数理解析研究所講究録 1248 巻 2002 年 18-23

NON-TRIVIAL LIMIT LAWS IN TOPOLOGICAL GROUPS: HOW SIMPLE CAN THEY BE?

S. MORRIS, P. NICKOLAS, AND D. SHAKHMATOV

ABSTRACT. A *limit law* is a map $f : (D, \leq) \to F(X)$ from a directed set $(D, \leq)$ to a free group $F(X)$ over some set X. A topological group G *satisfies* limit law f (we also say that f *holds* in G) provided that for every group homomorphism $\pi : F(X) \to G$ from $F(X)$ to G and each open set U containing the identity elelemt e_G of G there exists some $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$. For a group G a limit law $f : (D, \leq) \to F(X)$ is called G-*algebraic* provided that there exists $d \in D$ such that $\pi(f(c)) = e_G$ whenever $c \geq d$ and $\pi : F(X) \to G$ is a group homomorphism. A limit law that is not G-algebraic is called *essentially G-topological*. Main result: If a a Hausdorff group G satisfies some essentially G-topological limit law $f : (D, \leq) \to F(X)$ such that $(D, \leq)$ is either a linearly ordered set or a countable partially ordered set, then G also satisfies some essentially G-topological limit law $f' : (\mathbb{N}, \leq) \to F(X)$ having the usual set of integers $(\mathbb{N}, \leq)$ as its domain. It follows that if G is one of the three classical locally compact groups, $\mathbb{Z}$ (integers), $\mathbb{R}$ (reals) or $\mathbb{T}$ (unit circle), then every limit law with a linearly ordered domain that holds in G is G-algebraic.

As usual, the symbol $F(X)$ denotes the free group over a set X. If G is a group, then e_G denotes the identity element of G. The identity element of $F(X)$ will be simply denoted by e.

A *partially ordered set* (or shortly, *poset*) is a pair $(D, \leq)$ consisting of a set D together with a ralation $\leq$ which is:

(i) *reflexive*, i.e. $d \leq d$ for each $d \in D$, and

(ii) *transitive*, i.e. $d_0 \leq d_1$ and $d_1 \leq d_2$ implies $d_0 \leq d_2$.

A partially ordered set $(D, \leq)$ is *directed* provided that for every pair $d_0, d_1 \in D$ of elements of D there exists $d \in D$ such that $d_0 \leq d$ and $d_1 \leq d$.

Limit laws were introduced in [4] and recently studied extensively in [1, 3]. A *limit law* is a map $f : (D, \leq) \to F(X)$ from a directed set $(D, \leq)$ to a free group $F(X)$ over some set X. We say that a limit law $f : (D, \leq) \to F(X)$ *holds* in a topological group G, or that G *satisfies law* f, provided that for every group homomorphism $\pi : F(X) \to G$ from $F(X)$ to G the directed set $\{\pi(f(d)) : d \in D\}$ converges to the identity element e_G of G; that is, for every open set U containing e_G there exists $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$.

Let G be a group. A limit law $f : (D, \leq) \to F(X)$ will be called G-*algebraic* provided that there exists some $d \in D$ such that $\pi(f(c)) = e_G$ whenever $c \geq d$ and $\pi : F(X) \to G$ is a group homomorphism. If G is a topological group, then a G-algebraic limit law automatically holds in G for an obvious algebraic reason, thereby justifying its name.

Main results of this paper were obtained in March-April of 1997 when the third author was visiting Mathematical Analysis Research Group of University of Wollongong. He would like to thank cordially the first two authors and University of Wollongong for their generous hospitality and financial support.

This is an extended abstract of the talk presented by the third author at the Workshop on General and Geometric Topology and its Application held on October 17–19, 2001 at the Research Institute for Mathematical Sciences (RIMS) of Kyoto University (Kyoto, Japan).

Note that the topology of the group G plays absolutely no role in "deciding" whether a G-algebraic law holds in G or not; everything is determined by the algebraic structure of G. Therefore, from a topological point of view, G-algebraic laws are trivial and not particularly interesting. A limit law that is not G-algebraic will be called *essentially G-topological*. Contrary to G-algebraic laws, the topology of the group G plays a crucual role in (really!) deciding whether an essentially G-topological law holds in G or not; this both explains the choice of our terminology and indicates that essentially G-topological laws are of special interest from the topological point of view.

It appears to be natural to adopt the (luck of) complexity of a partially ordered set $(D, \leq)$ as a measure of "simplicity" of a limit law $f : (D, \leq) \to F(X)$. The main purpose of this article is to demonstrate that the three classical locally compact groups, the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers and the unit circle group $\mathbb{T}$, do not satisfy any "simple" essentially G-topological law (see Corollary 14).

Lemma 1. *Let $f : (D, \leq) \to F(X)$ be a limit law that holds in a Hausdorff topological group G. If $(D, \leq)$ has a biggest element (in particular, if the poset $(D, \leq)$ if finite), then f is G-algebraic.*

Proof. Let a be a biggest element of $(D, \leq)$. First suppose that there exists a group homomorphism $\pi : F(X) \to G$ such that $\pi(f(a)) \neq e_G$. Then $U = G \setminus \{\pi(f(a))\}$ is an open neighbourhood of e_G by Hausdorffness of G. Since f holds in G, there exists $c \in D$ with $\pi(f(d)) \in U$ for all $d \geq c$. Since a is the biggest element of $(D, \leq)$, it follows that $\pi(f(a)) \in U = G \setminus \{\pi(f(a))\}$, a contradiction. Therefore $\pi(f(a)) = e_G$ for every group homomorphism $\pi : F(X) \to G$. Since a is the biggest element of $(D, \leq)$, we also have that $\pi(f(d)) = e_G$ whenever $d \in D$, $d \geq a$ and $\pi : F(X) \to G$ is a group homomorphism. This means that f is G-algebraic. □

A subset C of a directed set $(D, \leq)$ is called *cofinal in* $(D, \leq)$ if for every $d \in D$ there exists $c \in C$ with $d \leq c$.

Let f and g be limit laws. We will write $f \Leftarrow g$ provided that f holds in every topological group in which g holds.

Lemma 2. *If $f : (D, \leq) \to F(X)$ is a limit law and C is a cofinal subset of $(D, \leq)$, then the restriction $f|_C : C \to F(X)$ of f to C is a limit law and $f|_C \Leftarrow f$.*

Proof. Being a cofinal subset of a directed set $(D, \leq)$, the partially ordered set $(C, \leq)$ is also directed, and so $f|_C$ is a limit law. Let G be a topological group in which f holds. We are going to prove that $f|_C$ also holds in G. Indeed, let $\pi : F(X) \to G$ be a homomorphism from $F(X)$ to G. Let U be an open subset of G which contains the identity element e_G. Since f holds in G, there exists some $d \in D$ such that $\pi(f(c)) \in U$ for all $c \geq d$. Since C is cofinal in $(D, \leq)$, one can find $c_0 \in C$ with $c_0 \geq d$. Clearly. $\pi(f(c)) \in U$ for all $c \geq c_0$. □

A *sequential law* is a limit law $f : (\mathbb{N}, \leq) \to F(X)$ with the set $(\mathbb{N}, \leq)$ of natural numbers as its directed set. A *countable law* is a limit law $f : (D, \leq) \to F(X)$ whose domain $(D, \leq)$ is a countable directed set.

In view of Lemma 1, the cardinality of the domain of an essentially G-topological law must be infinite, and thus countable laws are potentially the simplest possible essentially G-topological laws. This explains why our first theorem considers such laws.

Theorem 3. *Let G be a Hausdorff group. If G satisfies some essentially G-topological countable law, then it also satisfies some essentially G-topological sequential law.*

Proof. Let G be a Hausdorff group and let $f : (D, \leq) \to F(X)$ be an essentially G-topological countable limit law that holds in G. Let $D = \{d_n : n \in \mathbb{N}\}$ be an enumeration of D. According to Lemma 1 the poset $(D, \leq)$ does not have the biggest element. Using this fact, directedness of $(D, \leq)$ and the fact that f is essentially G-topological we can easily choose, by induction on n, an element $c_n \in D$ and a group homomorphism $\pi_n : F(X) \to G$ such that $d_n \leq c_n$, $c_{n-1} < c_n$ and $\pi_n(f(c_n)) \neq e_G$. By our construction, $C = \{c_n : n \in N\}$ is cofinal in $(D, \leq)$ and therefore $f|_C \Leftarrow f$ by Lemma 2. Since f holds in G, so does $f|_C$. By our construction, $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$ and $\pi_n(f(c)) \neq e_G$ for all $c \in C$. Thus $f|_C$ is an essentially G-topological sequential law. □

Recall that a cardinal τ is called *singular* provided that there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. A cardinal is *regular* if it is not singular.

If X is a set, G is a group and $\varphi : X \to G$ is a map, then $\widehat{\varphi} : F(X) \to G$ will denote the (unique) extension of φ over $F(X)$ that is a group homomorphism. If $y \in F(X)$ and $y \neq e$, then $\operatorname{supp} y$ denotes the smallest subset Y of X such that y belongs to the subgroup of $F(X)$ generated by Y. Note that $\operatorname{supp} y$ is always finite.

Our next lemma establishes an algebraic fact about free groups that is perhaps of some independent interest.

Lemma 4. *If X is a set and Z is a subset of $F(X)$ of uncountable regular cardinality, then there exist $Y \subseteq Z$, $y^* \in Y$ and a map $\varphi : X \to X$ such that $|Y| = |Z|$ and $\widehat{\varphi}(y) = y^*$ for all $y \in Y$.*

Proof. Without loss of generality we will assume that $z \neq e$ for each $z \in Z$. Note that $\{\operatorname{supp} z : z \in Z\}$ is a family of non-empty finite subsets of X, so by the Δ-system Lemma (see, for example, [2, Ch. II, Theorem 1.6]) there exists a finite (possibly empty) set $T \subseteq X$ and $Z' \subseteq Z$ such that $|Z'| = |Z|$ and $\operatorname{supp} z \cap \operatorname{supp} z' = T$ whenever $z, z' \in Z'$ and $z \neq z'$. For each $n \in \mathbb{N} \setminus \{0\}$ define $Z'_n = \{z \in Z' : |\operatorname{supp} z| = n\}$ and note that $Z' = \bigcup\{Z'_n : n \in \mathbb{N} \setminus \{0\}\}$. Since $|Z'| = |Z|$ is an uncountable regular cardinal, it follows that $|Z'_n| = |Z'|$ for some $n \in \mathbb{N} \setminus \{0\}$. Pick arbitrarily $z^* \in Z'_n$. For each $z \in Z'_n \setminus \{z^*\}$ choose a bijection $h_z : \operatorname{supp} z \to \operatorname{supp} z^*$ such that $h_z(t) = t$ for all $t \in T$, and let $\widehat{h_z} : F(\operatorname{supp} z) \to F(\operatorname{supp} z^*)$ be the natural homomorphic extension of h_z over $F(\operatorname{supp} z)$. Since the set $F(\operatorname{supp} z^*)$ is at most countable, and $|Z'_n \setminus \{z^*\}| = |Z'_n| = |Z'| = |Z|$ is an uncountable regular cardinal, there exist $g \in F(\operatorname{supp} z^*)$ and $Y \subseteq Z'_n$ such that $|Y| = |Z'_n|$ and $\widehat{h_y}(y) = g$ for all $y \in Y$. Pick $y^* \in Y$ arbitrarily. For each $y \in Y$ define the map $f_y : \operatorname{supp} y \to \operatorname{supp} y^*$ by $f_y = h_{y^*}^{-1} \circ h_y$ and note that the restriction of f_y to T is the idenity map of T. This allows us to define the map $\varphi : X \to X$ by $\varphi(x) = f_y(x)$ if $x \in \operatorname{supp} y$ for some $y \in Y$ and $\varphi(x) = x$ if $x \in X \setminus \bigcup\{\operatorname{supp} y : y \in Y\}$. Finally, by our construction

$$\widehat{\varphi}(y) = \widehat{f_y}(y) = \widehat{h_{y^*}}^{-1}(\widehat{h_y}(y)) = \widehat{h_{y^*}}^{-1}(g) = y^*$$

for each $y \in Y$. □

Another potential candidate for a "simple" limit law is the law with a linearly ordered domain. Recall that a poset $(D, \leq)$ is *linearly ordered* provided that for every pair d, d' of elements of D either $d \leq d'$ or $d' \leq d$ holds. A *linearly ordered law* is a limit law $f : (D, \leq) \to F(X)$ whose domain $(D, \leq)$ is a linearly ordered set. Sequential laws are particular types of linearly ordered laws.

Theorem 5. *If a Hausdorff topological group G satisfies some essentially G-topological linearly ordered law, then G also satisfies some essentially G-topological sequential law.*

Proof. The proof of this theorem will be split into a sequence of claims.

Let G be a Hausdorff topological group and $f : (D, \leq) \to F(X)$ be an essentially G-topological linearly ordered law which holds in G. Let τ be the smallest cardinality of a cofinal subset of $(D, \leq)$. Choose a cofinal subset $E = \{d_\alpha : \alpha < \tau\}$ of $(D, \leq)$ of cardinality τ.

Claim 6. *If $C \subseteq D$ and $|C| < \tau$, then there exists $d \in D$ such that $c < d$ for all $c \in C$.*

Proof. Since τ is a minimal cardinality of a cofinal subset of $(D, \leq)$, the set C cannot be cofinal in $(D, \leq)$. Therefore there exists some $d \in D$ such that for all $c \in C$ the inequality $d \leq c$ does *not* hold. It is precisely here where we use the fact that $(D, \leq)$ is a linearly ordered set to conclude that $c < d$ for all $c \in C$. □

By transfinite recursion we will choose points $\{c_\alpha : \alpha < \tau\} \subseteq D$ and a family $\{\pi_\alpha : \alpha < \tau\}$ of group homomorphisms from $F(X)$ to G in such way that, for every $\alpha < \tau$, one has $d_\alpha < c_\alpha$, $\pi_\alpha(f(c_\alpha)) \neq e_G$ and $c_\beta < c_\alpha$ for $\beta < \alpha$. Assume that $\alpha < \tau$ and that points $\{c_\beta : \beta < \alpha\} \subseteq D$ and group homomorphisms $\{\pi_\beta : \beta < \alpha\}$ from $F(X)$ to G have already been chosen. From Claim 6 it follows that there exists $d \in D$ such that $c_\beta < d$ for all $\beta < \alpha$. Since $(D, \leq)$ is directed, $d_\alpha \leq d'$ and $d \leq d'$ for some $d' \in D$. Now use the fact that f is essentially G-topological to pick $c_\alpha \in D$ and a group homomorphism $\pi_\alpha : F(X) \to G$ such that $d' \leq c_\alpha$ and $\pi_\alpha(f(c_\alpha)) \neq e_G$. Clearly c_α has all necessary properties.

Claim 7. *$\beta < \alpha < \tau$ implies $c_\beta < c_\alpha$.*

Proof. This was guaranteed as part of our inductive construction. □

Claim 8. *$C = \{c_\alpha : \alpha < \tau\}$ is a cofinal subset of $(D, \leq)$.*

Proof. $E = \{d_\alpha : \alpha < \tau\}$ is cofinal in $(D, \leq)$ and $d_\alpha \leq c_\alpha$ for all $\alpha < \tau$ implies that C is also cofinal in $(D, \leq)$. □

Claim 9. *If Γ is a cofinal subset of τ, then $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$.*

Proof. Suppose that Γ is cofinal in τ. Let $d \in D$. From Claim 8 it follows that $d \leq c_\beta$ for some $\beta < \tau$. Cofinality of Γ in τ yields $\gamma \in \Gamma$ such that $\beta < \gamma$. Now $d \leq c_\beta \leq c_\gamma$ by Claim 7. □

Claim 10. *τ is infinite.*

Proof. If τ is finite, then $(D, \leq)$ must have a biggest element a, and then f will be G-algebraic by Lemma 1. □

Claim 11. *τ is a regular cardinal.*

Proof. Assume the contrary, i.e. that τ is singular. Then there exists a cardinal $\kappa < \tau$ and a transfinite sequence $\{\tau_\beta : \beta < \kappa\}$ of cardinals such that $\sup\{\tau_\beta : \beta < \kappa\} = \tau$ and $\tau_\beta < \tau$ for each $\beta < \kappa$. For each $\beta < \kappa$ applying $\tau_\beta < \tau$ and Claim 6 to the set $C_\beta = \{d_\alpha : \alpha < \tau_\beta\}$ one can find $b_\beta \in D$ such that $d_\alpha < b_\beta$ for $\alpha < \tau_\beta$. We now claim that the set $\{b_\beta : \beta < \kappa\}$ is cofinal in $(D, \leq)$, thereby contradicting minimality of τ. Indeed, let $d \in D$. Since E is cofinal in $(D, \leq)$, one has $d \leq d_\alpha$ for some $\alpha < \tau$. Since $\sup\{\tau_\beta : \beta < \kappa\} = \tau$, there exists $\beta < \kappa$ with $\alpha < \tau_\beta$. It remains only to note that $d \leq d_\alpha < b_\beta$. □

Claim 12. *τ is countable.*

Proof. Assume the contrary. Then τ is an uncountable regular cardinal by Claims 10 and 11. We can now apply Lemma 4 to the set $Z = \{f(c_\alpha) : \alpha < \tau\}$ to find a subset $\Gamma \subseteq \tau$, an ordinal $\gamma^* \in \Gamma$ and a map $\varphi : X \to X$ such that $|\Gamma| = \tau$ and $\hat{\varphi}(f(c_\gamma)) = f(c_{\gamma^*})$. Recall now that the group homomorphism $\pi_{\gamma^*} : F(X) \to G$ satisfies $g = \pi_{\gamma^*}(f(c_{\gamma^*})) \neq e_G$. Since G is Hausdorff, $U = G \setminus \{g\}$ is an open neighbourhood of e_G in G. Define a group homomorphism $\pi : F(X) \to G$ via $\pi = \pi_{\gamma^*} \circ \hat{\varphi}$. Then $\pi(f(c_\gamma)) = \pi_{\gamma^*}(\hat{\varphi}(f(c_\gamma))) = \pi_{\gamma^*}(f(c_{\gamma^*})) = g$ for $\gamma \in \Gamma$, and therefore

$$\pi(f(c_\gamma)) \notin U \text{ for each } \gamma \in \Gamma. \tag{1}$$

Since $|\Gamma| = \tau$, Γ is cofinal in τ, and so the set $\{c_\gamma : \gamma \in \Gamma\}$ is cofinal in $(D, \leq)$ by Claim 9. Cofinality of $\{c_\gamma : \gamma \in \Gamma\}$ in $(D, \leq)$ and (1) imply that f does not hold in G, a contradiction. □

By Claim 8 $C = \{c_\alpha : \alpha < \tau = \omega\}$ is a cofinal subset of $(D, \leq)$, and so the restriction $h = f|_C$ of f to C is a limit law such that $h \Leftarrow f$ (see Lemma 2). Since f holds in G, so does h. From the choice of homomorphisms π_α it follows that h is essentially G-topological. Claim 7 implies that $(C, \leq)$ is order isomorphic to $(\mathbb{N}, \leq)$, i.e. that h is a sequential law. □

From Theorems 3 and 5 we immediately get the following

Corollary 13. *For a Hausdorff group G the following conditions are equivalent:*
(i) G satisfies some essentially G-topological linearly ordered law,
(ii) G satisfies some essentially G-topological countable law,
(iii) G satisfies some essentially G-topological sequential law.

If G is either the group $\mathbb{Z}$ of integer numbers, the group $\mathbb{R}$ of real numbers or the unit circle group $\mathbb{T}$, then each sequential law that holds in G is G-algebraic [1]. From this result and Corollary 13 we obtain

Corollary 14. *Let G be one of the groups $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$. Then all countable or linearly ordered laws that hold in G are G-algebraic.*

If a locally compact Abelian group G satisfies some essentially G-topological sequential law, then G is totally disconnected [1]. From this and Corollary 13 we get our last

Corollary 15. *If a locally compact Abelian group G satisfies either some essentially G-topological linearly ordered law or some essentially G-topological countable law, then G is totally disconnected.*

References

[1] R. D. Kopperman, M. W. Mislove, S. A. Morris, P. Nickolas, V. Pestov and S. Svetlichny, Limit laws for wide varieties of topological groups, Houston J. Math. 22 (1996), 307–328.

[2] K. Kunen, Set theory. An introduction to independence proofs (Studies in Logic and the Foundations of Mathematics, 102), North-Holland Publishing Co., Amsterdam-New York, 1980.

[3] S. A. Morris, P. Nickolas and V. Pestov, Limit laws for wide varieties of topological groups. II, Houston J. Math. 26 (2000), no. 1, 17–27.

[4] W. Taylor, Varieties of topological algebras, J. Austral. Math. Soc. (Series A), 23 (1977), 207–241.

(S. Morris)
SCHOOL OF INFORMATION TECHNOLOGY AND MATHEMATICAL SCIENCES
UNIVERSITY OF BALLARAT, P.O. BOX 663
BALLARAT, VICTORIA 3353, AUSTRALIA
E-mail address: S.Morris@ballarat.edu.au

(P. Nickolas)
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WOLLONGONG
NORTHFIELDS AVE., WOLLONGONG, NEW SOUTH WALES 2522, AUSTRALIA
E-mail address: peter_nickolas@uow.edu.au

(D. Shakhmatov)
DEPARTAMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE
EHIME UNIVERSITY, MATSUYAMA 790–8577, JAPAN
E-mail address: dmitri@dpc.ehime-u.ac.jp