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SOME EXAMPLES OF COUNTABLY COMPACT GROUPS

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ABSTRACT. We discuss some questions in the theory of countably compact groups motivated by results on compactness and pseudocompactness and discuss some progress obtained in recent years.

1. SOME BASIC DEFINITIONS AND RESULTS

All spaces are assumed to be Tychonoff and all the groups are assumed to be Abelian.

Compactness is a productive property, compact groups contain convergent sequences, the size of a compact group is of the form $2^{w(G)}$, where $w(G)$ is the weight of G, $w(X) \leq |X|$ for any compact space, the groups which carry compact group topologies have been classified and in particular, there are no compact group topologies on free Abelian groups.

These results motivated research on pseudocompact and countably compact groups. Pseudocompactness is productive for groups, they may not contain convergent sequences, there has been a through study of the relation between possible sizes and weights of pseudocompact groups, lots of papers have been published concerning the classification of the groups that may carry a pseudocompact group topology and there are pseudocompact group topologies on free Abelian groups.

The picture for countably compact groups is still blurred and we shall present some recent results concerning them. We shall also discuss some investigation done on groups whose every power is countably compact.

We start by reminding some basic definitions and well-known facts. All results without references are mentioned in Comfort's articles [3] and [4]. We recommend them for further details and references. The unitary circle group contained in the complex plane will be denoted by $\mathbb{T}$.

Definition 1. *1) A group is totally bounded if for every neighborhood U of the identity, there exists a finite number of translations that cover it.*

2) A topological space is pseudocompact if every real valued continuous function is bounded.

3)A space is countably compact if every countable open cover has a finite subcover.

We recall that a subset D of a topological space X is G_δ-dense if D meets every non-empty subset of X that is G_δ, that is, a countable intersection of open subsets of X.

Theorem 2. *1) A topological group is totally bounded if and only if it is a dense subgroup of a compact group. Every totally bounded Abelian group is a subgroup of a product of copies of $\mathbb{T}$.*

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2) A group is pseudocompact if and only if it is a G_δ-dense subset of some compact group. In particular, pseudcompact groups are totally bounded.

3) A space is countably compact if and only if every sequence has an accumulation point. Every countably compact space is pseudocompact and every normal pseudocompact space is countably compact.

We recall now the definition of ultrafilter which will be necessary to characterize spaces whose every power is countably compact.

Definition 3. *A subset p of the power set of ω is a free ultrafilter of ω if*

i) $A \in p \wedge \omega \supseteq B \supseteq A \rightarrow B \in p$,

ii) $A, B \in p \rightarrow A \cap B \in p$,

iii) $A \in p \rightarrow |A| = \omega$ *and*

iv) if $A \subseteq \omega$ *then either* $A \in p$ *or* $\omega \setminus A \in p$

The concept of p-limit is widely used on problems concerning countable compactness in product spaces. It was used implicitly in earlier works but defined explicitly by Berstein:

Definition 4. ([1]) *1) A sequence $\{x_n : n \in \omega\} \subseteq X$ has p-limit x if for every neighborhood U of x the set $\{n \in \omega : x_n \in U\} \in p$.*

2) Fixed an ultrafilter p, a space X is p-compact if every sequence in X has a p-limit in X.

Theorem 5. *A p-compact space is countably compact and p-compactness is a productive property. A space is p-compact for some p if and only if it all its powers are countably compact.*

Definition 6. *A space is ω-bounded if every countable subset has compact closure.*

Theorem 7. *A space is ω-bounded if and only if it is p-compact for every free ultrafilter p on ω.*

2. Products of countably compact groups

The following well-know result is due independently to Novak in 1953 [19] and Terasaka in 1952 [24]:

Example 8. *There exists a countably compact space whose square is not pseudocompact.*

Comfort and Ross showed that the situation for pseudocompact groups is different.

Theorem 9. ([6]) *The product of pseudocompact groups is pseudocompact.*

Thus, compactness and pseudocompactness are productive for topological groups. That motivated Comfort to ask:

Question 10. *Is the product of countably compact groups countably compact?*

E. van Douwen showed that consistently this is not true.

Example 11. ([8]) *Under Martin's Axiom, there exist two countably compact groups whose product is not countably compact.*

His example used countably compact groups without non-trivial convergent sequences to apply the trick of the small diagonal, which is used also in Frolik's construction. That motivated van Douwen to ask two questions in that paper:

Question 12. *1) Is there a countably compact group without non-trivial convergent sequences in ZFC? 2) Are there two countably compact groups whose product is not countably compact in ZFC?*

A CH group as in 1) has been first obtained by Hajnal and Juhasz [14] for other purposes and the main construction in van Douwen's paper is the construction of a MA example for 1). The example below concerns the second question of van Douwen.

Theorem 13. ([15]) *Under Martin's Axiom for countable posets, there exists a countably compact group whose square is not countably compact.*

Hart and van Mill reduced the need of MA using an ω-bounded group. However, ω-bounded groups contain convergent sequences, thus it does not touch question 1. Using an idea that came from elementary submodels an $MA_{\text{countable}}$ example was also obtained for 1) [16].

Question 14. (*Comfort,* [3])*For which cardinals κ is there a group G such that G^λ is countably compact for $\lambda < \kappa$ but G^κ is not.*

M. Hrusak showed recently that $\kappa \leq \mathfrak{c}$ if the Rudin Keisler order is downward directed, using a result of Yang [37]. It follows from a result of Ginsburg and Saks that $\kappa \leq 2^{\mathfrak{c}}$ in ZFC.

Theorem 15. *Under Martin's Axiom for countable posets,* [30] $\kappa = 3$ *is such a cardinal,* [26] *there is $\kappa \in]n, 2^n]$ for each positive integer n and* [33] *every finite κ is such a cardinal and the group witnessing it does not have non-trivial convergent sequences.*

Although pseudocompactness is not productive for topological spaces, if every countable subproduct is pseudocompact then the full product is countably compact. The following theorem of Frolik shows that ω is the best cardinal possible.

Example 16. ([10]) *There exists for each $\alpha \leq \omega$ a space X such that X^β is countably compact for $\beta < \alpha$ but X^α is not pseudocompact.*

The following theorem improved an earlier result of Scarborough and Stone [23], where the subproducts considered had size up to $2^{2^{\mathfrak{c}}}$.

Theorem 17. ([13]) *If every subproduct of size at most $2^{\mathfrak{c}}$ is countably compact then the full product is countably compact.*

Saks showed that this was consistently the best possible.

Example 18. ([22]) *Under Martin's Axiom, there exists a $2^{\mathfrak{c}}$-sized family of spaces whose every subproduct of size less than $2^{\mathfrak{c}}$ is countably compact but the full product is not.*

Such examples are not topological groups. It is natural to ask if for topological groups, Ginsburg and Saks theorem is still the best possible.

Example 19. ([12]) *There is a forcing model in which there exists a $2^{\mathfrak{c}}$-sized family of topological groups whose every subproduct of size less than $2^{\mathfrak{c}}$ is countably compact but the full product is not.*

3. Free Abelian groups, p-compact groups and size

Halmos asked for the classification of all groups which can be equipped with a compact group topology. The pseudocompact counterpart of this question was the object of study of Comfort, Remus, Tkacenko, Shakmatov and Dikranjan. In particular, they obtained pseudocompact topologies on free Abelian groups. The following are of particular interest, since Fucs showed that compact groups cannot be free Abelian.

Theorem 20. ([25], CH) ([29], MA) ([16], $MA_{\text{countable}}$) *The free Abelian group of size* $\mathfrak{c}$ *can be equipped with a countably compact group topology without non-trivial convergent sequences.*

Theorem 21. ([29]) *The* ω*-th power of a topological free Abelian group is not countably compact.*

Tkachenko's example is a modification of [14]. It was used by Robbie and Svetlichnii [20] to give a consistent solution to a problem due to Wallace. In fact, from their proof it is clear that the existence of a ZFC countably compact free Abelian group without non trivial convergent sequences would give a ZFC answer to Wallace's question.

Example 22. ([20], *under* CH) ([27], $MA_{\text{countable}}$) *There exists a countably compact topological semigroup which is both-sided cancellative but not a group.*

The motivation for Wallace's question was the fact that compact examples as above do not exist. The $MA_{\text{countable}}$ example mentioned above is a modification of [15] and could not be used to construct an example of a countably compact group topology on a free Abelian group, as it contains infinite compact subgroups. Recently the use of ideas from elementary submodels [16] reduced the use of MA to $MA_{\text{countable}}$ in many constructions of countably compact groups without non-trivial convergent sequences and made possible the use of countably closed forcing in such constructions.

Theorem 23. (*application of* [16] *in* [28] *and* [30], $MA_{\text{countable}}$) *There exists* $\mathfrak{c}^+$ *non-homeomorphic group topologies on the free Abelian group of size* $\mathfrak{c}$ *that make it countably compact and without non-trivial convergent sequences and its square not countably compact. There exists a semigroup and* $\mathfrak{c}^+$ *non-homeomorphic topologies that make it a counterexample to Wallace's question whose square is not countably compact.*

Recently Dikranjan and Tkachenko have obtained the classification of the groups of size $\mathfrak{c}$ that can carry a countably compact group topology under Martin's Axiom. For this, it was essential to modify Tkachenko's construction of countably compact free Abelian groups. The classification of groups of larger size start with a basic problem:

Question 24. (*Dikranjan and Shakmatov,* [5] *and* [7]) *Which are the sizes of free Abelian groups that can carry a countably compact group topology?*

Theorem 25. ([16]) *It is consistent with forcing that there exists a countably compact group topology on the free Abelian group of size* $2^{\mathfrak{c}}$. *In particular, there is such a topology for every infinite cardinal* $\kappa = \kappa^\omega \leq 2^{\mathfrak{c}}$.

The example above was also the first countably compact group without non-trivial convergent sequences whose size is bigger than $\mathfrak{c}$.

A standard closing off argument shows that there are countably compact groups of size κ^ω for any infinite cardinal κ. The following shows that these may be the only possiblities for the cardinality of such groups.

Theorem 26. ([9]) *If* α *is a cardinal of countable cofinality and* $2^\lambda < \alpha$ *whenever* $\lambda < \alpha$ *then there are no pseudcompact homogeneous spaces of size* α. *In particular, under* GCH, *there are no pseudocompact groups whose size has countable cofinality.*

E. van Douwen also showed that with appropriate cardinal arithmetic, there are pseudocompact groups whose size has countable cofinality. The same trick could not be used for the countable compact case. That led to the following question:

Question 27. *If G is a countably compact group (or homogeneous space) does it imply that $|G|^\omega = |G|$? At least that $|G|$ has countable cofinality?*

The following example combines ideas from [16] and [27].

Example 28. ([32]) *There is a forcing model in which there exists a countably compact group of size $\aleph_\omega$.*

The example above is not a free Abelian group and contains non-trivial convergent sequences. It seems more difficult but still possible to obtain a group of size $\aleph_\omega$ which is free Abelian, countably compact and without non-trivial convergent sequences.

So far, all countably compact free Abelian groups do not have non-trivial convergent sequences and Dikranjan mentioned this question during his invited talk at the International Topology Conference in Istambul in 2000. The importance of this question lies on the fact that the construction of countably compact groups without non-trivial convergent sequences is hard. However, there are differences between constructing free Abelian groups and groups without non-trivial convergent sequences if we require that every power is countably compact.

Saks [22] showed under Martin's Axiom that there are two spaces whose every power is countably compact but the product is not and Garcia Ferreira [11] showed that the class of spaces whose every power is countably compact is finitely productive in a model from [2]. These results motivated the following question:

Question 29. *(Garcia-Ferreira, [4]) Does Martin's Axiom imply the existence of two groups whose every power is countably compact but the product is not countably compact?*

Theorem 30. ([35]) *There exist two such groups under* $\mathrm{MA}_{\mathsf{countable}}$.

The construction of the example above uses ultraproducts and the vector space structure of $[\mathfrak{c}]^{<\omega}$ over the field $\{0,1\}$. A variation of this method has been used to produce groups whose n-th power is countably compact but whose $n+1$-st is not. This technique can be used to produce arbitrarily large countably compact groups without non-trivial convergent.

4. Weight of pseudocompact groups without non-trivial convergent sequences

Compact groups are dyadic, thus they do contain non-trivial convergent sequences. Arhangelskii, asked whether pseudocompact groups could contain non-trivial convergent sequences.

Theorem 31. ([21]) *There exists (in ZFC) a pseudocompact group of size α without non-trivial convergent sequences whenever $\alpha = \alpha^\omega$.*

Malykhin and Sapiro showed that the cardinals above are consistently the only ones that can be the weight of such groups.

Theorem 32. ([18]) *If α is a cardinal of countable cofinality and $2^\lambda < \alpha$ for every $\lambda < \alpha$ then there is no totally bounded group of weight α. In particular, under GCH, there is no totally bounded group topology of whose weight α satisfies $\alpha < \alpha^\omega$.*

Example 33. ([18]) *There exists a forcing model in which there exists a pseudocompact group without non-trivial convergent sequences whose weight is* $\omega_1 < \mathfrak{c}$.

Since ω_1 is a small cardinal and has uncountable cofinality, it is natural to consider an improvement from this point of view.

Theorem 34. [34] *If* α *has countable cofinality and there exists* $\lambda < \alpha$ *such that* $\lambda^\omega < \alpha < 2^\lambda$*, then there exists a pseudocompact group of weight* α*. In particular, there is a model in which the class* $\{\alpha : \alpha$ *has countable cofinality and there is a group of weight* $\alpha\}$ *is proper.*

The previous theorems show that the existence of pseudocompact group topologies without non-trivial convergent sequences whose weight has countable cofinality is quite related to cardinal arithmetic. We recall the Singular Hipothesis Cardinal: if α has uncountable cofinality then $\alpha^\omega = \alpha$. GCH implies SCH.

Corollary 35. *Under* SCH, *a cardinal* α *is the weight of a pseudocompact group without non-trivial convergent sequences if and only if either* α *has uncountable cofinality or* α *has countable cofinality and it is not strong limit, that is, there exists* $\lambda < \alpha$ *with* $\alpha < 2^\lambda$.

In the previous sections we discussed some countably compact groups without non-trivial convergent sequences. All of them had weight of the form α^ω, for some $\alpha \leq 2^\mathfrak{c}$.

Theorem 36. ([32]) $\mathrm{MA}_{\text{countable}} + \mathfrak{c} < \aleph_\omega < 2^\mathfrak{c}$ *implies the existence of a countably compact free Abelian group without non-trivial convergent sequences whose weight is* $\aleph_\omega$. *It is consistent that there exists a countably compact free Abelian group without non-trivial convergent sequences whose weight is* $\aleph_\omega$*, with* $2^\mathfrak{c} < \aleph_\omega < 2^{2^\mathfrak{c}}$.

The basic idea in this constructions is to raise the weight. If one allows convergent sequences, it suffices to use a dense set and closing off arguments to obtain groups whose weight is larger than the size of the group . That has been done by Comfort and van Douwen, but closing off arguments may add convergent sequences so the construction requires more care. There is a relation between the size and the weight of regular spaces, thus we first see how to construct large countably compact groups without non-trivial convergent sequences. The methods used in [8] and [16] do not seem to work for cardinals bigger than $2^\mathfrak{c}$. We apply ultraproducts to obtain these examples.

Example 37. ([35]) *Assume* $\mathrm{MA}_{\text{countable}}$. *There exists a free ultrafilter* p *such that for each infinite cardinal* $\kappa = \kappa^\omega$*, there exists a* p*-compact group of size* κ *without non-trivial convergent sequences..*

Analysing the construction of the example above, it is possible to raise the weight and obtain the following example:

Example 38. ([34]) *Under* SCH, *there exists a countably compact group without non-trivial convergent sequences of weight* α *if and only if* α *has countable cofinality and it is not strong limit or* α *has uncountable cofinality. It is consistent that the class of all cardinals of countable cofinality that are the weight of a countably compact group without non-trivial convergent sequences is a proper class.*

It seems fairly likely that via forcing a group of size $\aleph_\omega$ whose every power is countably compact can be constructed, but it is not clear if it could be made without non-trivial convergent sequences. It is not also clear if one could raise the weight of such examples as in the last example the size being $\kappa^\omega = \kappa$ was necessary in the argument.

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