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RECENT PROGRESS IN SOME ASPECTS OF STAR COVERING PROPERTIES

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ABSTRACT. In the last ten years, the study of star covering properties of a topological space has attracted many general topologists. The main purpose of the present paper is to outline recent progress in some aspects of this area and present some open questions.

1. INTRODUCTION

The study of covering properties is one of the major topics in General Topology. In the last ten years, a new type of covering properties, namely star covering proerties, has attracted many general topologists. A relatively comprehensive survey is Matveev [M3]. The main purpose of the present paper is to outline recent progress in some aspects of this area, and the author has no intention to cover everything happened for the area recently. Let X be a topological space, $\mathcal{U} \subseteq \mathcal{P}(X)$, $A \in \mathcal{P}(X)$. The *star of A with respect to* $\mathcal{U}$ is defined by

$$\mathrm{st}^1(A,\mathcal{U}) = \mathrm{st}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}.$$

Inductively, one can define $\mathrm{st}^{n+1}(A,\mathcal{U}) = \mathrm{st}(\mathrm{st}^n(A,\mathcal{U}),\mathcal{U})$ for all $n \in \mathbb{N}$. It is well-known that stars of open covers of spaces can be applied to characterize many important topological properties and classes of spaces, for instance, paracompactness, normality, connectedness and many classes of generalized metric spaces (see [E], [NP] and [KV] etc). But, it seems that the study of star covering properties of general topological spaces does not have a very long history. In 1967, Aquaro [A] studied the problem when every point countable open cover of a space X has a countable subcover. He considered the following property $(*)$ in [A]:

$(*)$ any discrete family of nonempty closed sets in X is countable.

Aquaro observed that all countably compact spaces and all Lindelöf spaces have the property $(*)$, and if a space X has the property $(*)$ then every point countable open cover of X has a countable subcover. It was Fleischman who first used the term "starcompact" in 1970 and proved that every countably compact space is starcompact. After that, starcompactness and its generalizations were further considered by Ikenaga, Matveev and Sarkhel et al in 1980s. For example, Ikenaga [I1] studied n-star-Lindelöf spaces under the name *ω-n-star spaces*. Matveev [M1] generalized the theorem of Miščenko and also proved that 3-starcompact = pseudocompact for Tychonoff spaces. The first systematic investigation on star covering properties was done by van Douwen et al in [DRRT].

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We shall give some basic defintions in Section 2, then discuss the cardinal invariants related to star covering proerties, and iterated starcompact spaces in Section 3, 4, 5. In the last section, we list some open questions. For undefined notations and concepts, see references of this paper.

2. Prelimaries

First of all, for a T_1-space, the property $(*)$ can be characterized in term of stars of open covers in the next lemma.

Lemma 1. *For a T_1-space X, $(*)$ is equivalent to that for every open cover $\mathcal{U}$ of X there exists a countable subset $A \subseteq X$ such that* $\mathrm{st}(A, \mathcal{U}) = X$.

Proof. $(\Rightarrow)$ Suppose the contrary. Then there exists an open cover $\mathcal{U}$ of X such that $\mathrm{st}(A, \mathcal{U}) \neq X$ for any countable set $A \subseteq X$. By transfinite induction, one can select an uncountable sequence $(x_\alpha : \alpha < \omega_1)$ in X such that for all $\alpha < \omega_1$, $x_\alpha \notin \mathrm{st}(\{x_\beta : \beta < \alpha\}, \mathcal{U})$. Then $\{\{x_\alpha\} : \alpha < \omega_1\}$ is uncountable discrete. This contradicts with $(*)$.

$(\Leftarrow)$ Suppose that $(*)$ does not hold. Then there exists an uncountable discrete family $\{A_\alpha : \alpha < \omega_1\}$ of nonempty closed sets of X. Pick a point $z_\alpha \in A_\alpha$ for each $\alpha < \omega_1$ and an open set U_α such that $U_\alpha \cap Z = \{z_\alpha\}$, where $Z = \{z_\alpha : \alpha < \omega_1\}$. It follows that $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{X \setminus Z\}$ is an open cover of X such that $\mathrm{st}(A, \mathcal{U}) \neq X$ for any countable set $A \subseteq X$. This is a contradiction. □

Lemma 1 provides us some "new" thought of studying covering properties, which is considering whether $\{\mathrm{st}(x, \mathcal{U}) : x \in X\}$ has certain"special subcover" rather than considering the existence of subcovers of an open cover $\mathcal{U}$ of X. Specifically, we have the following definitions.

Definition 2. A space X is *n-starcompact* (*n-star-Lindelöf*) ($n \in \mathbb{N}$) if for every open cover $\mathcal{U}$ of X there exists a finite (countable) $A \subseteq X$ such that $\mathrm{st}^n(A, \mathcal{U}) = X$ (when $n = 1$, it is simply called *starcompact* (*star-Lindelöf*)).

Definition 3. A space X is *$n\frac{1}{2}$-starcompact* (*$n\frac{1}{2}$-star-Lindelöf*) ($n \in \mathbb{N}$) if for every open cover $\mathcal{U}$ of X there exists a finite (countable) $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

It is known that for Hausdorff spaces, starcompactness is equivalent to countable compactness. van Douwen et al [DRRT] proved that for regular spaces, both n-starcompactness (n-star-Lindelöfness) ($n \geq 3$) and $n\frac{1}{2}$-starcompactness ($n\frac{1}{2}$-star-Lindelöfness) ($n \geq 2$) are equivalent to DFCC (DCCC), the discrcte finite (countable) chain condition. Thus, the interesting new classes of spaces are: $1\frac{1}{2}$-starcompact spaces, 2-starcompact spaces, star-Lindelöf spaces, $1\frac{1}{2}$-star-Lindelöf spaces and 2-star-Lindelöf spaces. So far, behaviors of 2-starcompact (2-star-Lindelöf) spaces are not completely clear. For instance, it is not known if the product of a 2-starcompact (2-star-Lindelöf) space with a compact factor is 2-starcompact (2-star-Lindelöf).

3. Some questions of Bonanzinga and Matveev

By counting how many stars of open covers of a space could essentially cover the space, we can extend the concepts in Definition 2 and Definition 3 by defining some cardinal invariants. In this section, we shall study and compare some of them.

The *Aquaro number* $a(X)$ [M5] of a T_1-space X is defined as the smallest infinite cardinal κ such that for each open cover $\mathcal{U}$ of X there exists a closed and discrete subset

$A \subseteq X$ such that $|A| \leq \kappa$ and $\mathrm{st}(A, \mathcal{U}) = X$. Moreover, X is *discretely star-Lindelöf* (or *in countable discrete web* by Gao and Yasui [YG]) iff $a(X) = \omega$.

The *star-Lindelöf number st-l*(X) [BM1] of a space X is defined as the smallest infinite cardinal κ such that for each open cover $\mathcal{U}$ of X there exists a subset $A \subseteq X$ such that $|A| \leq \kappa$ and $\mathrm{st}(A, \mathcal{U}) = X$. Furthermore, the ***absolute star-Lindelöf number a-st-l***(X) [B] of a space X is defined as the smallest infinite cardinal κ such that for each open cover $\mathcal{U}$ of X and each dense subset $D \subseteq X$, there exists a subset $A \subseteq D$ such that $|A| \leq \kappa$ and $\mathrm{st}(A, \mathcal{U}) = X$. If $a\text{-}st\text{-}l(X) = \omega$, we shall call X ***absolutely star-Lindelöf.***

Clearly, $st\text{-}l(X) \leq \min\{a(X), a\text{-}st\text{-}l(X)\}$ and $a(X) \leq e(X)$ for any space, where $e(X)$ is the extent of X. These relations can be summarized in the following diagram.

$$\begin{tikzcd}
 & & l(X) \arrow[dl] \arrow[ddrr] & & \\
 & e(X) \arrow[dl] & & & \\
a(X) \arrow[drr] & & & & a\text{-}st\text{-}l(X) \arrow[dll] \\
 & & s\text{-}l(X) & &
\end{tikzcd}$$

In [B], Bonanzinga constructed an example ([B], Example 3.18) demonstrating that the extent of an absolutely star-Lindelöf Hausdorff space can be $2^{\mathfrak{c}}$. Then she asked the following question.

Question 4. [B] *Is it true that the extent of absolutely star-Lindelöf Hausdorff (regular) spaces cannot be greater than $2^{\mathfrak{c}}$ $(\mathfrak{c})$?*

Matveev [M4] proved that for any uncountable cardinal κ, there exists a Tychonoff space X such that $a(X) = \omega$ and $e(X) = \kappa$. Since his space is not pseudocompact, Matveev asked the following question.

Question 5. [M4] *How big can be the extent of a pseudocompact and discretely star-Lindelöf Tychonoff space?*

These two questions were solved in [CS] recently, by comparing the Aquaro number and the absolute star-Lindelöf number for Tychonoff spaces. In fact, the following results of Cao and Song provide more information than the above questions asked.

Theorem 6. [CS] *For any cardinal κ, there exists an absolutely star-Lindelöf Tychonoff space X such that $a(X) \geq \kappa$.*

Theorem 7. [CS] *For any cardinal κ, there exists a pseudocompact and discretely star-Lindelöf Tychonoff space X such that $a\text{-}st\text{-}l(X) \geq \kappa$ and $e(X) \geq k$.*

4. More on cardinal invariants

In this section, we shall consider more cardinal invariants associated with star covering properties. Let X be any space. The 2-*star-Linelöf number st*$_2$-*l*(X) of X is defined as the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of X, there exists $A \subseteq X$

such that $|A| \leq \kappa$ and $\mathrm{st}^2(A, \mathcal{U}) = X$. In a similar way, the *$n\frac{1}{2}$-star-Lindelöf number* $st_{n\frac{1}{2}}\text{-}l(X)$ $(n = 1, 2)$ of X is defined as the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$. It turns out that for regular spaces, $st_{2\frac{1}{2}}\text{-}l(X)$ is not a new cardinal invariant. In fact, it can be shown that $st_{2\frac{1}{2}}\text{-}l(X) = dc(X)$ for a regular space X, where $dc(X)$ is the smallest infinite cardinal κ such that every family of discrete open sets of X is at most κ. In the literature, $dc(X)$ is called the *discrete cellularity* of X. By definition, for any regular space X, we have

$$dc(X) \leq st_2\text{-}l(X) \leq st_{1\frac{1}{2}}\text{-}l(X) \leq st\text{-}l(X).$$

A natural question is to decide how large the gaps between these cardinal invariants are for Tychonoff spaces. The following theoem, which improves some results of Matveev in [M6], was established in [CKNS] recently.

Theorem 8. [CKNS] *For each regular cardinal κ, there exists a Tychonoff space X such that $st_{n\frac{1}{2}}\text{-}l(X) = \omega$ and $st_n\text{-}l(X) \geq \kappa$, where $n \in \{1, 1\frac{1}{2}, 2\}$.*

Now, we shall have a look at possible bounds for the absolute star-Lindelöf number of a space. For a Hausdorff space X, by a result of de Groot in 1965, we have

$$a\text{-}st\text{-}l(X) \leq 2^{d(X)}$$

In the class of Tychnoff spaces, we can show that the bound $2^{d(X)}$ is always attainable.

Theorem 9. *For each cardinal κ, there exists a Tychonoff space X such that $d(X) = \kappa$ and $a\text{-}st\text{-}l(X) = 2^\kappa$.*

A classical result of Hajnal and Juhász claims that for any Hausdorff space X, $|X| \leq 2^{c(X)\cdot\chi(X)}$, where $c(X)$ and $\chi(X)$ denote the Souslin number (or the cellularity) and the character of X respectively. To establish an analogy to this result for the absolute star-Lindelöf number of a space X, let $ccc\text{-}l(X)$ (called *ccc-Lindelöf number* of X in [BM2]) denote the smallest infinite cardinal κ such that every open cover of X has an open refinement whose pairwise disjoint subfamilies have cardinality at most κ. It is easy to see that $ccc\text{-}l(X) \leq \min\{st\text{-}l(X), c(X)\}$ for any space X.

Theorem 10. *For any Hausdorff space X, $a\text{-}st\text{-}l(X) \leq 2^{\pi\chi(X)\cdot cccc\text{-}l(X)}$, where $\pi\chi(X)$ denotes the π-character of X.*

For a normal space X, it is known that the following hold
(a) $dc(X) = st_2\text{-}l(X)$;
(b) $e(X) \leq 2^{st\text{-}l(X)}$ [M4] ;
(c) For any cardinal κ, there exists a $1\frac{1}{2}$-star-Lindedlöf normal space X such that $st\text{-}l(X) \geq \kappa$ [CS] ;
(d) For any cardinal κ, there esists a countably compact normal space X such that $a\text{-}st\text{-}l(X) \geq \kappa$ [CS].

Furthermore, it is also interesting to consider if the bound $2^{st\text{-}l(X)}$ in (b) is attainable in the class of normal spaces. Recently, Levy [L] announced that the existence of a star-Lindelöf normal space X with $|X| = e(X) = \mathfrak{c}$ is independent of ZFC. More precisely, he proved that, on one hand, CH implies that there is no star-Lindelöf normal space X with $|X| = e(X) = \mathfrak{c}$; on the other hand, if $\mathfrak{c}$ is a limit cardinal such that $\omega \leq \alpha < \mathfrak{c}$ implies $2^\alpha = \mathfrak{c}$, then there exists a normal separable space X with $|X| = e(X) = \mathfrak{c}$.

5. Iterated starcompact spaces

A space X is called *n-$\mathcal{P}$-starcompact* ($n \in \mathbb{N}$), where $\mathcal{P}$ is some topological property, if for each open cover $\mathcal{U}$ of X there exists some $A \subseteq X$ with the property $\mathcal{P}$ such that $\mathrm{st}^n(A, \mathcal{U}) = X$. Clearly, n-starcompactness is n-$\mathcal{P}$-starcompactness for $\mathcal{P}$="finiteness". Ikenaga [I2], Hiremath [H] and Song [So] have studied the notion of 1-$\mathcal{P}$-starcompactness, when $\mathcal{P}$= compactness, σ-compactness, or Lindelöfness. In the literature, 1-compactness-starcompact (1-Lindelöfness-starcompact) spaces are also called *$\mathcal{K}$-starcompact* (*$\mathcal{L}$-starcompact*). Recently, Kim [K] has considered n-$\mathcal{P}$-starcompactness when $\mathcal{P}$ = k-starcompactness, and he called them *iterated star covering properties*. By definition, a space X is said to be *(n, k)-starcompact* [M3] if for every open cover $\mathcal{U}$ of X there exists an k-starcompact subspace $A \subseteq X$ such that $\mathrm{St}^n(A, \mathcal{U}) = X$.

Basic properties and inter-relationships of (n, k)-starcompact spaces were discussed in [K]. In particular, it is shown that a $(1, 1)$-starcompact Tychonoff space may not be $1\frac{1}{2}$-starcompact, and a $(1, 2)$-starcompact Hausdorff space may not be $(2, 1)$-starcompact. However, every $(1, 1)$-starcompact meta-Lindelöf T_1-space is $1\frac{1}{2}$-starcompact.

6. Some open questions

To conclude this short article, we list some open questions in this area in this section. First of all, it is shown in [DRRT] that under CH or $\mathfrak{b} = \mathfrak{c}$, every $1\frac{1}{2}$-starcompact Moore space is compact and metrizable. Since, every Moore space has a G_δ-diagonal, and every countably compact space with a G_δ-diagonal is compact and metrizable, it is a question to decide if every $1\frac{1}{2}$-starcompact Moore space is countably compact.

Question 11. *In ZFC, is every $1\frac{1}{2}$-starcompact Moore space countably compact?*

Question 12. *Is every 2-star-Lindelöf normal space $1\frac{1}{2}$-star-Lindelöf? If the answer is negative, how large could be the $1\frac{1}{2}$-star-Lindelöf number of a 2-star-Lindelöf normal space?*

Question 13. *Is every star-Lindelöf collectionwise normal space absolutely star-Lindelöf? If the answer is negative, how large could be the absolute star-Lindelöf number of a star-Lindelöf collectionwise normal space?*

Question 14. *Does there exist an absolutely star-Lindelöf normal space which is not discretely star-Lindelöf?*

A *Q-set* is an uncountable set of reals such that in the subspace topology, every subset of it is an F_σ. It is well-known that the existence of a Q-set is independent of ZFC. Assume the existence of a Q-set B. Consider the space

$$X = (B \times \{0\}) \cup (\mathbb{R} \times (0, +\infty))$$

with the subspace topology of the Niemytzki upper half plane. Then X is a discretely star-Lindelöf and absolutely star-Lindelöf normal space. But, the extent $e(X) = |B| \geq \omega_1$.

Question 15. *Does there exist a discretely star-Lindelöf normal space with uncountable extent in ZFC?*

Question 16. *Does there exist an absolutely star-Lindelöf normal space with uncountable extent in ZFC?*

Question 17. *Is every star-Lindelöf normal space discretely star-Lindelöf?*

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