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A Singular Limit arising in Combustion Theory: Fine Properties of the Free Boundary

G. S. Weiss*
Graduate School of Math. Sciences, University of Tokyo
3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan

This is an announcement of results to appear.
Let us consider the family of non-negative solutions for the initial-value problem

$$\partial_t u_\epsilon - \Delta u_\epsilon = -\beta_\epsilon(u_\epsilon) \text{ in } (0,\infty)\times\mathbf{R}^n\,,\ u_\epsilon(0,\cdot) = u_\epsilon^0 \text{ in } \mathbf{R}^n\,. \tag{1}$$

Here $\epsilon \in (0,1)$, $\beta_\epsilon(z) = \frac{1}{\epsilon}\beta(\frac{z}{\epsilon})$, $\beta \in C_0^1([0,1])$, $\beta > 0$ in $(0,1)$ and $\int \beta = \frac{1}{2}$. We assume the initial data $(u_\epsilon^0)_{\epsilon\in(0,1)}$ to be bounded in $C^{0,1}(\mathbf{R}^n)$ and to satisfy $u_\epsilon^0 \to u^0$ in $H^{1,2}(\mathbf{R}^n)$ and $\bigcup_{\epsilon\in(0,1)} \operatorname{supp} u_\epsilon^0 \subset B_S(0)$ for some $S < \infty$.
Formally, each limit u with respect to a sequence $\epsilon_m \to 0$ will be a solution of the free boundary problem

$$\partial_t u - \Delta u = 0 \text{ in } \{u>0\}\cap(0,\infty)\times\mathbf{R}^n,\ |\nabla u| = 1 \text{ on } \partial\{u>0\}\cap(0,\infty)\times\mathbf{R}^n\,. \tag{2}$$

The singular limit problem (1) has been derived as a model for the propagation of equidiffusional premixed flames with high activation energy ([4]);

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here $u = \lambda(T_c - T)$, T_c is the flame temperature, which is assumed to be constant, T is the temperature outside the flame and λ is a normalization factor.

Let us shortly summarize the mathematical results directly relevant in this context, beginning with the limit problem (2): In the excellent paper [1], H.W. Alt and L.A. Caffarelli proved via minimization of the energy $\int(|\nabla u|^2 + \chi_{\{u>0\}})$ – here $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\}$ – existence of a stationary solution of (2) in the sense of distributions. They also derived regularity of the free boundary $\partial\{u > 0\}$ up to a set of vanishing $n - 1$-dimensional Hausdorff measure. The question of the existence of classical solutions in three dimensions stands still exposed. Existence would however follow by [13], once the non-existence of singular minimizing cones has been established. *Non-minimizing* singular cones *do* in fact appear for $n = 3$ (cf. [1, example 2.7]). Moreover it is known, that solutions of the Dirichlet problem in two space dimensions are not unique (cf. [1, example 2.6]).

For the time-dependent (2), a "trivial non-uniqueness" complicates the matter further, as the positive solution of the heat equation is always another solution of (2). Even for flawless initial data, classical solutions of (2) develop singularities after a finite time span; consider e.g. the example of two colliding traveling waves

$$\begin{aligned} u(t,x) &= \chi_{\{x+t>1\}}(\exp(x+t-1)-1) \\ &+ \chi_{\{-x+t>1\}}(\exp(-x+t-1)-1) \text{ for } t \in [0,1) \,. \end{aligned} \tag{3}$$

Let us now turn to results concerning the singular perturbation (1): For the stationary problem (1) H. Béréstycki, L.A. Caffarelli and L. Nirenberg obtained in [3] uniform estimates and – assuming the existence of a minimal solution – further results.

L.A. Caffarelli and J.L. Vazquez contributed in [8] among other things the corresponding uniform estimates for the time-dependent case and a convergence result: for initial data u^0 that is strictly mean concave in the interior of its support, a sequence of ϵ-solutions converges to a solution of (2) in the sense of distributions.

Let us finally mention several results on the corresponding two-phase problem, which are relevant as solutions of the one-phase problem are automatically solutions of the corresponding two-phase problem. In [6] and [7], L.A. Caffarelli, C. Lederman and N. Wolanski prove convergence to a sort of barrier solution in the case that $\{u = 0\}^\circ = \emptyset$. These results deal quite well with the *true two-phase behavior* of limits, but have – as will become more plain in the examples below – to largely ignore the one-phase behavior. One of the reasons for this is that the limit cannot be expected to be close to a monotone function near free boundary points that are not true two-phase points.

Our result: As an intermediate result we obtain that each limit u of (1) is a solution *in the sense of domain variations*, i.e. u is smooth in $\{u > 0\}$ and satisfies

$$\int_0^\infty \int_{\mathbf{R}^n} [-2\partial_t u \nabla u \cdot \xi + |\nabla u|^2 \text{div}\, \xi - 2\nabla u D\xi \nabla u] = -\int_0^\infty \int_{R(t)} \xi \cdot \nu \, d\mathcal{H}^{n-1} \, dt \tag{4}$$

for every $\xi \in C_0^{0,1}((0,\infty) \times \mathbf{R}^n; \mathbf{R}^n)$. Here

$$R(t) := \{x \in \partial\{u(t) > 0\} \ : \ \text{there is } \nu(t,x) \in \partial B_1(0) \text{ such that } u_r(s,y) = \frac{u(t + r^2 s, x + ry)}{r} \to \max(-y \cdot \nu(t,x), 0) \text{ locally uniformly in } (s,y) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}$$

is for a.e. $t \in (0,\infty)$ a countably $n-1$-rectifiable subset of the free boundary. Let us remark that already this equation contains information (apart from the rectfiability of $R(t)$) that cannot be inferred from the viscosity notion of solution in [11, Definition 4.3] (the stationary case): whereas any function of the form $\alpha \max(x_n, 0) + \beta \max(-x_n, 0)$ with $\alpha, \beta \in (0,1]$ is a viscosity solution in the sense of [11, Definition 4.3], positive α and β have to be equal

in order to satisfy (4).
Our main result is then that each limit of (1) – no additional assumptions are necessary – satisfies for a.e. $t \in (0,\infty)$

$$\begin{aligned} \int_{\mathbf{R}^n} (\partial_t u(t)\phi + \nabla u(t) \cdot \nabla\phi) &= -\int_{R(t)} \phi \, d\mathcal{H}^{n-1} \\ - \int_{\Sigma_*(t)} 2\theta(t,\cdot)\, \phi \, d\mathcal{H}^{n-1} &- \int_{\Sigma_z(t)} \phi \, d\lambda(t) \end{aligned} \tag{5}$$

for every $\phi \in C_0^1(\mathbf{R}^n)$, that the non-degenerate singular set

$$\Sigma_*(t) := \{x \in \partial\{u(t) > 0\} : \text{there is } \theta(t,x) \in (0,1] \text{ and } \xi(t,x) \in \partial B_1(0) \text{ such}$$

$$\text{that } u_r(s,y) = \frac{u(t + r^2 s, x + ry)}{r} \to \theta(t,x)|y \cdot \xi(t,x)| \text{ locally uniformly}$$

$$\text{in } (s,y) \in \mathbf{R}^{n+1} \text{ as } r \to 0\}$$

is for a.e. $t \in (0,\infty)$ a countably $n-1$-rectifiable subset of the free boundary whereas $\lambda(t)$ is for a.e. $t \in (0,\infty)$ a Borel measure such that the $n-1$ dimensional Hausdorff measure is on

$$\Sigma_z(t) := \{x \in \partial\{u(t) > 0\} \ : \ r^{-n-2} \int_{Q_r(t,x)} |\nabla u|^2 \to 0 \text{ as } r \to 0\}$$

totally singular with respect to $\lambda(t)$, i.e. $r^{1-n}\lambda(t)(B_r(x)) \to 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma_z(t)$. Up to a set of vanishing $\mathcal{H}^{n-1}$ measure, $\partial\{u(t) > 0\} = R(t) \cup \Sigma_*(t) \cup \Sigma_z(t)$.

In the two-dimensional stationary case one can prove that λ does not appear in the equation. On the other hand there seem to exist very bad distributional solutions that are not solutions in the sense of domain variations. This suggests that the solution in the sense of domain variations is a better notion of solution than that in the sense of distributions.

Let us shortly describe relevant parts of the proof:
As a first step, we prove convergence of $2B_{\epsilon_m}(u_{\epsilon_m})$ to a characteristic function. We also need some control over the set of *horizontal points,* i.e. the set of points at which the solution's behaviour in the time direction is dominant.

A crucial tool in the local analysis at the free boundary is the *monotonicity formula*

Theorem 1 **(ϵ-Monotonicity Formula)** *Let* $(t_0, x_0) \in (0,\infty)\times\mathbf{R}^n, T_r^-(t_0) = (t_0 - 4r^2, t_0 - r^2) \times \mathbf{R}^n$, $0 < \rho < \sigma < \frac{\sqrt{t_0}}{2}$ *and*

$$G_{(t_0,x_0)}(t,x) = 4\pi(t_0 - t)\,|4\pi(t_0-t)|^{-\frac{n}{2}-1}\exp\left(-\frac{|x-x_0|^2}{4(t_0-t)}\right) .$$

Then

$$\Psi^\epsilon_{(t_0,x_0)}(r) = r^{-2}\int_{T_r^-(t_0)}\left(|\nabla u_\epsilon|^2 + 2B_\epsilon(u_\epsilon)\right) G_{(t_0,x_0)} + \\ - \frac{1}{2}r^{-2}\int_{T_r^-(t_0)}\frac{1}{t_0-t}\,u_\epsilon^2\,G_{(t_0,x_0)}$$

satisfies the monotonicity formula

$$\Psi^\epsilon_{(t_0,x_0)}(\sigma) - \Psi^\epsilon_{(t_0,x_0)}(\rho) \geq \int_\rho^\sigma r^{-1-2}\int_{T_r^-(t_0)}\frac{1}{t_0-t}\Big(\nabla u_\epsilon\cdot(x-x_0) \\ - 2(t_0-t)\partial_t u_\epsilon - u_\epsilon\Big)^2 G_{(t_0,x_0)}\,dr \geq 0 .$$

The key to our result is then an *estimate for the parabolic mean frequency.*

Proposition 1 *On the closed set* $\Sigma := \{(t,x)\in(0,\infty)\times\mathbf{R}^n : \Psi_{(t,x)}(0+) = 2H_n\}$ *the parabolic mean frequency*

$$2\left(\int_{T_r^-(t)}\frac{1}{t-s}u^2\,G_{(t,x)}\right)^{-1}\int_{T_r^-(t)}|\nabla u|^2\,G_{(t,x)} \geq 1 .$$

The function $r \mapsto r^{-2}\int_{T_r^-(t)}\frac{1}{t-s}u^2\,G_{(t,x)}$ *is non-decreasing and has a right limit* $\theta^2(t,x)\int_{T_1^-(0)}\frac{1}{-s}|x_1|^2 G_{(0,0)}$ *. The function* θ *is upper semicontinuous on* Σ *. At each* $(t,x)\in\Sigma$

$$\int_0^r s^{-3}\int_{T_s^-(t)}(1-\chi)G_{(t,x)}\,ds \to 0 \text{ as } r \to 0 .$$

It is a surprising fact that the parabolic mean frequency is bounded from below at each point of highest density, which includes the set Σ_* . As a consequence we obtain *unique tangent cones* for a.e. time and at $\mathcal{H}^{n-1}$-a.e. point of the graph of u , whence GMT-tools lead to our result.

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