数理解析研究所講究録 1248 巻 2002 年 100-106

Certain covering-maps and k-networks

東京学芸大学 田中祥雄 (Yoshio Tanaka)

The characterization for nice images of metric spaces is one of the most important problems in General Topology. Various kinds of characterizations have been obtained by means of certain k-networks. For a survey in this field, see [T5], for example.

In this paper, we shall introduce a general type of covering-maps, *σ-(P)-maps associated with certain covering properties* (P), in terms of σ-maps defined by [L1]. Then, we unify lots of characterizations and obtain new ones by means of these maps.

All spaces are regular and T_1, and all maps are continuous and onto.

Let $\mathcal{P}$ be a cover of a space X. Let (P) be a certain covering-property of $\mathcal{P}$. Let us say that $\mathcal{P}$ has property σ-(P) if $\mathcal{P}$ can be expressed as $\cup\{\mathcal{P}_i : i \in N\}$, where each $\mathcal{P}_i$ is a cover of X having the propery (P) such that $\mathcal{P}_i \subset \mathcal{P}_{i+1}$, and $\mathcal{P}_i$ is closed under finite intersections. (Sometimes, we may assume that $X \in \mathcal{P}_i$). When $\mathcal{P} = \mathcal{P}_i = \mathcal{P}_{i+1}$ for all $i \in N$, we shall say that $\mathcal{P}$ has property (P) (instead of σ-(P)).

In this paper, we shall restrict (P) to the covering-property which is (*): *Locally finite; Countable; Locally countable; Star-countable; or Point-countable.*

Let us say that a map $f : X \to Y$ is a σ-(P)-map (resp. (P)-map) if, for some base $\mathcal{B} = \{B_\alpha : \alpha\}$ in X, the family $f(\mathcal{B}) = \{f(B_\alpha) : \alpha\}$ has property σ-(P) (resp. (P)).

Remark 1. In the above definition, we assume that the family $f(\mathcal{B}) = \{f(B_\alpha) : \alpha\}$ is to be interpreted in the strict " indexed " sense, hence, the sets $f(B_\alpha)$ are *not requied to be different.* Thus, by the restriction (*), the base $\mathcal{B} = \{B_\alpha : \alpha\}$ must be at least point-countable, and f be an s-map (i.e., every $f^{-1}(y)$ is separable). When $f(\mathcal{B})$ is σ-locally finite, then X is a metrizable space with the σ-locally finite base $\mathcal{B}$; Y is a σ-space with the σ-locally finite network $f(\mathcal{B})$); and $f^{-1}(L)$ is Lindelöf for every Lindelöf subset L of Y. When $f(\mathcal{B})$ is locally countable or star-countable, then X is a locally separable, metrizable space with the locally countable base $\mathcal{B}$.

For map $f : X \to Y$, the following hold in view of the above.

(a) If f is a σ-(locally finite)-map, then X is metrizable.

(b) If f a (locally countable)-map or a (star-countable)-map, then X is locally separable, metrizable.

(c) (i) f is a (countable)-map iff X is separable metric.

(ii) f is a (locally-finite)-map iff X and Y are discrete.

We do not consider a trivial case of (locally finite)-maps.

S. Lin [L1] introduced the concept of σ-maps; that is, a map is a σ-map if it is a σ-(locally finite)-map. Related to σ-maps, let us review certain maps which are useful in the theory of networks. K. Nagami [N] introduced a σ-map $f : X \to Y$ in the following sense: For every σ-locally finite open cover $\mathcal{G}$ of X, $f(\mathcal{G})$ has a refinement $\mathcal{F}$ such that $\mathcal{F}$ is a σ-locally finite closed cover of Y. Let us call such a map f a *weak σ-map* here, but we need not the closedness of the cover $\mathcal{F}$. Related to σ-maps of [N], E. Michael [E1] (or [E2]) defined a σ-locally finite map $f : X \to Y$ as follows: Every σ-locally finite (not necessarily open) cover of X has a refinement $\mathcal{P}$ such that $f(\mathcal{P})$ is a σ-locally finite cover of Y.

The following implication holds: σ-maps $\rightarrow$ σ-locally finite maps $\rightarrow$ weak σ-maps, but each converse need not hold; see Remark 2 below.

For a cover $\mathcal{P}$ of a space X, we recall the following definitions. These are generalizations of bases. For a survey around k-networks, see [T5], for example.

$\mathcal{P}$ is a k-network if, for any compact set K and for any open set U such that $K \subset U$, $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. (When K is a single point, such a cover $\mathcal{P}$ is called a network (or net)). As is well-known, a space X is called an $\aleph$-space (resp. $\aleph_0$-space) if X has a σ-locally finite k-network (resp. countable k-network).

$\mathcal{P}$ is a cs-network (resp. $cs*$-network) if, for each $x \in X$, each nbd V of x, and each convergent sequence L with the limit point x, there exists $P \in \mathcal{P}$ such that $x \in P \subset V$, and P contains L eventually (resp. frequently).

$\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ with each $\mathcal{P}_x$ closed under finite intersections is a weak base if (a) each $P \in \mathcal{P}_x$ contains x; (b) for each $x \in X$, and each nbd G of x, there exists $P(x) \in \mathcal{P}_x$ such that $P(x) \subset G$; and (c) $G \subset X$ is open in X if, for each $x \in G$, there exists $P(x) \in \mathcal{P}_x$ such that $P(x) \subset G$. A space X is called g-metrizable [S2] if X has a σ-locally finite weak base.

$\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ satisfying the above (a) and (b) is an sn-network [L2] if, for each $x \in X$, any $P \in \mathcal{P}_x$ is a sequential neighborhood of x (i.e., any sequence converging to x is eventually contained in P).

Remark 2. (i) A map $f : X \rightarrow Y$ is a weak σ-map if the following (a) or (b) holds.

(a) f is a closed map such that X is a σ-space.

(b) f is an open map such that Y is subparacompact.

(In fact, for case (a), every open cover $\mathcal{G}$ of X has a refinement $\mathcal{P}$ which is a σ-locally finite closed network for X. But, $f(\mathcal{P})$ is a σ-closure preserving closed network for Y. Thus, $f(\mathcal{P})$ has a refinement which is a σ-discrete closed network $\mathcal{F}$ in view of the proof of [SNa; Theorem]. Then, $\mathcal{F}$ is a σ-locally finite refinement of $f(\mathcal{G})$).

(ii) Let $f : X \rightarrow Y$ be a map. If (a) or (b) below holds, then f is σ-locally finite ([M1] or [M2]). Conversely, if $f : X \rightarrow Y$ is σ-locally finite, then for any closed, and ω_1-compact subset L of Y (i.e., every uncountable subset of L has an accumulation point), $f^{-1}(L)$ is ω_1-compact.

(a) f is a closed map with every $f^{-1}(y)$ Lindelöf, and X or Y is subparacompact.

(b) $f(\mathcal{P})$ is σ-locally finite for some network $\mathcal{P}$ in X. (Thus, X and Y must be σ-spaces).

(iii) Let $f : X \rightarrow Y$ be a map such that X is a σ-space. Then (a) $\leftrightarrow$ (b) $\rightarrow$ (c) holds. When f is closed, (a), (b), and (c) are equivalent, and (a) and (c) are equivalent under X being subparacompact. (In fact, these hold by means of (ii) and [E2; Proposition 2.2]).

(a) f is a σ-locally finite map.

(b) $f(\mathcal{P})$ is σ-locally finite for some *network* $\mathcal{P}$ in X.

(c) Every $f^{-1}(y)$ is Lindelöf.

The above shows that every σ-locally finite image of a σ-space is a σ-space. But, every weak σ-image (actually, open s-image) of a metric space need not be a σ-space (by the Michael-Line).

For closed maps, we have the following. In (a) or (b), f can not been weaken to be a weak σ-map in view of (i).

(iv) For a closed map $f : X \to Y$ with X metric, the following are equivalent.

(a) f is a σ-map.

(b) f is a σ-locally finite map.

(c) f is an s-map.

(d) X has a point-countable k-network consisting of closed subsets.

(e) X is an $\aleph$-space.

(Indeed, (a) $\to$ (b) $\to$ (c) is already shown. For (c) $\leftrightarrow$ (d), see [T2], For (c) $\to$ (a), since f is a closed s-map with Y paracompact, every σ-locally finite base for X has a refinement $\mathcal{B}$ such that $\mathcal{B}$ is a base for X and $f(\mathcal{B})$ is σ-locally finite in Y. (c) $\to$ (e) holds by [Ga; Theorem 1]).

Concerning characterizations for σ-spaces by means of maps, the following holds. (a) $\leftrightarrow$ (b); (a) $\leftrightarrow$ (d) $\leftrightarrow$ (e); and (a) $\leftrightarrow$ (c) is respectively due to [L1]; [N]; and [E1] or [E2].

(v) For a space X, the following are equivalent. In (b), (c), and (e), the map can be chosen to be one-to-one. In (d) and (e), the condition of the weak σ-map is essential; see (iii).

(a) X is a σ-space.

(b) X is the image of a metric space under a σ-map.

(c) X is the image of a metric space under a σ-locally finite map.

(d) X is the image of a metric space under a one-to-one, weak σ-map.

(e) X is the image of a metric space under a weak σ-map f such that $f^{-1}(x)$ is compact for every $x \in X$.

Proposition: For a map $f : X \to Y$, (1), (2), and (3) below hold.

(1) The following are equivalent.

(a) f is a (point-countable)-map.

(b) X has a point-countable base, and f is an s-map.

(c) X has a point-countable base, and $f(\mathcal{B})$ is point-countable for any point-countable base $\mathcal{B}$ in X.

(2) Let X be locally separable, metric. Then the following are equivalent.

(a) f is a (locally countable)-map (resp. (star-countable)-map).

(b) Each point $y \in Y$ has a nbd V_y with $f^{-1}(V_y)$ (resp. each point $x \in X$ has a nbd W_x with $f^{-1}(f(W_x)))$ separable in X.

(c) $f(\mathcal{B})$ is locally countable (resp. star-countable) for any locally countable (resp. star-countable) base $\mathcal{B}$ in X.

(d) $f(\mathcal{B})$ is locally countable (resp. star-countable) for any star-countable base $\mathcal{B}$ in X.

(3) Let X be locally separable, metric. Then the implications (a) $\to$ (b) $\to$ (c); and (d) $\to$ (e) $\to$ (b) and (c) hold. When f is quotient, (a) $\sim$ (f) are equivalent.

(a) f is a (locally countable)-map.

(b) $f^{-1}(L)$ is Lindelöf for every Lindelöf subset L of Y.

(c) f is a (star-countable)-map.

(d) f is a σ-map.

(e) f is a σ-locally finite map.

(f) $f^{-1}(L)$ is separable for every separable subset L of Y.

(Indeed, (1) holds in view of Remark 1(i). (2) would be routinely shown (cf. [TX;

Proposition 1.1], but note that any star-countable base for X is locally countable. We show (3) holds, but the implication (a) $\to$ (b) $\to$ (c) is routine, and (d) $\to$ (e) is already shown. (e) $\to$ (b) holds by Remark 2(ii). For (e) $\to$ (c), let f be a σ-locally finite map, and let $\mathcal{B}$ be a σ-locally finite base for X consisting of hereditarily Lindelöf subsets. Then, $\mathcal{B}$ has a refinement $\mathcal{F}$ such that $f(\mathcal{F})$ is σ-locally finite. For each $B \in \mathcal{B}$, $f(B)$ meets only countably many $f(F_n) \in f(\mathcal{F})$ with $F_n \in \mathcal{F}$, for $f(B)$ is Lindelöf. While, each Lindelöf subset F_n meets only countably many elements of $\mathcal{B}$. Hence, each $f(B)$ meets only countably many elements of $f(\mathcal{B})$. Then, $f(\mathcal{B})$ is a star-countable cover of Y. Thus, f is a (star-countable)-map. For the latter part, let (c) hold. Since f is quotient, Y is determined by a star-countable cover $\mathcal{C} = f(\mathcal{B})$ for some base $\mathcal{B}$ in X. Thus, as in the proof of [T3; Theorem 1], Y is the topological sum of subspaces, where each subspace is a countable union of elements of $\mathcal{C}$. Thus, the cover $\mathcal{C}$ is locally countable and σ-locally finite in Y. Thus (c) implies (a), (d), and (f). (f) $\to$ (c) would be routine).

Remark 3. In view of (a) $\leftrightarrow$ (d) in (2), (locally countable)-maps (resp. (star-countable)-maps) coincide with locally countable maps (resp. star-countable maps) discussed in [TX].

We note that it is impossible to replace " any star-countable base " by " any locally countable base " in (d) for the parenthetic part.

Corollary 1. For a quotient map $f : X \to Y$ such that X is a locally separable, metric space, the following are equivalent.

(a) f is a (locally countable)-map.
(b) f is a (star-countable)-map.
(c) f is a σ-map.
(d) f is a σ-locally finite map.
(e) $f^{-1}(L)$ is Lindelöf for every Lindelöf subset L of Y.
(f) $f^{-1}(S)$ is separable for every separable subset S of Y.

For a map $f : X \to Y$, let us recall the following definitions around compact-covering maps.

f is sequence-covering [S1], if each convergent sequence in Y is the image of some convergent sequence in X.

f is sequence-covering of [GMT], if each convergent sequence L in Y is the image of some compact subset of X. In this paper, let us call such a sequence-covering map of [GMT] *pseudo-sequence-covering* as in [ILuT]. (When " convergent sequence L " is replaced by " compact set L ", as is well-known, such a map f is called compact-covering).

f is subsequence-covering [LLuD], if for each $y \in Y$, and each sequence L in Y converging to y, there exists a convergent sequence K in X such that $f(K)$ is a subsequence of L.

f is 1-sequence-covering [L3], if for each $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each sequence K converging to y, there exists a sequence L converging to x such that $f(L) = K$. For 1-sequence-covering maps, see [LY], for example.

Let $f : X \to Y$ be a map such that X is sequential. If f is pseudo-sequence-covering, then f is subsequence-covering. Also, f is quotient iff f is subsequence-covering such that Y is sequential ([T4]).

Lemma: Let $f : X \to Y$ be a σ-(P)-map. Then the following hold.

(i) If f is quotient, then Y has a k-network having property σ-(P).

(ii) If f is subsequence-covering (resp. sequence-covering; 1-sequence covering), then Y has a $cs*$-network (resp. cs-network; sn-network) having property σ-(P).

(Indeed, for (i), let $f(\mathcal{B})$ have property σ-(P) for some base $\mathcal{B}$ in X. Let $K \subset U$ with K compact and U open in Y. Since $f|f^{-1}(U)$ is quotient, U is determined by a point-countable cover $\mathcal{U} = \{f(B) : B \in \mathcal{B}, f(B) \subset U\}$. Thus, $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{U}$ by [GMT: Proposition 2.1]. This shows that $f(\mathcal{B})$ is a k-network. (ii) is routine).

Every σ-image of a metric space is a σ-space, but need not be an $\aleph$-space in view of Remark 2(v). But, we have the following by the previous lemma and Corollary 1.

Corollary 2. (1) Every quotient σ-image of a metric space is an $\aleph$-space.

(2) Every quotient σ-locally finite image of a locally separable, metric space is an $\aleph$-space.

Remark 4. (i) Every (1-sequence-covering) quotient σ-locally finite image of a metric space need not be an $\aleph$-space (by the open finite-to-one image of a metric space in Example 3.2 in [T1]). This shows that the local separability of the domain is essential in Corollary 2(2).

(ii) Every quotient, finite-to-one, weak σ-image of a locally compact, metric space need not be an $\aleph$-space, and need not satisfy each of (e) $\sim$ (f) in Corollary 1, even if the range is a paracompact σ-space (by the example in [LT; Remark 14(2)]). Hence, we can not replace " σ-locally finite " by " weak σ " in Corollary 1 and Corollary 2(2).

The nice characterization for quotient s-images of metric spaces was obtained by [GMT], in 1984. Since then, lots of characterizations for certain images of metric spaces have been obtained by many topologists by using the analogous methods to the proof of [GMT; Theorem 6.1]. To unify these characterizations, we have General Theorem below. This theorem (resp. its latter part) could be shown by modifying the proof of [Li; Lemma 2.1] (resp. [L2; Theorem]). But, we shall omit the proof here.

General Theorem: For a space X, the following are equivalent. Also, it is possible to replace " subsequence-covering " by " pseudo-sequence-covering " in (b).

(a) X has a $cs*$-network (resp. cs-network; sn-network) having property σ-(P).

(b) X is the subsequence-covering (resp. sequence-covering; 1-sequence-covering) σ-(P)-image of a metric space.

The following is due to [Li]. Also, an analogous result for a σ-(locally countable)-property could be valid.

Corollary 3. A space X is an $\aleph$-space iff X is the sequence-covering σ-image of a metric space. Also, it is possible to replace " sequence-covering " by " subsequence-covering " or " pseudo-sequence-covering " (cf. [L1]).

In the following, (a) $\leftrightarrow$ (b) is due to [L2] (resp. [LLu]; [L3]).

Corollary 4. For a space X, the following are equivalent. Also, it is possible to replace " subsequence-covering " by " pseudo-sequence-covering " in (b) and (c).

(a) X has a point-countable $cs*$-network (resp. cs-network; sn-network).

(b) X is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), s-image of a metric space.

(c) X is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), (point-countable)-image of a metric space.

In the following, (1) is (well) known, and some parts of (2) are shown in [TX].

Corollary 5. For a space X, the following hold. Also, it is possible to replace " subsequence-covering " by " pseudo-sequence-covering " in (1) and (2), and to replace " locally countable " by " star-countable " in (2).

(1) X has a countable $cs*$-network (resp. cs-network; sn-network) iff X is the subsequence-covering (resp. sequence-covering; 1-sequence-covering) image of a separable metric space.

(2) X has a locally countable $cs*$-network (resp. cs-network; sn-network) iff X is the subsequence-covering (resp. sequence-covering; 1-sequence-covering), (locally-countable)-image of a locally separable metric space.

Remark 5. Related to (1), let us recall a result that, for a space X, X has a countable $cs*$-network $\leftrightarrow$ X has a countable cs-network $\leftrightarrow$ X is an $\aleph_0$-space. Concerning (2), when X is sequential, then X has a locally countable $cs*$-network $\leftrightarrow$ X has a locally countable cs-network $\leftrightarrow$ X is the topological sum of $\aleph_0$-spaces. Also, we can replace " locally countable " by " star-countable." (cf. [T5]).

Corollary 6. (1) A space X is a sequential space with a point-countable $cs*$-network iff X is the quotient s-image of a metric space ([T4] or [L2]).

(2) A space X is a sequential space with a point-countable cs-network iff X is the sequence-covering, quotient s-image of a metric space ([LLu]).

(3) A space X has a point-countable weak base iff X is the 1-sequence-covering, quotient s-image of a metric space ([L2]).

Corollary 7. For a space X, the following are equivalent. It is possible to replace " locally countable " by " star-countable " in (a) or (b). Moreover, if we replace " $cs*$-network " by " cs-network (resp. sn-network) " in (a), then the same equivalence holds by adding the prefix " sequence-covering (resp. 1-sequence-covering) " before " quotient " in (b) $\sim$ (e).

(a) X is a sequential space with a locally countable $cs*$-network.

(b) X is the quotient (locally-countable)-image of a locally separable metric space.

(c) X is the quotient σ-image of a locally separable metric space.

(d) X is the quotient σ-locally finite image of a locally separable metric space.

(e) X is the image of a locally separable metric space under a quotient map f such that $f^{-1}(S)$ is separable for every separable (or Lindelöf) subset S of Y.

Corollary 8. (1) A space X is a k-and-$\aleph$-space iff X is the (sequence-covering) quotient σ-image of a metric space.

(2) A space X is g-metrizable iff X is the quotient, 1-sequence-covering, σ-image of a metric space.

REFERENCES

[Ga] Zhi Min Gao, ℵ-space is invariant under perfect mappings, Questions and answers in General Topology, 5(1987), 271-279.

[GMT] G. Gruenhage, E. Michael and Y. Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.

[ILuT] Y. Ikeda, C. Liu and Y. Tanaka, Quotient compact images of metric spaces, and related matters, to appear in Topology Appl., (2001).

[Li] Z. Li, A mapping theorem on g-metrizable spaces, (pre-print).

[L1] S. Lin, σ-maps and Alexandorff's question, Proc. First Academic Annual Meeting for Fujian Association of Science and Technology, Fuzhou, 1992, 5-8. (Chinese)

[L2] S. Lin, The sequence-covering s-images of metric spaces, Northeastern Math. J., 9(1993), 81-85.

[L3] S. Lin, On sequence-covering s-mappings, Adv. Math., 26(1996), 548-551.

[LT] S. Lin and Y. Tanaka, Point-countable k-networks, closed maps, and related results, Topology Appl., 59(1994), 79-86.

[LLu] S. Lin and C. Liu, On spaces with point-countable cs-networks, Topology Appl., 74(1996), 51-60).

[LLuD] S. Lin, C. Liu, and M. Dai, Images on locally separable spaces, Acta. Math. Sinica, 13(1997), 1-8.

[LY] S. Lin and P. Yan, Sequence-covering maps of metric spaces, Topology Appl., 109(2001), 301-314.

[M1] E. Michael, On Nagami's Σ-spaces and some related matters, Proc. Washington State Univ., Conf. on General Topology, 1970, 13-19.

[M2] E. Michael, σ-locally finite maps, Proc. Amer. Math. Soc., 65(1977), 159-164.

[N] K. Nagami, σ-spaces and product spaces, Math. Ann., 181(1969), 109-118.

[S1] F. Siwiec, Sequence-covering and countably bi-quotient mappings, General Topology Appl., 1(1971), 143-154.

[S2] F. Siwiec, On defining a space by a weak base, Pacific, J. Math., 52(1974), 233-245.

[SNa] F. Siwiec and J. Nagata, A note on nets and metrization, Proc. Japan Acad., 44(1968), 623-627.

[T1] Y. Tanaka, On open finite-to-one maps, Bull. Tokyo Gakugei Univ., Ser., IV, 25(1973), 1-13.

[T2] Y. Tanaka, Closed maps on metric spaces, Topology Appl., 11(1980), 87-92.

[T3] Y. Tanaka, Point-countable k-systems and products of k-spaces, Pacific J. Math., 101(1982), 199-208.

[T4] Y. Tanaka, Point-countable covers and k-networks, Topology Proc., 12(1987), 327-349).

[T5] Y. Tanaka, Theory of k-networks II, Questions and Answers in General Topology, 19(2001), 27-46.

[TX] Y. Tanaka and S. Xia, Certain s-images of locally separable metric spaces, Questions and Answers in General Topology, 14(1996), 217-231.

Department of Mathematics, Tokyo Gakugei University, Koganei, Tokyo, 184-8501, JAPAN
e-mail address: ytanaka@u-gakugei.ac.jp