n is quotient for each $n \in \mathbb{N}$, then $F(X)$ has the above description. On the other hand, since the mapping i_n is continuous, if $F(X)$ has the above description, then $F(X)$ has the inductive limit topology. Now, we can prove the following.

Proposition 3.1 *Let X be a space. If $F(X)$ has the above description, then i_n is quotient for each $n \in \mathbb{N}$. The same is true for $A(X)$.*

As a consequence, we obtain the following results.

Theorem 3.2 *For a Diedonné complete space X, in particular, for a paracompact space X, the following are equivalent:*

(1) *$F(X)$ is a k-space,*

(2) *$F(X)$ has the inductive limit topology and the mapping i_n is quotient for each $n \in \mathbb{N}$,*

(3) *a set $U \subseteq F(X)$ is open in $F(X)$ if and only if $i_n^{-1}(U \cap F_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$.*

The same is true for $A(X)$.

Furthermore, from Theorem 2.1 and Theorem 2.2, we can obtain a characterization of a metrizable space X such that $F(X)$ and $A(X)$ has the above simple description, respectively.

Theorem 3.3 *For a metrizable space X the following are equivalent:*

(1) *a set $U \subseteq F(X)$ is open in $F(X)$ if and only if $i_n^{-1}(U \cap F_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$,*

(2) *X is locally compact separable or discrete.*

Theorem 3.4 *For a metrizable space X the following are equivalent:*

(1) *a set $U \subseteq A(X)$ is open in $A(X)$ if and only if $i_n^{-1}(U \cap A_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each $n \in \mathbb{N}$,*

(2) *X is locally compact and the set of all nonisolated points of X is separable.*

4 The mapping i_3 and $F_3(X)$

In the last section, we shall obtain a characterization of a metrizable space X such that i_3 is quotient, and hence $F_3(X)$ is a k-space. To obtain it, we need another neighborhood of e in $F_n(X)$ which is defined by the author in [12].

Let X be a space and $\overline{X} = X \oplus \{e\} \oplus X^{-1}$, where e is the identity of $F(X)$. Fix an arbitrary $n \in \mathbb{N}$. For a subset U of $\overline{X}^2$ which includes the diagonal of $\overline{X}^2$, let $W_n(U)$ be a subset of $F_{2n}(X)$ which consists of the identity e and all words g satisfying the following conditions;

(1) g can be represented as the reduced form $g = x_1 x_2 \cdots x_{2k}$, where $x_i \in \overline{X}$ for $i = 1, 2, \ldots, k$ and $1 \leq k \leq n$,

(2) there is a partition $\{1, 2, \dots, 2k\} = \{i_1, i_2, \dots, i_k\} \cup \{j_1, j_2, \dots, j_k\}$,

(3) $i_1 < i_2 < \cdots < i_k$ and $i_s < j_s$ for $s = 1, 2, \dots, k$,

(4) $(x_{i_s}, x_{j_s}^{-1}) \in U$ for $s = 1, 2, \dots, k$ and

(5) $i_s < i_t < j_s \Longleftrightarrow i_s < j_t < j_s$ for $s, t = 1, 2, \dots, k$.

The author proved in [12] that $W_n(U)$ is a neighborhood of e in $F_{2n}(X)$ for every $U \in \mathcal{U}_X$ and $n \in \mathbb{N}$. Furthermore, we need the following lemma.

Lemma 4.1 *Let X be a space and $m, n \in \mathbb{N}$ with $n \leq m$. If B is a neighborhood of e in $F_{m+n}(X)$ and $g \in F_n(X)$, then $gB \cap F_m(X)$ is a neighborhood of g in $F_m(X)$.*

Applying the above neighborhood $W_2(X)$ and Lemma 4.1 as $n = 1$ and $m = 3$, we can prove the following.

Proposition 4.2 *If X is a locally compact metrizable space, then $F_3(X)$ is a k-space, and hence i_n is quotient.*

Proposition 4.3 *For a metrizable space X, if the set of all nonisolated points is compact, then $F_3(X)$ is a k-space, and hence i_3 is quotient.*

Consequently, joining to Theorem 2.4, we have the following result.

Theorem 4.4 *For a metrizable space X the following are equivalent:*

(1) *$F_3(X)$ is a k-space,*

(2) *$A_3(X)$ is a k-space,*

(3) *i_3 is quotient (for both case),*

(4) *X is locally compact or the set of all nonisolated points is compact.*

We remark that the author proved in [12] that a metrizable space X has to be compact or discrete in order to i_3 is closed (for both case).

Reference

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