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Bifurcation of Transition Layers in Reaction-Diffusion Systems[1]

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Another title of this article may be:

Searching for Stable Multi-Dimensional Patterns
in Reaction-Diffusion Systems.

This is a story of my research activities during the last few years, in searching for multi-dimensional patterns for reaction-diffusion systems. As regard to stable patterns, this is not a success story. It tells us, however, about the intricacies one faces in dealing with multi-dimensional transition layers. In retrospect, one space-dimension was a fortunate exception in which there is no extra dimension for instability to set in. As soon as the dimension of the domain becomes higher than one, instabilities creep in to transition layers from the extra dimension along interfaces.

1. Reaction-Diffusion Equation and Interface Equation

1.1. **Reaction Diffusion System.** A two component system of reaction-diffusion equations, such as

$$\text{(R-D)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + f(u,v) & \\ & (x\in\Omega,\ t>0) \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + g(u,v) & \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 = \dfrac{\partial v}{\partial \mathbf{n}} & (x\in\partial\Omega,\ t>0) \end{cases}$$

has been widely employed to model various pattern formation phenomena [4, 7]. The domain $\Omega\subset\mathbb{R}^N$ here is assumed to be bounded and smooth, and $\mathbf{n}$ stands for outward unit normal on $\partial\Omega$.

According to various types of nonlinearity (f,g), the system above, dispite of its simplicity, is capable of modelling multitude of pattern formation phenomena. In this article, we deal with two types of nonlinearity. Prototypical examples of these are:

$$\text{(AI)}\qquad f(u,v)=u-u^3-v,\qquad g(u,v)=u-v$$

and

$$\text{(CO)}\qquad f(u,v)=u-u^3+v,\qquad g(u,v)=u-v.$$

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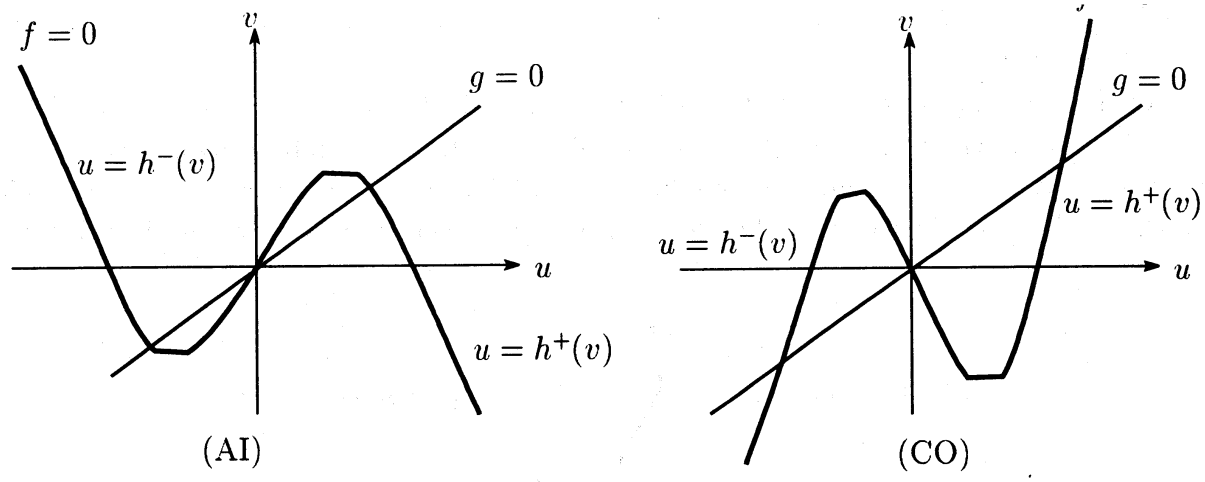

FIGURE 1. Nullclines of (f, g) for (AI) and (CO)

When does the system (R-D) produce patterns? Here, a *pattern* means a spatially inhomogeneous solution. The following theorem gives us an insight in answering this question.

Theorem 1 (Conway-Hoff-Smoller [3]). *There exits a positive constant* $d_0 = d_0(\Omega, f, g)$ *such that if* $\min\{d_1, d_2\} > d_0$ *then the solutions of* (R-D) *behaves approximately similar to those of the ordinary differential equations*

$$\text{(ODE)} \qquad u_t = f(u, v), \qquad v_t = g(u, v),$$

after initial transients.

The theorem says that the diffusion forces the spatial homogenization of the solution of (R-D). In this sense, the diffusion acts in accordance with our intuition, and when both diffusion coefficients d_1 and d_2 are rather large, the system (R-D) does not create any pattern. This in turn suggests that one of the diffusion rates must be *small* in order for (R-D) to produce patterns.

Therefore, we assume in the sequel that the diffusion rate d_1 of u is small, while d_2 remains of order $O(1)$ as $d_1 \to 0$:

$$0 < d_1 = \epsilon^2 \ll 1, \qquad 0 < d_2 = D = O(1) \quad (\text{as } \epsilon \to 0).$$

Then (R-D) is rewritten as

$$\text{(1.1)} \qquad \begin{cases} u_t = \epsilon^2 \Delta u + f(u, v) \\ \\ v_t = D\Delta v + g(u, v) \end{cases} \qquad x \in \Omega \ \ t > 0.$$

In (1.1) (and also in the sequel), the reference to the boundary conditions is omitted, because we always deal with the homogeneous Neumann boundary conditions. The general behavior of solutions to (R-D) with appropriate initial conditions

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x))$$

is well known [1]. For example, if the second component $v_0(x)$ of the initial condition satisfies $|v_0(x)| \leq 2/(3\sqrt{3})$ for $x \in \overline{\Omega}$, then the solution $(u(x,t), v(x,t))$ of (R-D) develops internal layers as $t \to \infty$.

Namely, for large t, $u(x,t) \approx h^-(v(x,t))$ for $x \in \Omega^-(t)$ and $u(x,t) \approx f^+(v(x,t))$ for $x \in \Omega^+(t)$, thus creating a sharp transition of $u(x,t)$ from the left branch $u = h^-(v)$ to the right branch $u = h^+(v)$ of the nullclien $\{f = 0\}$ (cf. Figure 1) near the set $\Gamma(t)$, called interface. The development of the sharp transition layer is caused by the bistability of

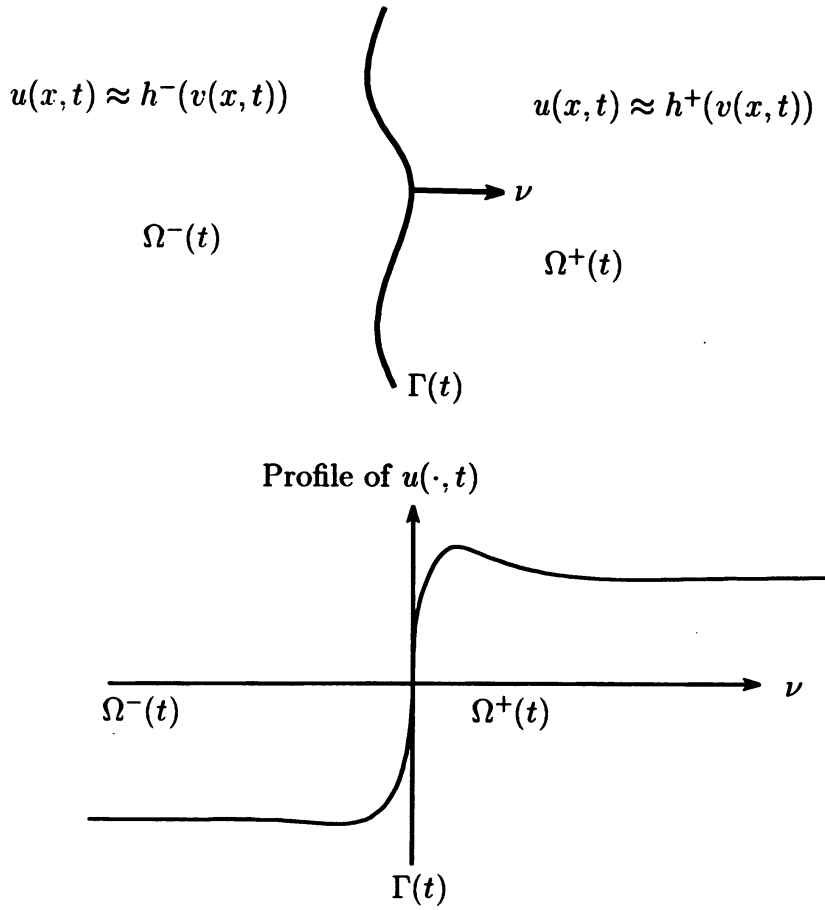

FIGURE 2. Development of interface

the scalar ordinary differential equation

$$\frac{du}{dt} = f(u,v)$$

which is the first equation in (R-D) with the diffusion term being neglected. Note that $u = h^-(v), h^+(v)$ are stable equilibria of the scalar ordinary differential equation for $|v| < 2/(3\sqrt{3})$.

1.2. **Interface Equation.** When the transition layer becomes so sharp that the diffusion term $\epsilon\Delta u$ cannot be neglected, the location of the layer $\Gamma(t) \subset \Omega$, called the *interface* at $t > 0$ which divides Ω into two regions $\Omega^{\pm}(t)$ (cf. Figure 3), starts migrating according

to a certain law of motion, where

$$\Omega^-(t) = \{x \in \Omega \mid u(x,t) \approx h^-(v(x,t))\}, \qquad \Omega^+(t) = \{x \in \Omega \mid u(x,t) \approx h^+(v(x,t))\}.$$

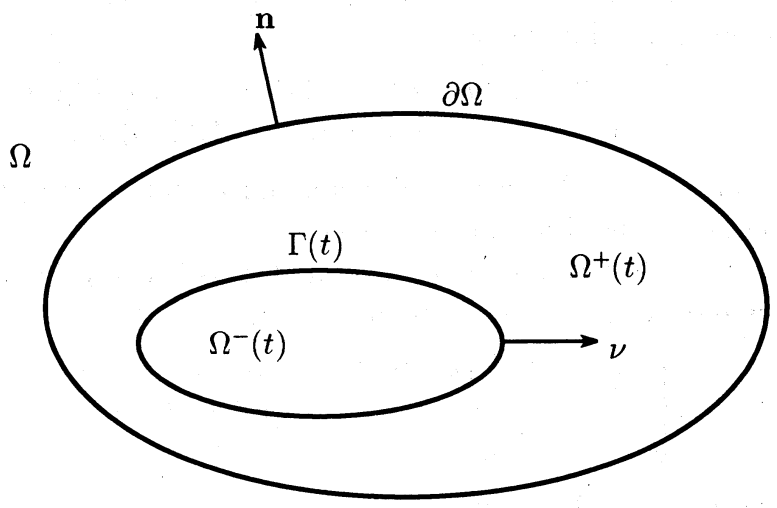

FIGURE 3. $\Gamma(t)$ divides Ω into two parts, $\Omega^-(t)$ and $\Omega^+(t)$.

In order to describe the motion law, let us consider the following nonlinear eigenvalue problem:

$$\phi''(z)+c\phi'(z) + f(\phi(z), v) = 0 \qquad (z \in \mathbb{R}),$$
$$\lim_{z\to-\infty} \phi(z) = h^-(v), \quad \lim_{z\to+\infty} \phi(z) = h^+(v), \quad \phi(0) = 0.$$

It is known that the problem has a unique solution pair $(\phi(z), c(v))$ for $v \in (\underline{v}, \overline{v})$. The *wave speed* satisfies:

(WS)
$$\begin{cases} c'(v) > 0 & \text{for (AI)} \\ c'(v) < 0 & \text{for (CO)} \end{cases}$$

It turns out that the sign of $c'(v)$ plays a crucial role in the following discussions.

When the u-component develops internal layer it is either $u(x,t) \approx h^-(v(x,t))$ or $u(x,t) \approx h^+(v(x,t))$, on $\Omega^-(t)$ or $\Omega^+(t)$, respectively. So it is natural to define g^* by

$$g^*(v,x;\Gamma(t)) = \begin{cases} g(h^-(v),v) & \text{if } x \in \Omega^-(t) \\ g(h^+(v),v) & \text{if } x \in \Omega^+(t). \end{cases}$$

The motion-law of the interface under the time scale of (1.1) is described by the following system of equations:

(1.2-a)
$$\mathbf{v}(x;\Gamma(t)) = 0 \qquad (x \in \Gamma(t),\ t > 0),$$

(1.2-b)
$$v_t = D\Delta v + g^*(v,x;\Gamma(t)) \qquad (x \in \Omega\backslash\Gamma(t),\ t > 0),$$

(1.2-c)
$$v(\cdot,t) \in C^1(\overline{\Omega}) \cap C^2(\Omega\backslash\Gamma(t)).$$

In (1.2-a), $\mathbf{v}(x;\Gamma(t))$ is the speed of $\Gamma(t)$ at $x \in \Gamma(t)$ along the unit normal vector $\nu(x,t)$ pointing into $\Omega^+(t)$. The condidion (1.2-c) is called a C^1-matching condition, which is of crucial importance.

The interface equation (1.2-a) above says that $\Gamma(t)$ does not move under the time scale of (1.1), $\Gamma(t) \equiv \Gamma(0)$. Therefore, (1.2-b) is a gradient system and its solutions approach equilibrium solutions, i.e., solutions of

$$\text{(1.2-Equil.)} \qquad \begin{cases} 0 = D\Delta v + g^*(v,x;\Gamma(0)) & (x \in \Omega\backslash\Gamma(0)) \\ v(\cdot) \in C^1(\overline{\Omega}) \cap C^2(\Omega\backslash\Gamma(0)). \end{cases}$$

Does (2.1-Equil.) have a solution for a reasonable initial interface $\Gamma(0)$? The answer is yes. We do not, however, dwell on this issue here.

The reason why the interface does not move in (1.2) is because the time scale of (1.1) is too slow. Let us rescale time by $t \to t/\epsilon$ to obtain:

$$\text{(1.3)} \qquad \begin{cases} \epsilon u_t = \epsilon^2 \Delta u + f(u,v), \\ \qquad\qquad\qquad\qquad x \in \Omega \;\; t > 0 \\ \epsilon v_t = D\Delta v + g(u,v). \end{cases}$$

The interface equation for (1.3) is given by

$$\text{(1.4-a)} \qquad \mathbf{v}(x;\Gamma(t)) = c(v(x,t)) \qquad (x \in \Gamma(t),\ t > 0),$$

$$\text{(1.4-b)} \qquad 0 = D\Delta v + g^*(v,x;\Gamma(t)) \qquad (x \in \Omega\backslash\Gamma(t),\ t > 0),$$

$$\text{(1.4-c)} \qquad v(\cdot,t) \in C^1(\overline{\Omega}) \cap C^2(\Omega\backslash\Gamma(t)).$$

Note that the left hand side of (1.4-b) is equal to 0, but not to v_t. This can be understandable if we recall that the limit $\epsilon \to 0$ in (1.3) can have an effect of the limit $t \to \infty$ in (1.2) because of the rescaling of time.

Theorem 2 (Nishiura [8] for $N = 1$, Chen [1] for $N \geq 2$)

(1) *Let* $(v_0(x), \Gamma(0))$ *be a smooth initial condition for* (1.4). *There exist* $T > 0$ *and a unique solution* $(v(x,t), \Gamma(t))$ *of* (1.4) *on* $[0,T]$.

(2) *There exist a family of solutions* $(u^\epsilon(x,t), v^\epsilon(x,t))$ *of* (1.3) *for sufficiently small* $\epsilon > 0$ *such that*

$$\lim_{\epsilon \to 0} v^\epsilon(x,t) = v(x,t) \qquad \textit{uniformly on } \overline{\Omega} \times [0,T]$$

$$\lim_{\epsilon \to 0} u^\epsilon(x,t) = \begin{cases} h^-(v(x,t)) & \textit{uniformly on } \cup_{t\in[0,T]} \Omega^-(t)\backslash\Gamma_\delta(t) \times \{t\} \\ h^+(v(x,t)) & \textit{uniformly on } \cup_{t\in[0,T]} \Omega^+(t)\backslash\Gamma_\delta(t) \times \{t\} \end{cases}$$

for each $\delta > 0$, *where*

$$\Gamma_\delta(t) := \{x \in \Omega \mid \operatorname{dist}(x, \Gamma(t)) < \delta\}$$

is the δ*-neighborhood of* $\Gamma(t)$.

The last theorem says that the interface equation (1.4) does approximate the reaction-diffusion system (1.3) on finite time intervals. Even if the solution $(v(x,t), \Gamma(t))$ of (1.4) exists on $[0,\infty)$, the approximation in the sense of Theorem 2 (2) above is valid only on a finite time interval $[0, T_\epsilon]$ (although $T_\epsilon \to \infty$ as $\epsilon \to 0$ may be the case). Therefore

some of asymptotic information on the solutions of (1.3) may not be captured by only analyzing the behavior of solutions of (1,4).

2. Free Interface Problem and Equilibrium Transition Layers

A first step to analyze asymptotic (as $t \to \infty$) behaviors of solutions to (1.3) is to deal with equilibrium solutions, namely, solutions of the semilinear singularly perturbed elliptic system:

$$\text{(2.1)} \qquad \begin{cases} 0 = \epsilon^2 \Delta u + f(u,v), & \\ & x \in \Omega \\ 0 = D\Delta v + g(u,v), & \\ \\ 0 = \partial u/\partial \mathbf{n} = \partial v/\partial \mathbf{n} & x \in \partial\Omega. \end{cases}$$

As mentioned earlier, the result in Theorem 2 guarantees the approximation of (1.3) by (1.4) only on finite time intervals, and hence do not answer the following question:

> *If* (1.4) *has an equilibrium solution* $(\Gamma_0, v(x;\Gamma_0))$, *then, does* (2.1) *have a corresponding equilibrium solutions for small* $\epsilon > 0$*?*

The answer to the question is affirmative when the space dimension $N = 1$. Mimura, Tabata and Hosono [6] proved the existence of equilibrium transition layers, and Nishiura and Fujii [9] established their stability property. According to Nishiura and Fujii [9], the equilibrium transition layers are stable for (AI)-nonlinearity and ustable for (CO)-nonlinearity.

An answer to the question above for a higher dimensional case ($N \geq 2$) is given in [10] in a general situation, which we now describe by using interface equation.

The equilibrium solution of (1.4) gives rise to the following *free interface problem*:

$$\text{(2.2-a)} \qquad 0 = D\Delta V^* + g^*(V^*, x; \Gamma_0) \qquad (x \in \Omega \backslash \Gamma_0),$$

$$\text{(2.2-b)} \qquad V^*(x) = 0 \quad \text{on } \Gamma_0 \text{ and } \partial V^*(x)/\partial \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

$$\text{(2.2-c)} \qquad V^*(\cdot) \in C^1(\overline{\Omega}) \cap C^2(\Omega \backslash \Gamma_0),$$

where the unknown is a pair $(V^*(x), \Gamma_0)$. Note that the nonlinearity $g^*(v, x; \Gamma_0)$ has a jump discontinuity along Γ_0. The problem (2.2) is called a *free interface* problem because the equilibrium interface Γ_0 is unknown. The C^1-matching condition (2.2-c) forces that the problem cannot have a solution for an arbitrarily given interface Γ_0.

Remark 1. *Note that the free interface problem* (2.2-a, b, c) *is different from the problem* (1.2-Equil.) *in which* $\Gamma(0)$ *is arbitrarily given. In* (2.2-b), *the Dirichlet condition* $V^* = 0$ *is to be satisfied on* Γ_0, *while in* (1.2-Equil.) *no such condition is imposed.*

To the best of our knowledge, the existence of solutions of the free interface problem (2.2) is not known in a general situation. We hereafter assume that (2.2) has a smooth solution $(V^*(x), \Gamma_0)$. Our question then is: *Does this* $(V^*(x), \Gamma_0)$ *give rise to a transition layer solution of* (2.1)?

It turns out that (1.4) is not a *correct* interface equation for (1.3), at least as regard to equilibrium solutions. The correct one is given by replacing (1.4-a) by a curvature dependent version

(1.4-a') $$\mathbf{v}(x;\Gamma(t)) = c(v(x,t)) - \epsilon\kappa(x;\Gamma(t)) \qquad (x \in \Gamma(t),\ t > 0),$$

where $\kappa(x;\Gamma)$ stands for the sum of principal curvatures of Γ at $x \in \Gamma$. Let us linearize (1.4-a',b,c) at the solution $(V^*(x), \Gamma_0)$ of (2.2). Let v^* be such that $v(v^*) = 0$. Then, the associated *linearized eigenvalue problem* is given by

(2.3-a) $$\lambda p = \epsilon\left(\Delta^{\Gamma_0} + \sum_{j=1}^{N-1}\kappa_j(x)^2\right)p + c'(v^*)\frac{\partial V^*(x)}{\partial\nu(x)}\Big|_{\Gamma_0} p + c'(v^*)q|_{\Gamma_0} \qquad x \in \Gamma_0,$$

(2.3-b) $$0 = D\Delta q + g_v^*(V^*(x), x; \Gamma_0)q - [g^*]p \otimes \delta_{\Gamma_0} \qquad x \in \Omega$$

for $p(x)$ (defined for $x \in \Gamma_0$) and $q(x)$ (defined for $x \in \Omega$), where Δ^{Γ_0} is the Laplace-Beltrami operator on Γ_0, $\kappa_j(x)$ $(j = 1, \ldots, N-1)$ are principal curvatures at $x \in \Gamma_0$ and $[g^*]$ is the jump of g^* across Γ_0:

$$[g^*] = g(h^+(v^*), v^*) - g(h^-(v^*), v^*) \qquad (v^* = 0 \text{ in our examples (AI) and (CO)}).$$

In the second equation above, the symbol δ_{Γ_0} stands for the Dirac-delta function supported on Γ_0, and hence the equation should be interpreted in distributional sense. Therefore, by writing it in weak form:

$$0 = -D\int_\Omega \nabla q(x)\cdot\nabla\eta(x)dx + \int_\Omega g_v^*(V^*(x), x; \Gamma_0)q(x)\eta(x)dx - [g^*]\int_{\Gamma_0} p(x)\eta(x)dS_x^{\Gamma_0}$$

(with η being a test function and $dS_x^{\Gamma_0}$ standing for the volume element on Γ_0), and integrating by parts, one can recast (2.3-b) as concisely as

(2.3-b') $$\Pi_0 q|_{\Gamma_0} + \frac{[g^*]}{D}p = 0 \qquad x \in \Gamma_0.$$

The operator Π_0 in the last equation is the Dirichlet-to-Neumann map defined by

$$\Pi_0 q(x) := \frac{\partial v_0^-(x)}{\partial\nu} - \frac{\partial v_0^+(x)}{\partial\nu} \qquad (x \in \Gamma_0)$$

in which $v_0^\pm(x)$ are solutions of the following problem:

$$\begin{aligned} D\Delta v^\pm + g_v^*(V^*(x), x; \Gamma_0)v^\pm &= 0 \qquad (x \in \Omega^\pm),\\ v^\pm(x) &= q(x) \qquad (x \in \Gamma_0),\\ \frac{\partial v^\pm(x)}{\partial \mathbf{n}} &= 0 \qquad (x \in \partial\Omega), \end{aligned}$$

where $\Omega^- \cup \Omega^+ = \Omega\backslash\Gamma_0$.

Lemma 3 ([10]). *Assume that $g_v^* < 0$ on $\Omega\backslash\Gamma_0$.*

(1) *The operator $\Pi_0 : C^{2+\alpha}(\Gamma_0) \to C^{1+\alpha}(\Gamma_0)$ is invertible for $0 < \alpha < 1$, and extends to a self-adjoint operator on $L^2(\Gamma_0)$.*

(2) *Eigenvalues of* Π_0 *are all positive:*

$$0 < \pi_0 < \pi_1 < \ldots < \pi_j \to \infty \quad (j \to \infty),$$

where only distinct eigenvalues are listed.

For both of our nonlinearities (AI) and (CO), the condition $g_v^* < 0$ is satisfied. Therefore, thanks to Lemma 3, we can solve (2.3-b') in $q|_{\Gamma_0}$ and substitute it into (2.3-a) to reduce the eigenvalue problem (2.3) to

$$\lambda p = \mathcal{A}^\epsilon p \qquad \text{on } \Gamma_0, \tag{2.4}$$

where $\mathcal{A}^\epsilon$ is defined by

$$\mathcal{A}^\epsilon p := \epsilon \left(\Delta^{\Gamma_0} + \sum_{j=1}^{N-1} \kappa_j(x)^2 \right) p + c'(v^*) \frac{\partial V^*(x)}{\partial \nu(x)}\Big|_{\Gamma_0} p - c'(v^*) \frac{[g^*]}{D} \Pi_0^{-1} p \qquad \text{on } \Gamma_0. \tag{L}$$

Theorem 4 ([10]).

(1) *Let* (V^*, Γ_0) *be a smooth solution of* (2.2). *Suppose that the operator*

$$\mathcal{A}^\epsilon : C^{2+\alpha}(\Gamma_0) \to C^\alpha(\Gamma_0) \qquad (0 < \alpha < 1)$$

is invertible uniformly in $\epsilon \in (0, \epsilon_0]$ *for some* ϵ_0. *Then* (2.1) *has a family of solutions* (u^ϵ, v^ϵ) *such that*

$$\lim_{\epsilon \to 0} v^\epsilon(x) = V^*(x) \qquad \textit{uniformly in } \overline{\Omega}$$

$$\lim_{\epsilon \to 0} u^\epsilon(x) = \begin{cases} h^-(V^*(x)) & \textit{uniformly on } \Omega_\delta^- \\ h^+(V^*(x)) & \textit{uniformly on } \Omega_\delta^+ \end{cases}$$

for each $\delta > 0$, *where* $\Omega_\delta^\pm$ *are defined by*

$$\Omega_\delta^\pm = \{x \in \Omega^\pm \mid \operatorname{dist}(x, \Gamma_0) \geq \delta\}.$$

(2) *When* $c'(v) < 0$, *the operator* $\mathcal{A}^\epsilon$ *above is invertible uniformly in* $\epsilon > 0$ *small.*

(3) *The solutions in* (1) *are unstable.*

Outline of Proof. The proof of Theorem 4 (1) consists of two steps.

Step 1 is due to Ikeda [5]. It was shown in [5] that there exist two families of boundary layer solutions $(u^{\epsilon,\pm}, v^{\epsilon,\pm})$ on $\Omega^\pm$ with transition layers along the common interface Γ_0.

In Step 2, we match the two families of solutions as follows.

$$(u^{\epsilon,-}, v^{\epsilon,-}) = (u^{\epsilon,-}, v^{\epsilon,+}) \qquad \left(\frac{\partial u^{\epsilon,-}}{\partial \nu}, \frac{\partial v^{\epsilon,-}}{\partial \nu}\right) = \left(\frac{\partial u^{\epsilon,+}}{\partial \nu}, \frac{\partial v^{\epsilon,+}}{\partial \nu}\right) \quad \text{on } \Gamma_0. \tag{2.5}$$

It turns out that the matching conditions are equivalent to

$$\mathcal{A}^\epsilon p = \text{a known function} \in C^\alpha(\Gamma_0) \quad (0 < \alpha < 1). \tag{2.6}$$

The condition on the invertibility of $\mathcal{A}^\epsilon$ enables us to solve (2.6), which in turn allows us to establish the matching conditions (2.5), completing the proof of (1).

The idea of proof for (2) and (3) will be explained below when we prove Theorems 6

In (WS) in §1.2, we have shown that $c'(v) < 0$ for the nonlinearity (CO). Therefore, Theorem 4 applies to this case and the existence of a family of transition layer solutions of (2.1) is established.

What is going on when the nonlinearity is of (AI)-type? When $c'(v) > 0$, one can show that the eigenvalue problem (2.4) has small eigenvalues. More pricisely, there exists a sequence $\{\epsilon_j\}$ with $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_j \to 0$ as $j \to \infty$ such that for each $\epsilon = \epsilon_j$, 0 is an eigenvalue of the problem (2.4). This suggests that an infinite series of static bifurcations of transition layer solutions may be taking place at each $\epsilon = \epsilon_j$. To prove the last statement in a general situation is not so easy. So let us deal with a special case in the next Section.

3. Bifurcation of Transition Layers

We treat in this section the case where the domain Ω is the unit disk in $\mathbb{R}^N$:

$$\Omega = \{x \in \mathbb{R}^N \mid |x| < 1\}.$$

Theorem 5 ([11]).

(1) *There exists $D_0 > 0$ such that the free interface problem* (2.2) *has a radially symmetric solution $(V^*(|x|), \Gamma_0)$ with $\frac{d}{dr}V^*(r) > 0$ for each $D \in [D_0, \infty)$, where*

$$\Gamma_0 = \{x \in \mathbb{R}^N \mid |x| = R_*\} \qquad (0 < R_* < 1).$$

(2) *For both nonlinearities* (AI) *and* (CO), (2.1) *has a family of radially symmetric solutions $(u^\epsilon(|x|), v^\epsilon(|x|))$ with the same limiting behavors as in* Theorem 4 *for each $D \in [D_0, \infty)$.*

For a radially symmetric pair $(V^*(r), \Gamma_0)$, the free interface problem (2.2-a, b, c) reduces to a problem described by an ordinary differential equation (ode). Based upon a detailed analysis of the (ode), the proof of Theorem 5 (1) is rather elementary.

The proof of Theorem 5 (2) goes as follows. In the same manner as in the proof of Theorem 4, the existence is equivalent to

$$\mathcal{A}_0^\epsilon p = \text{a known constant},$$

where $\mathcal{A}_0^\epsilon$ is a constant given by

$$\mathcal{A}_0^\epsilon = c'(0)\left[V_r^*(R_*) - \frac{1}{D}\pi_0^{-1}\right] + O(\epsilon) \neq 0$$

in which π_0 is the first eigenvalue of the Dirichlet-to-Neumann map Π_0.

Theorem 6 ([11]). *The equilibrium solutions $(u^\epsilon(|x|), v^\epsilon(|x|))$ of* Theorem 5 *are unstable with respect to* (1.3). *Moreover, there are (cf. Figures 4 and 5)*

- *two unstable eigenvalues $\lambda_0 > \lambda_1 > 0$ for* (CO)*-nonlinearity;*
- *many unstable eigenvalues for* (AI)*-nonlinearity,*

$$\lambda_0 < \lambda_1 < 0 < \lambda_2, \ldots, \lambda_{j_\epsilon},$$

where $j_\epsilon = O(\epsilon^{-1/2})$.

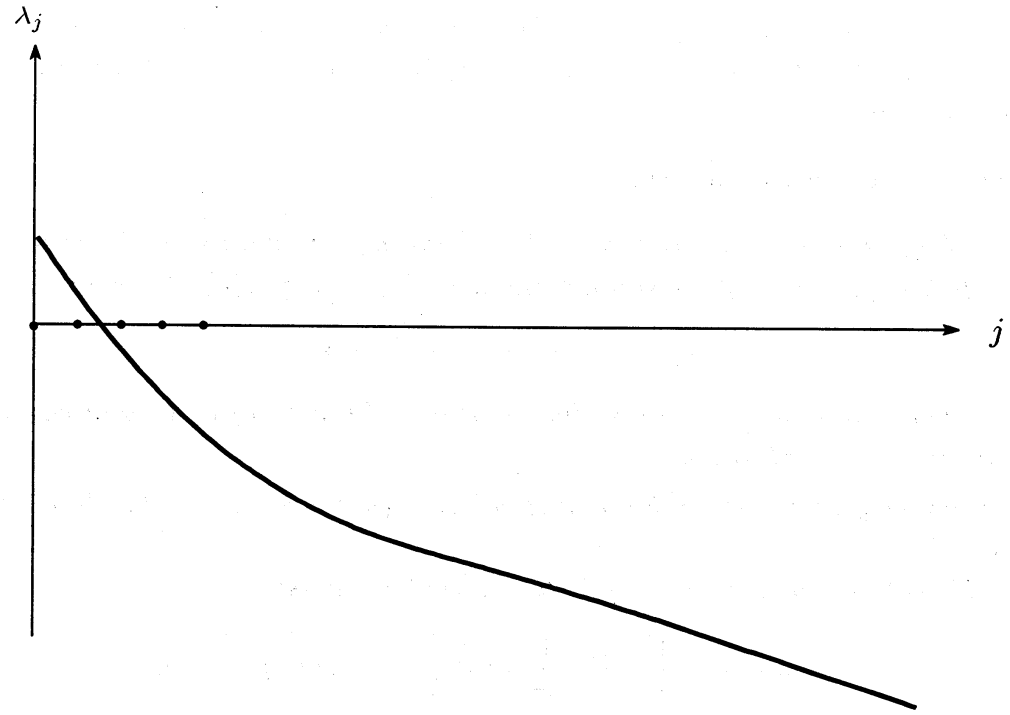

FIGURE 4. Eigenvalues for (CO).

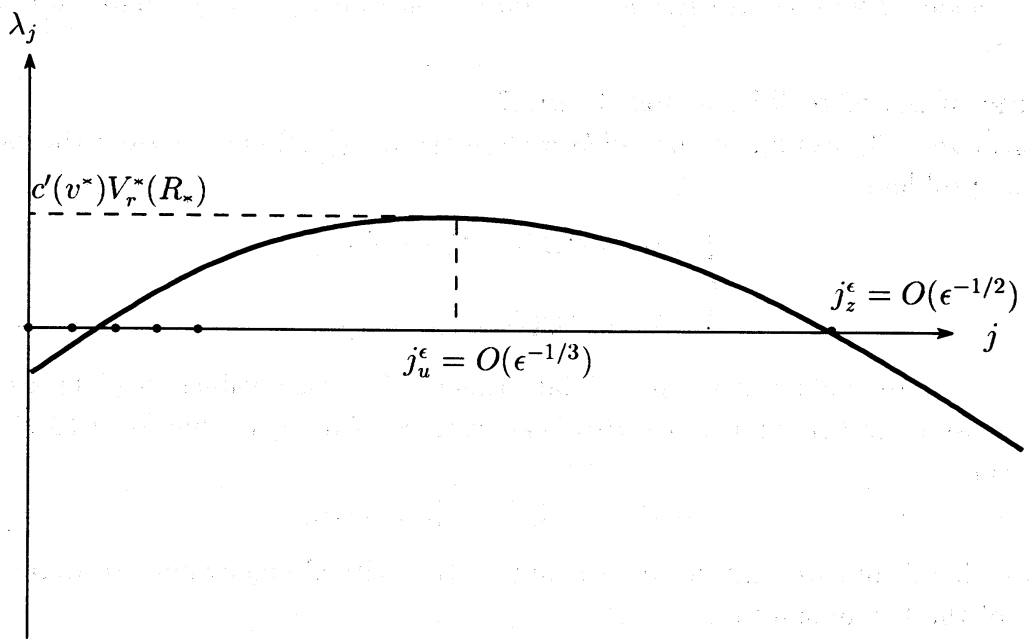

FIGURE 5. Eigenvalues for (AI).

In the above, the multiplicity m_j of the eigenvalue λ_j is given by

$$m_j = \frac{(2j+N-2)(j+N-3)!}{j!(N-2)!}$$

which is the dimension of the space of spherical harmonics of degree j. Moreover, eigenfunction associated with λ_j are of the form $p(|x|)\Theta(y)$ with $|y| = 1$ and Θ being a spherical harmonics of degree j.

We now state a bifurcation result.

Theorem 7 ([11]). *Assume that $N = 2$ and the nonlinearity in (2.1) is of (AI)-type. There exists a sequence $\{\epsilon_j\}$ for sufficiently large j, say $j \geq O(\epsilon_0^{-1/2})$, with*

$$\epsilon_{j-1} > \epsilon_j \to 0 \quad (as \ j \to \infty)$$

such that when ϵ passes ϵ_j, a non-radial solution $(u^{\epsilon,j}, v^{\epsilon,j})$ of (2.1) bifurcates from the trivial branch (u^ϵ, v^ϵ). Moreover,

(1) *the symmetry group of the bifurcated solution $(u^{\epsilon,j}, v^{\epsilon,j})$ is the dihedral group $\mathbf{D}_j$ of order $2j$;*

(2) *the bifurcation points ϵ_j are explicitly characterized as:*

$$\epsilon_j = c'(v^*)\frac{dV^*}{dr}(R_*)\frac{1}{j^2} + O(\frac{1}{j^4}) \qquad (as \ j \to \infty).$$

Remark 2. *The restriction $N = 2$ in Theorem 7 is only for the sake of avoiding the algebraic complication in identifying subgroups of the orthogonal group $O(N)$ which have a one-dimensional fixed point subspace. Similar results hold for $N \geq 3$ with more intricate statements.*

Outline of proof of Theorems 6 and 7.

We linearize (1.3) around the *trivial branch* $(u^\epsilon(|x|), v^\epsilon(|x|))$ and consider the associated eigenvalue problem:

$$(3.1) \qquad \begin{cases} \epsilon\lambda\phi = \epsilon^2\Delta\phi + f_u^\epsilon\phi + f_v^\epsilon\psi \\ \\ \epsilon\lambda\psi = D\Delta\psi + g_v^\epsilon\psi + g_u^\epsilon\phi, \end{cases}$$

where f_u^ϵ etc. are evaluated at the trivial branch. The eigenvalues of (3.1) are divided into two classes, *critical* and *non-critical* eigenvalues. An eigenvalue λ^ϵ of (3.1) is called non-critical if

$$\text{Re}\,\lambda^\epsilon \to -\infty \qquad (\text{as } \epsilon \to 0).$$

From this definition, we only need to examine the critical eigenvalues to determine the stability of the trivial solutions (u^ϵ, v^ϵ).

The key is the following:

$$\{\text{critical eigenvalues of (3.1)}\} \approx \sigma(\mathcal{A}^\epsilon).$$

Namely, the critical eigenvalues are well approximated by the eigenvalues of the operator $\mathcal{A}^\epsilon$. In our radially symmetric case, the eigenvalues of $\mathcal{A}^\epsilon$ are explicitly given by

$$(3.2) \qquad \lambda_j = c'(v^*)\left[\frac{dV^*}{dr}(R_*) - \frac{1}{D}\frac{[g^*]}{\pi_j}\right] - \frac{\epsilon}{R_*^2}(j-1)(j-1+N) \qquad (j = 1, 2, \ldots),$$

$$\frac{-1}{R_*^2}(j-1)(j-1+N)$$

is the j-th eigenvalue of the Jacobi operator

$$\Delta^{\Gamma_0} + \sum_{k=1}^{N-1} \kappa(x)^2 \qquad \text{on } \Gamma_0 = \{x \in \mathbb{R}^N \mid |x| = R_* \}.$$

Let us recall here that π_j is the j-th eigenvalue of the Dirichlet-to-Neumann map Π_0, and it is asymptotically characterized ([10]) as

$$\lim_{j\to\infty} \frac{\pi_j}{\sqrt{j(j+N-2)}} = \frac{2}{R_*}. \tag{3.3}$$

Moreover, thanks to the maximum principle ([10]), one can prove that

$$\frac{dV^*}{dr}(R_*) - \frac{1}{D}\frac{[g^*]}{\pi_j} \begin{cases} < 0 & \text{if } j = 1,2 \\ > 0 & \text{if } j \geq 3. \end{cases} \tag{3.4}$$

From (3.2), (3.3) and (3.4), the statements in Theorem 6 follows immediately.

The proof of Theorem 7 is furnished by the equivariant branching lemma due to Vanderbauwhede [12] and Cicogna [2]. The characterization of the critical eigenvalues as in (3.2)-(3.4) plays a decisive role in verifying the conditions of the branching lemma.

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