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HOMOTOPY TYPES OF DIFFEOMORPHISM GROUPS OF NONCOMPACT 2-MANIFOLDS

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1. INTRODUCTION

This is a report on the study of topological properties of the diffeomorphism groups of noncompact smooth 2-manifolds endowed with the compact-open C^∞-topology [18].

When M is a compact smooth 2-manifold, the diffeomorphism group $\mathcal{D}(M)$ with the compact-open C^∞-topology is a smooth Fréchet manifold [6, Section I.4], and the homotopy type of the identity component $\mathcal{D}(M)_0$ has been classified by S. Smale [15], C. J. Earle and J. Eell [4], et. al. In the C^0-category, for any compact 2-manifold M, the homeomorphism group $\mathcal{H}(M)$ with the compact-open topology is a topological Fréchet manifold [3, 11, 19], and the homotopy type of the identity component $\mathcal{H}(M)_0$ has been classified by M. E. Hamstrom [7].

Recently we have shown that $\mathcal{H}(M)_0$ is a topological Fréchet-manifold even if M is a noncompact connected 2-manifold, and have classified its homotopy type [17]. The argument in [17] is based on the following ingredients: (i) the ANR-property and the contractibility of $\mathcal{H}(M)_0$ for compact M, (ii) the bundle theorem connecting the homeomorphism group $\mathcal{H}(M)_0$ and the embedding spaces of submanifolds into M [16, Corollary 1.1], and (iii) a result on the relative isotopies of 2-manifolds [17, Theorem 3.1]. The same strategy based on the C^∞-versions of these results implies a corresponding conclusion for the diffeomorphism groups of noncompact smooth 2-manifolds.

Suppose M is a smooth 2-manifold and X is a closed subset of M. We denote by $\mathcal{D}_X(M)$ the group of C^∞-diffeomorphisms h of M onto itself with $h|_X = id_X$, endowed with the compact-open C^∞-topology [9, Ch.2, Section 1], and by $\mathcal{D}_X(M)_0$ the identity connected component of $\mathcal{D}_X(M)$.

The following is our main result:

Theorem 1.1. *Suppose M is a noncompact connected smooth 2-manifold without boundary.*
(1) $\mathcal{D}(M)_0$ is a topological ℓ_2-manifold.
(2) (i) $\mathcal{D}(M)_0 \simeq \mathbb{S}^1$ if $M =$ a plane, an open Möbius band or an open annulus.
(ii) $\mathcal{D}(M)_0 \simeq *$ in all other cases.

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Any separable infinite-dimensional Fréchet space is homeomorphic to the Hilbert space $\ell_2 \equiv \{(x_n) \in \mathbb{R}^\infty : \sum_n x_n^2 < \infty\}$. A topological ℓ_2-manifold is a separable metrizable space which is locally homeomorphic to ℓ_2. Topological types of ℓ_2-manifolds are classified by their homotopy types. Theorem 1.1 implies the following conclusion:

Corollary 1.1. *(i) $\mathcal{D}(M)_0 \cong \mathbb{S}^1 \times \ell_2$ if M = a plane, an open Möbius band or an open annulus.*
(ii) $\mathcal{D}(M)_0 \cong \ell_2$ in all other cases.

For the subgroup of diffeomorphisms with compact supports, we have the following results: Let $\mathcal{D}(M)_0^c$ denote the subgroup of $\mathcal{D}(M)_0$ consisting of $h \in \mathcal{D}(M)$ which admits a C^∞-isotopy with a compact support, $h_t : M \to M$ such that $h_0 = id_M$ and $h_1 = h$.

We say that a subspace A of a space X has the homotopy negligible (h.n.) complement in X if there exists a homotopy $\varphi_t : X \to X$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ $(0 < t \leq 1)$. In this case, the inclusion $A \subset X$ is a homotopy equivalence, and X is an ANR iff A is an ANR.

Theorem 1.2. *Suppose M is a noncompact connected smooth 2-manifold without boundary. Then $\mathcal{D}(M)_0^c$ has the h.n. complement in $\mathcal{D}(M)_0$*

Corollary 1.2. *(1) $\mathcal{D}(M)_0^c$ is an ANR.*
(2) The inclusion $\mathcal{D}(M)_0^c \subset \mathcal{D}(M)_0$ is a homotopy equivalence.

Section 2 contains fundamental facts on diffeomorphism groups of 2-manifolds and ℓ_2-manifolds. Section 3 contains a sketch of proofs of Theorems 1.1 and 1.2.

2. Fundamental properties of diffeomorphism groups

In this preliminary section we list fundamental facts on diffeomorphism groups of 2-manifolds (general properties, bundle theorem, homotopy type, relative isotopies, etc) and basic facts on ANR's and ℓ_2-manifolds. Throughout the paper all spaces are separable and metrizable and maps are continuous.

2.1. General property of diffeomorphism groups.

Suppose M is a smooth n-manifold possibly with boundary and X is a closed subset of M.

Lemma 2.1. *(c.f.* [9, Ch 2., Section 4]*, etc)*
$\mathcal{D}_X(M)$ is a topological group, which is separable, completely metrizable, infinite-dimensional and not locally compact.

When N is a smooth submanifold of M, the symbol $\mathcal{E}_X(N, M)$ denotes the space of C^∞-embeddings $f : N \hookrightarrow M$ with $f|_X = id_X$ with the compact-open C^∞-topology, and $\mathcal{E}_X(N, M)_0$ denotes the connected component of the inclusion $i_N : N \subset M$ in $\mathcal{E}_X(N, M)$.

Lemma 2.2. *(i) Suppose M is a smooth manifold without boundary, N is a comapct smooth submanifold of M and X is a closed subset of N. Then $\mathcal{E}_X(N, M)$ is a Fréchet manifold.*
(ii) Suppose M is a compact smooth n-manifold and X is a closed subset of M with $\partial M \subset X$ or $\partial M \cap X = \emptyset$. Then $\mathcal{D}_X(M)$ is a Fréchet manifold.

In Lemma 2.2 $\mathcal{E}_X(N, M)_0$ and $\mathcal{D}_X(M)_0$ are path-connected. Thus any $h \in \mathcal{D}_X(M)_0$ can be joined with id_M by a path h_t $(t \in [0, 1])$ in $\mathcal{D}_X(M)_0$.

2.2. **Bundle theorems.**

The bundle theorem asserts that the natural restriction maps from diffeomorphism groups to embedding spaces are principal bundles [2, 12]. This has been used to study the homotopy types of diffeomorphism groups. This theorem also plays an essential role in our argument.

Suppose M is a smooth m-manifold without boundary, N is a compact smooth n-submanifold of M and X is a closed subset of N.

Case 1: $n < m$ [2, 12]

Let U be any open neighborhood of N in M.

Theorem 2.1. *For any $f \in \mathcal{E}_X(N, U)$ there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{E}_X(N, U)$ and a map $\varphi : \mathcal{U} \to \mathcal{D}_{X \cup (M \setminus U)}(M)_0$ such that $\varphi(g)f = g$ $(g \in \mathcal{U})$ and $\varphi(f) = id_M$.*

Corollary 2.1. *The restriction map $\pi : \mathcal{D}_{X \cup (M \setminus U)}(M)_0 \to \mathcal{E}_X(N, U)_0$, $\pi(h) = h|_N$, is a principal bundle with fiber $\mathcal{D}_{X \cup (M \setminus U)}(M)_0 \cap \mathcal{D}_N(M)$.*

Case 2: $n = m$

In this case we have a weaker conclusion: Suppose N' is a compact smooth n-submanifold of M obtained from N by attaching a closed collar $\partial N \times [0, 1]$ to ∂N. Let U be any open neighborhood of N' in M. We can apply Theorem 2.1 to $\partial N'$ to obtain the following result:

Theorem 2.2. *For any $f \in \mathcal{E}_X(N', U)$ there exist a neighborhood $\mathcal{U}'$ of f in $\mathcal{E}_X(N', U)$ and a map $\varphi : \mathcal{U}' \to \mathcal{D}_{X \cup (M \setminus U)}(M)_0$ such that $\varphi(g)f|_N = g|_N$ $(g \in \mathcal{U}')$ and $\varphi(f) = id_M$.*

For the sake of simplicity, we set $\mathcal{D}_0 = \mathcal{D}_{X \cup (M \setminus U)}(M)_0$, $\mathcal{E}_0 = \mathcal{E}_X(N, U)_0$, $\mathcal{E}'_0 = \mathcal{E}_X(N', U)_0$.

Consider the restriction map $p : \mathcal{E}'_0 \to \mathcal{E}_0$, $p(f) = f|_N$ and $\pi : \mathcal{D}_0 \to \mathcal{E}_0$, $\pi(h) = h|_N$. We have the pullback diagram:

$$\begin{array}{ccc} p^*(\mathcal{D}_0) & \xrightarrow{p_*} & \mathcal{D}_0 \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{E}_0' & \xrightarrow{p} & \mathcal{E}_0, \end{array}$$

where $p^*\mathcal{D}_0 = \{(f,h) \in \mathcal{E}_0' \times \mathcal{D}_0 \mid f|_N = h|_N\}$, $p_*(f,h) = h$ and $\pi_*(f,h) = f$. The map p_* admits a natural right inverse $q : \mathcal{D}_0 \to p^*\mathcal{D}_0$, $q(h) = (h|_{N'}, h)$. The group $\mathcal{D}_0 \cap \mathcal{D}_N(M)$ acts on $p^*\mathcal{D}_0$ by $(f,h)g = (f,hg)$ $(g \in \mathcal{D}_0 \cap \mathcal{D}_N(M))$.

Corollary 2.2.
(1) $\pi_* : p^*(\mathcal{D}_0) \to \mathcal{E}_0'$ is a principal bundle with fiber $\mathcal{D}_0 \cap \mathcal{D}_N(M)$.
(2) $p_* : p^*(\mathcal{D}_0) \to \mathcal{D}_0$ is a homotopy equivalence with the homotopy inverse $q : \mathcal{D}_0 \to p^*(\mathcal{D}_0)$.
(3) $p : \mathcal{E}_0' \to \mathcal{E}_0$ is a homotopy equivalence if $X \subset \operatorname{int} N$.

The statements (2) and (3) exhibit a close relation between the restriction map π and the pullback π_*.

2.3. Diffeomorphism groups of 2-manifolds.

Next we recall fundamental facts on diffeomorphism groups of compact 2-manifolds. The following theorem shows that $\mathcal{D}_X(M)_0 \simeq *$ except a few cases. The symbols $\mathbb{S}^1$, $\mathbb{S}^2$, $\mathbb{T}$, $\mathbb{P}$, $\mathbb{K}$, $\mathbb{D}$, $\mathbb{A}$ and $\mathbb{M}$ denote the 1-sphere, 2-sphere, torus, projective plane, Klein bottle, disk, annulus and Möbius band respectively.

Theorem 2.3. *([4, 15] etc.) Suppose M is a compact connected smooth 2-manifold. Then the homotopy type of $\mathcal{D}(M)_0$ is classified as follows:*

M	$\mathcal{D}(M)_0$
$\mathbb{S}^2$, $\mathbb{P}$	$SO(3)$
$\mathbb{T}$	$\mathbb{T}$
$\mathbb{K}$, $\mathbb{D}$, $\mathbb{A}$, $\mathbb{M}$	$\mathbb{S}^1$
all other cases	$*$

- $\mathcal{D}_\partial(\mathbb{D}) \simeq *$, $\mathcal{D}_\partial(\mathbb{M}) \simeq *$.
- *If X is a disjoint union of a compact smooth 2-submanifold and finitely many smooth circles and points in M and $\partial M \subset X$, then $\mathcal{D}_X(M)_0 \simeq *$.*

For 2-manifolds there is no difference among the conditions: homotopic, C^0-isotopic, C^∞-isotopic and joinable by a path in the diffeomorphism group. By [4] and a C^∞-analogue of [5] we have

Proposition 2.1. *Suppose M is a compact smooth 2-manifold.*
(1) Suppose N is a closed collar of ∂M. If $h \in \mathcal{D}_N(M)$ is homotopic to id_M rel N, then h is C^∞-isotopic to id_M rel N.
(2) Suppose N is a compact smooth 2-submanifold of M with $\partial M \subset N$. For $h \in \mathcal{D}_N(M)$, the

following conditions are equivalent:

(a) h is C^0-isotopic to id_M rel N.

(b) h is C^∞-isotopic to id_M rel N.

(c) $h \in \mathcal{D}_N(M)_0$.

In Corollaries 2.1 and 2.2 we have a principal bundle with fiber $\mathcal{G} \equiv \mathcal{D}_X(M)_0 \cap \mathcal{D}_N(M)$. The next theorem gives us a sufficient condition that $\mathcal{G} = \mathcal{D}_N(M)_0$. The symbol $\#X$ denotes the cardinal of a set X.

Theorem 2.4. *Suppose M is a compact connected smooth 2-manifold, N is a compact smooth 2-submanifold of M with $\partial M \subset N$, X is a subset of N. Suppose (M, N, X) satisfies the following conditions:*

(i) *$M \neq \mathbb{T}, \mathbb{P}, \mathbb{K}$ or $X \neq \emptyset$.*

(ii) *(a) if H is a disk component of N, then $\#(H \cap X) \geq 2$,*

(b) if H is an annulus or Möbius band component of N, then $H \cap X \neq \emptyset$,

(iii) *(a) if L is a disk component of $cl(M \setminus N)$, then $\#(L \cap X) \geq 2$,*

(b) if L is a Möbius band component of $cl(M \setminus N)$, then $L \cap X \neq \emptyset$.

Then we have:

(1) If $h \in \mathcal{D}_N(M)$ is C^0-isotopic to id_M rel X, then h is C^∞-isotopic to id_M rel N.

(2) $\mathcal{D}(M)_0 \cap \mathcal{D}_N(M) = \mathcal{D}_N(M)_0$.

Theorem 2.4 follows from [17, Theorem 3.1] and Proposition 2.1.

2.4. Basic properties of ANR's and ℓ_2-manifolds.

The ANR-property of diffeomorphism groups and embedding spaces is also essential in our argument. Here we recall basic properties of ANR's [8, 10, 13] and a topological characterization theorem of ℓ^2-manifolds.

A metrizable space X is called an ANR (absolute neighborhood retract) for metric spaces if any map $f : B \to X$ from a closed subset B of a metrizable space Y admits an extension to a neighborhood U of B in Y. If we can always take $U = Y$, then X is called an AR. It is known that X is an AR (an ANR) iff it is a retract of (an open subset of) a normed space. Any ANR has a homotopy type of CW-complex. An AR is exactly a contractible ANR.

We apply the following criterion of ANR's:

Lemma 2.3. *(1) A space X is an ANR iff every point of X has an ANR neighborhood in X.*
(2) If $X = \cup_{i=1}^\infty U_i$, U_i is open in X and $U_i \subset U_{i+1}$ and if each U_i is an AR, then X is also an

(3) In a fiber bundle, the total space is an ANR iff both the base space and the fiber are ANR's.

(4) A metric space X is an ANR iff for any $\varepsilon > 0$ there is an ANR Y and maps $f : X \to Y$ and $g : Y \to X$ such that gf is ε-homotopic to id_X.

Since any Fréchet space is an AR, every Fréchet manifold is an ANR.

Finally we recall a characterization of ℓ_2-manifold topological groups [3, 19].

Theorem 2.5. *A topological group is an ℓ_2-manifold iff it is a separable, non locally compact, completely metrizable ANR.*

The diffeomorphism group $\mathcal{D}(M)_0$ satisfies all conditions except the ANR property (Lemma 2.1). Thus the proof of Theorem 1.1 (1) reduces to the verification of ANR property of $\mathcal{D}(M)_0$. The latter follows from the ANR property of the diffeomorphism groups and embedding spaces of compact 2-manifolds (Lemma 2.2).

3. Proof of Main Theorems

In this section we give a sketch of proofs of Theorems 1.1 and 1.2 in the case where $M \neq$ a plane, an open Möbius band, an open annulus. Below we assume that M is a noncompact connected smooth 2-manifold without boundary and that $M \neq$ a plane, an open Möbius band, an open annulus.

We can write as $M = \cup_{i=0}^{\infty} M_i$, where $M_0 = \emptyset$ and for each $i \geq 1$

(a) M_i is a nonempty compact connected smooth 2-submanifold of M and $M_{i-1} \subset int\, M_i$,

(b) for each component L of $cl\,(M \setminus M_i)$, L is noncompact and $L \cap M_{i+1}$ is connected.

Note that M is a plane (an open Möbius band, an open annulus) iff infinitely many M_i's are disks (Möbius bands, annuli respectively). Since $M \neq$ a plane, an open Möbius band, an open annulus, passing to a subsequence, we may assume that

(c) $M_i \neq$ a disk, an annulus, a Möbius band.

For each $i \geq 1$ let $U_i = int\, M_i$, and choose a small closed collar E_i of ∂M_i in $U_{i+1} \setminus U_i$, and set $M_i' = M_i \cup E_i \subset U_{i+1}$.

3.1. Proof of Theorem 1.1.

[1] For each $j > i > k \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc} (p^i_{k,j})^*(\mathcal{D}_{M_k\cup(M\setminus U_j)}(M)_0) & \xrightarrow{(p^i_{k,j})_*} & \mathcal{D}_{M_k\cup(M\setminus U_j)}(M)_0 \\ {\scriptstyle (\pi^i_{k,j})_*}\downarrow & & \downarrow{\scriptstyle \pi^i_{k,j}} \\ \mathcal{E}_{M_k}(M_i', U_j)_0 & \xrightarrow{p^i_{k,j}} & \mathcal{E}_{M_k}(M_i, U_j)_0, \end{array}$$

$\pi^i_{k,j}$, $p^i_{k,j}$: the restriction maps,
$\mathcal{G}^i_{k,j} \equiv \mathcal{D}_{M_k\cup(M\setminus U_j)}(M)_0 \cap \mathcal{D}_{M_i}($

Lemma 3.1. *(1) $(\pi^i_{k,j})_*$ is a principal bundle with fiber $\mathcal{G}^i_{k,j}$.*
(2) $\mathcal{G}^i_{k,j}$ is an AR.
(3) $(\pi^i_{k,j})_*$ is a trivial bundle.
(4) $\mathcal{E}_{M_k}(M_i', U_j)_0$ is an AR.

In (2) we apply Theorem 2.4 to deduce $\mathcal{G}^i_{k,j} \cong \mathcal{D}_{M_i\cup E_j}(M_j')_0$. The latter is an AR (Lemma 2.2 (ii), Theorem 2.3).

[2] For each $i > k \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc} (p^i_k)^*(\mathcal{D}_{M_k}(M)_0) & \xrightarrow{(p^i_k)_*} & \mathcal{D}_{M_k}(M)_0 \\ {\scriptstyle (\pi^i_k)_*}\downarrow & & \downarrow{\scriptstyle \pi^i_k} \\ \mathcal{E}_{M_k}(M_i', M)_0 & \xrightarrow{p^i_k} & \mathcal{E}_{M_k}(M_i, M)_0, \end{array}$$

π^i_k, p^i_k : the restriction maps,
$\mathcal{G}^i_k \equiv \mathcal{D}_{M_k}(M)_0 \cap \mathcal{D}_{M_i}(M)$.

Lemma 3.2. *(1) $(\pi^i_k)_*$ is a principal bundle with fiber $\mathcal{G}^i_k$.*
(2) $\mathcal{E}_{M_k}(M_i', M)_0$ is an AR.
(3) $(\pi^i_k)_*$ is a trivial bundle.
(4) $\mathcal{G}^i_k = \mathcal{D}_{M_i}(M)_0$ and $\mathcal{D}_{M_k}(M)_0$ strongly deformation retracts onto $\mathcal{D}_{M_i}(M)_0$.

The assertion (2) follows from Lemma 2.3 (2), Lemma 3.1 (4) and the fact that $\mathcal{E}_{M_k}(M_i', M)_0 = \cup_{j>i}\mathcal{E}_{M_k}(M_i', U_j)_0$.

Proof of Theorem 1.1.

(A) $\mathcal{D}(M)_0 \simeq *$:

$\mathcal{D}_{M_i}(M)_0$ strongly deformation retracts onto $\mathcal{D}_{M_{i+1}}(M)_0$ for each $i \geq 0$ (Lemma 3.2 (4)). Since $\operatorname{diam} \mathcal{D}_{M_i}(M)_0 \to 0$ $(i \to \infty)$, it follows that $\mathcal{D}(M)_0$ strongly deformation retracts onto $\{id_M\}$.

(B) $\mathcal{D}(M)_0$ is an ℓ_2-manifold:

By Theorem 2.5 and Lemma 2.1 it remains to show that $\mathcal{D}(M)_0$ is an ANR. We apply Lemma 2.3 (4): For each $i \geq 0$, we have the following pullback diagram:

$$\begin{array}{ccc} (p_i)^*(\mathcal{D}(M)_0) & \xrightarrow{(p_i)_*} & \mathcal{D}(M)_0 \\ {\scriptstyle (\pi_i)_*}\downarrow & & \downarrow{\scriptstyle \pi_i} \\ \mathcal{E}(M_i', M)_0 & \xrightarrow{p_i} & \mathcal{E}(M_i, M)_0, \end{array}$$

π_i, p_i : the restriction maps,
$q_i : \mathcal{D}(M)_0 \to (p_i)^*(\mathcal{D}(M)_0)$
$q_i(h) = (h|_{M_i'}, h)$.

Since $(\pi_i)_*$ is a trivial principal bundle with the contractible fiber $\mathcal{D}_{M_i}(M)_0$ (Lemma 3.2 (3),(4), (A)), it follows that $(\pi_i)_*$ admits a section s_i and $s_i(\pi_i)_*$ is $(\pi_i)_*$-fiber preserving homotopic to id. Consider the two maps

$$\varphi = (\pi_i)_* q_i : \mathcal{D}(M)_0 \to \mathcal{E}(M_i, M)_0 \quad \text{and} \quad \psi = (p_i)_* s_i : \mathcal{E}(M_i, M)_0 \to \mathcal{D}(M)_0.$$

Then $\mathcal{E}(M_i, M)_0$ is an ANR (Lemma 2.2 (i)) and $\psi\varphi : \mathcal{D}(M)_0 \to \mathcal{D}(M)_0$ is π_i-fiber preserving homotopic to id. Since diam (fibers of π_i) $\to 0$ $(i \to \infty)$, Lemma 2.3 (4) implies that $\mathcal{D}(M)_0$ is an ANR. □

3.2. Proof of Theorem 1.2.

We use the following notations:

$$\mathcal{D}_j = \mathcal{D}_{M \setminus U_j}(M)_0, \quad \mathcal{U}_{i,j} = \mathcal{E}(M_i, U_j)_0, \quad \mathcal{U}_{i,j}' = \mathcal{E}(M_i', U_j)_0 \quad (j > i \geq 1).$$

We have the pullback diagram:

$$\begin{array}{ccc} (p_{i,j})^*\mathcal{D}_j & \xrightarrow{(p_{i,j})_*} & \mathcal{D}_j \\ {\scriptstyle (\pi_{i,j})_*}\downarrow & & \downarrow{\scriptstyle \pi_{i,j}} \\ \mathcal{U}_{i,j}' & \xrightarrow{p_{i,j}} & \mathcal{U}_{i,j}, \end{array}$$

$\pi_i' : \mathcal{D}(M)_0 \to \mathcal{E}(M_i', M)_0$,

$\pi_{i,j}$, $p_{i,j}$, π_i' : the restriction maps.

Lemma 3.3. *(i) $(\pi_{i,j})_*$ is a trivial bundle with AR fiber.*
(ii) $\pi_{i,j}$ has the following lifting property:

> (*) *If Y is a metric space, B is a closed subset of Y and $\varphi : Y \to \mathcal{U}_{i,j}'$ and $\varphi_0 : B \to \mathcal{D}_j$ are map with $p_{i,j}\varphi|_B = \pi_{i,j}\varphi_0$, then there exists a map $\Phi : Y \to \mathcal{D}_j$ such that $\pi_{i,j}\Phi = p_{i,j}\varphi$ and $\Phi|_B = \varphi_0$.*

For each $j > i \geq 1$, we regard as $\mathcal{U}_{i,j}' \subset \mathcal{E}(M_i', M)_0$ and set $\mathcal{V}_{i,j}' = (\pi_i')^{-1}(\mathcal{U}_{i,j}') \subset \mathcal{D}(M)_0$. For each $i \geq 1$ we have:

(i) $\mathcal{E}(M_i, M)_0 = \cup_{j>i} cl\,\mathcal{U}_{i,j}'$ ($\mathcal{U}_{i,j}'$ is open in $\mathcal{E}(M_i, M)_0$, $cl\,\mathcal{U}_{i,j}' \subset \mathcal{U}_{i,j+1}'$)

(ii) $\mathcal{D}(M)_0 = \cup_{j>i} cl\,\mathcal{V}_{i,j}'$ ($\mathcal{V}_{i,j}'$ is open in $\mathcal{D}(M)_0$, $cl\,\mathcal{V}_{i,j}' \subset \mathcal{V}_{i,j+1}'$, $\mathcal{V}_{i+1,j}' \subset \mathcal{V}_{i,j}'$ $(j > i+1)$)

(iii) $\mathcal{D}(M)_0^c = \cup_{j>i} \mathcal{D}_j$ ($\mathcal{D}_j \subset \mathcal{D}_{j+1}$)

Proof of Theorem 1.2.

We construct a homotopy

$$F : \mathcal{D}(M)_0 \times [1, \infty] \to \mathcal{D}(M)_0 \text{ such that } F_\infty = id \text{ and } F_t(\mathcal{D}(M)_0) \subset \mathcal{D}(M)_0^c \ (1 \leq t < \infty).$$

(1) F_i $(i \geq 1)$: Using Lemma 3.3 (ii), inductively we can construct a map $s_j^i : cl\,\mathcal{U}_{i,j}' \to \mathcal{D}_{j+1}$ such that $s_j^i(f)|_{M_i} = f|_{M_i}$ $(f \in cl\,\mathcal{U}_{i,j}')$ and $s_{j+1}^i|_{cl\,\mathcal{U}_{i,j}'} = s_j^i$ $(j > i)$. Define a map

$s^i : \mathcal{E}(M_i', M)_0 \to \mathcal{D}(M)_0^c$ by $s^i|_{cl\,\mathcal{U}_{i,j}'} = s^i_j$, and set $F_i = s^i \pi_i'$. We have $F_i(cl\,\mathcal{V}_{i,j}') \subset \mathcal{D}_{j+1}$ and $F_i(h)|_{M_i} = h|_{M_i}$.

(2) F_t $(i \le t \le i+1)$: Inductively we can construct a sequence of homotopies $G^j : cl\,\mathcal{V}_{i+1,j}' \times [i, i+1] \to \mathcal{D}_{j+1}$ $(j > i+1)$ such that $G^j_i = F_i$, $G^j_{i+1} = F_{i+1}$, $G^{j+1}|_{cl\,\mathcal{V}_{i+1,j}' \times [i,i+1]} = G^j$ and $G^j_t(h)|_{M_i} = h|_{M_i}$. If G^j is given, then G^{j+1} is obtained by applying Lemma 3.3 (ii) to the diagram:

$$\begin{array}{ccccc} B & & \xrightarrow{\quad \varphi_0 \quad} & & \mathcal{D}_{j+2} \\ \cap & & & & \downarrow \\ Y & \xrightarrow{\varphi} & \mathcal{U}_{i,j+2}' & \longrightarrow & \mathcal{U}_{i,j+2}, \end{array} \qquad \varphi(h,t) = h|_{M_i'}, \qquad \varphi_0(h,t) = \begin{cases} G_j(h,t) & (h \in cl\,\mathcal{V}_{i+1,j}') \\ F_t(h) & (t = i, i+1) \end{cases}$$

$$(Y, B) = (cl\,\mathcal{V}_{i+1,j+1}' \times [i, i+1], (cl\,\mathcal{V}_{i+1,j}' \times [i, i+1]) \cup (cl\,\mathcal{V}_{i+1,j+1}' \times \{i, i+1\})).$$

Define $F : \mathcal{D}(M)_0 \times [i, i+1] \to \mathcal{D}(M)_0^c$ by $F = G^j$ on $cl\,\mathcal{V}_{i+1,j}' \times [i, i+1]$.

(3) F_∞: Since $F_t(h)|_{M_i} = h|_{M_i}$ for $t \ge i$, we can continuously extend F by $F_\infty = id$. This completes the proof. □

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