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Singular limit of a reaction-diffusion system with resource-consumer interaction

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Abstract

We consider a two component reaction-diffusion system with a small parameter ϵ

$$\begin{cases} u_t = d_u\Delta u + \dfrac{1}{\epsilon}(u^m v - au^n), \\ v_t = d_v\Delta v - \dfrac{1}{\epsilon}u^m v, \end{cases}$$

where m and n are positive integers, together with zero-flux boundary conditions. It is known that any nonnegative solution becomes spatially homogeneous for large time. In particular when $m \geq n \geq 1$, there exists some positive constant v_∞^ϵ such that $(u^\epsilon, v^\epsilon)(x,t) \to (0, v_\infty^\epsilon)$ as t tends to infinity. In order to approximate the value of v_∞^ϵ, we derive a limiting problem when $\epsilon \downarrow 0$, which in turn enables us to determine the limiting value v_∞ of v_∞^ϵ under some conditions on the values of m, n and on the initial functions $(u,v)(x,0)$.

1 Introduction

Among many classes of reaction-diffusion (RD) systems, we restrict ourselves to the following rather specific two component RD system :

$$\begin{cases} u_t = d_u\Delta u + ku^m v - au^n, \\ v_t = d_v\Delta v - ku^m v, \end{cases} \tag{1.1}$$

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where d_u, d_v are diffusive rates for u and v respectively and m, n are positive integers. System (1.1) is a model for cubic autocatalytic chemical reaction processes

$$\begin{cases} mU + V \longrightarrow (m+1)U \\ nU \stackrel{\cdot}{\longrightarrow} P, \end{cases}$$

where u, v are the concentrations of U, V, respectively, k and a are the reaction rates which are positive constants and m, n are some positive integers. In the specific case where $m = n = 1$, (1.1) is a diffusive epidemic model where u and v are respectively the population densities of infective and susceptable species [KM]. When $m = 2$, $n = 1$, it is the Gray-Scott model without feeding process [GS]. Fundamental problems for (1.1) involve the global existence, uniqueness and asymptotic behavior of nonnegative solutions. Let us consider (1.1) in a smooth bounded domain Ω (in $\mathbb{R}^N$) together with the boundary and initial conditions

$$\frac{\partial u}{\partial \nu}(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 \quad x \in \Omega, \tag{1.3}$$

where ν stands for the outward normal unit vector to $\partial\Omega$. If $a = 0$, (1.1) reduces to

$$\begin{cases} u_t = d_u \Delta u + k u^m v, \\ v_t = d_v \Delta v - k u^m v, \end{cases} \tag{1.4}$$

which is called a consumer and resource system with balance law. There are many papers devoted to the system (1.4), (1.2), (1.3) (e.g. [Al], [Ma], [HK], [HY], [HMP], [Pa], [Ba], [Ho1]). Indeed, we know that as $t \to \infty$, $(u,v)(t)$ converges uniformly in $\bar{\Omega}$ to $(u_\infty, 0)$ where u_∞ is explicitly given by $u_\infty = < u_0 + v_0 >$. Here $< w >$ is the spatial average of w over Ω. Furthermore, it is proved by [Ho1] that for $m > 1$ there exists some constant $K > 0$ such that

$$\|(u(t) - u_\infty, v(t))\|_{L^\infty(\Omega)} \leq K t^{-\frac{1}{m-1}} \quad \text{as } t \to \infty.$$

On the other hand, if $a > 0$ [Ho2], the asymptotic state depends on the values of m and n. If $n > m \geq 1$, $(u,v)(t)$ converges to $(0,0)$ uniformly in $\bar{\Omega}$ as $t \to \infty$. On the contrary, if $m \geq n \geq 1$, there exists a positive constant v_∞ such that $(u,v)(t)$ converges to $(0, v_\infty)$ uniformly in $\bar{\Omega}$ as $t \to \infty$. Therefore every solution of (1.1)-(1.2) becomes spatially homogeneous and the fundamental problems have been already solved. However, we still have the following questions :

(*i*) When $m \geq n \geq 1$, how does the asymptotic state v_∞ depend on the initial functions u_0, v_0, on k, a and on the domain Ω ?

(ii) How is the transient behavior of solutions (u, v) of (1.1)-(1.3) ?

We have not yet been able to completely answer these questions, except in some special cases. Consider first a limiting situation where the reaction rates k and a are both sufficiently small (or, in other words, the diffusion rates are very large), so that (1.1) can be rewritten as

$$\begin{cases} u_t = \dfrac{1}{\epsilon} d_u \Delta u + u^m v - a u^n, \\ \\ v_t = \dfrac{1}{\epsilon} d_v \Delta v - u^m v. \end{cases} \tag{1.5}$$

Here we may set $k = 1$. For sufficiently small $\epsilon > 0$, the two-timing method reveals that the solution (u, v) becomes immediately spatially homogeneous and then its time evolution is described by the solution of the initial value problem for the following system of ordinary differential equations :

$$\begin{cases} U_t = U^m V - a U^n, \\ V_t = -\, U^m V, \end{cases} \tag{1.6}$$

together with the initial conditions

$$(U, V)(0) = (< u_0 >, < v_0 >). \tag{1.7}$$

The phase plane analysis shows that there exists some positive constant V_∞ such that as $t \to \infty$ the solution $(U, V)(t)$ of (1.6), (1.7) converges to $(0, V_\infty)$, where V^∞ approximately gives the value v_∞ for the original problem (1.1)-(1.3). For more precise discussion, we refer to the papers by [CHS], [EM]. Another limiting situation is the opposite case when k and a are both very large. Let us rewrite (1.1) as

$$\begin{cases} u_t = d_u \Delta u + \dfrac{1}{\epsilon}(u^m v - a u^n), \\ \\ v_t = d_v \Delta v - \dfrac{1}{\epsilon} u^m v. \end{cases} \tag{1.8}$$

We first present some numerical simulations of the one-dimensional problem corresponding to (1.8) with small but not zero ϵ in the interval $I = (0, L)$, where the corresponding boundary and initial conditions are given by (1.2) and (1.3) respectively, and where the initial functions satisfy

$$\begin{cases} u(x, 0) = u_0(x) \geq 0 & \text{(the support is near } x = 0\text{, as in Fig.1-1)} \\ v(x, 0) = v_0 > 0 & \text{which is constant.} \end{cases} \tag{1.9}$$

Here we suppose that $m = n = 1$. If v_0 is relatively small, $u(x,t)$ becomes uniformly zero and then $v(x,t)$ becomes spatially homogeneous and eventually tends to some positive constant v_∞ (Fig. 1-1). On the other hand, if v_0 is relatively large, the situation is changed, that is, when the interval L is very long, u and v form a pulse and a front wave respectively, and propagate fast to the right direction, as if they were a traveling wave, and the pulse u annihilates on hitting the boundary $x = L$, so that u tends to zero and v tends to some constant v_∞ (Fig.1-2). It turns out that there are two kinds of transient behavior for solutions (u,v) of (1.8), (1.2), (1.3). In order to understand these behaviors, the information about traveling wave solutions of (1.1) is very useful. When $m = n = 1$, Hosono and Ilyas [HI] showed that if $a < v_0$, then there are traveling wave solutions $(u,v)(z)$ $(z = x - ct)$ with velocity $c \geq c^* = 2\sqrt{d_u(v_0 - a)}$, while if $a \geq v_0$, there are no traveling wave solutions. This indicates that the transient behavior of solutions can be classified according to the critical value $v_0 = a$.

The transient behavior of solutions in higher space dimension is not so simple, sensitively depending on the values of m, n and a, even if they eventually become spatially homogeneous. In fact, it was numerically observed in the previous paper [FHMW] that when $m = 2$, $n = 1$, there appear very complex transient patterns for the behavior of (u,v), if one chooses suitable values of the ratio $d = d_v/d_u$ and of v_0.

Our aim is to answer question (i). To that purpose we study the asymptotic behavior as $\epsilon \to 0$ of solutions (u^ϵ, v^ϵ) of System (1.8) together with the boundary and initial conditions (1.2) and (1.3). We assume that the initial functions u_0 and v_0 satisfy the hypothesis $\|u_0\|^{m-n}_{L^\infty(\Omega)}\|v_0\|_{L^\infty(\Omega)} < a$ and derive the limiting system corresponding to (1.8) as ϵ tends zero, which in turn yields the asymptotic limit of the constant v^ϵ_∞ as $\epsilon \to 0$. We refer to [HMW] for the complete proofs of the results which we present below.

2 Results

We may use a space rescaling which amounts to setting $d_u = 1$ and $d_v = d$ and consider the following ϵ-family of parabolic problems :

$$
(P^\epsilon)\quad \begin{cases} u_t = \Delta u + \dfrac{1}{\epsilon}(u^m v - a u^n) & \text{in } Q := \Omega \times (0,\infty) \\ v_t = d\Delta v - \dfrac{1}{\epsilon} u^m v & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{for all } x \in \Omega, \end{cases}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $m \geq n \geq 1$, d and a are positive constants and $u_0, v_0 \in C^1(\bar{\Omega})$ are two nonnegative functions. In the sequel we use the notation $Q_T := \Omega \times (0,T)$.

It is well known (see [HY], [Ho2]) that there exists an unique global bounded non negative smooth solution pair (u^ϵ, v^ϵ) of Problem (P^ϵ). We make the hypothesis

$$H_a \quad : \quad \|u_0\|^{m-n}_{L^\infty(\Omega)} \|v_0\|_{L^\infty(\Omega)} < a,$$

and set

$$M_1 := \|u_0\|_{L^\infty(\Omega)} \quad \text{and} \quad M_2 := \|v_0\|_{L^\infty(\Omega)},$$

so that Hypothesis H_a can be rewritten as

$$M_1^{m-n} M_2 < a.$$

The main result of this paper is the following :

Theorem 1. *Let $T > 0$ be arbitrary. As $\epsilon \to 0$*

$$u^\epsilon \to 0 \quad \text{in } C(\bar{\Omega} \times [\mu, \infty)) \cap L^2(Q_T), \tag{2.1}$$

for all $\mu > 0$ and there exists a function $v \in L^2(Q_T)$ such as

$$v^\epsilon \to v \quad \text{in } L^2(Q_T). \tag{2.2}$$

Moreover the function v is the unique classical solution of the problem

$$(P^0) \quad \begin{cases} v_t = d\Delta v & \text{in } Q, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ v(x, 0) = \bar{V}(x) & \text{for all } x \in \Omega, \end{cases}$$

and

$$\bar{V}(x) = \lim_{t \to \infty} V(x, t),$$

where (U, V) is the unique solution of the initial value problem (Q^0)

$$(Q^0) \quad \begin{cases} U_t = U^m V - aU^n & \text{in } Q, \\ V_t = -U^m V & \text{in } Q, \\ U(x, 0) = u_0(x) \quad V(x, 0) = v_0(x) & \text{for all } x \in \Omega. \end{cases}$$

In order to prove this result, we set $\tau = \dfrac{t}{\epsilon}$ and introduce the functions

$$U^\epsilon(x, \tau) := u^\epsilon(x, t) \quad V^\epsilon(x, \tau) := v^\epsilon(x, t),$$

which satisfy the problem

$$(Q^\epsilon)\quad \begin{cases} U_t = \epsilon\Delta U + U^m V - aU^n & \text{in } Q, \\ V_t = \epsilon d\Delta V - U^m V & \text{in } Q, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ U(x,0) = u_0(x) \quad V(x,0) = v_0(x) & \text{for all } x \in \Omega. \end{cases}$$

We recall [Ho2] that

$$(u^\epsilon, v^\epsilon)(t) \to (0, v^\epsilon_\infty) \quad \text{in } C(\bar{\Omega}) \text{ as } t \to \infty, \tag{2.3}$$

where v^ϵ_∞ is a constant satisfying

$$v^\epsilon_\infty > 0,$$

and, if $m = n$

$$v^\epsilon_\infty < a.$$

The second result which we prove is the following

Theorem 2. *We have that*

$$v^\epsilon_\infty \to \frac{1}{|\Omega|}\int_\Omega \bar{V}(x)dx \quad \text{as } \epsilon \to 0. \tag{2.4}$$

Remark. *In the case that $m = n$, the condition H_a becomes $\|v_0\|_{L^\infty(\Omega)} < a$. Suppose that it is not satisfied ; then Theorem 2 does not hold. As a counter example, choose u_0 with support in $[0, \frac{1}{2}]$ and $v_0 = 3a$ on $\Omega = (0,1)$. Then, the study of the ODE system shows that $V(x,t) = v_0 = 3a$ for $x \in (\frac{1}{2}, 1]$ and all $t > 0$ so that*

$$\int_0^1 \bar{V}(x)dx \geq \frac{3a}{2},$$

whereas

$$v^\epsilon_\infty < a.$$

Finally we study two special cases *without assuming Hypothesis H_a*. As the first one we take $a = 0$. Then the $L^1(\Omega)$ norm of $(u^\epsilon + v^\epsilon)(t)$ is preserved in time and equal to the average over Ω of $(u_0 + v_0)$. Thus the asymptotic behavior of $(u^\epsilon, v^\epsilon)(t)$ as $t \to \infty$ is well known. More precisely, we prove the following result:

Theorem 3. *Let (u^ϵ, v^ϵ) be the solution of (P^ϵ) with $a = 0$. Then*

$$v^\epsilon \to 0 \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0, \tag{2.5}$$

and

$$u^\epsilon \to u \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0,$$

where u is the unique solution of the problem

$$\begin{cases} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) + v_0(x) & \text{for all } x \in \Omega. \end{cases}$$

The second case which we consider is the case that $n > m \geq 1$. Then we have that (see [Ho2])

$$(u^\epsilon, v^\epsilon)(t) \to (0,0) \quad \text{as } t \to \infty.$$

We prove the following result.

Theorem 4. *Fix $T > 0$ arbitrarily and suppose that $n > m \geq 1$ and that $u_0(x) > 0$ for all $x \in \Omega$. Then*

$$u^\epsilon(t),\ v^\epsilon(t) \to 0 \quad \text{in } L^2(Q_T) \text{ as } \epsilon \to 0. \tag{2.6}$$

In this paper, we have addressed the question of determing how the asymptotic state v_∞ depends on the initial functions. To that purpose, we have introduced a small parameter ϵ such that the reaction terms are very strong, compared with diffusion terms and then derived a singular limit equation as ϵ tends to zero, under the restriction that the initial functions satisfy the hypothesis H_a. We have then been able to derive an approximate value for v_∞. In the situation where H_a is violated, a different type of singular limit equation should be derived. We plan to perform this derivation in future work.

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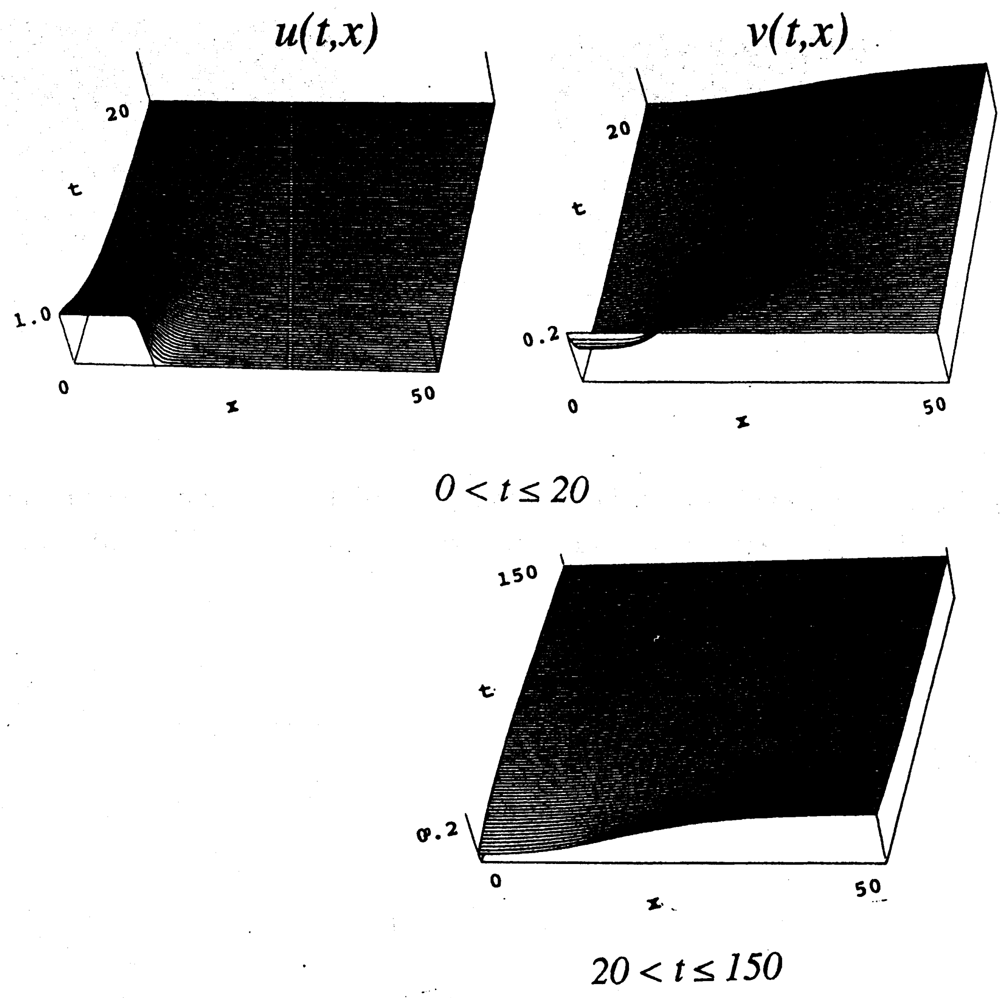

Fig. 1-1 Time evolution of the solution (u,v) of the one dimensional problem (1.1)-(1.3) with $m = n = 1$ where $d_u = 1.0$, $d_v = 11.5$, $k = 1$ and $v_0 = 0.2$

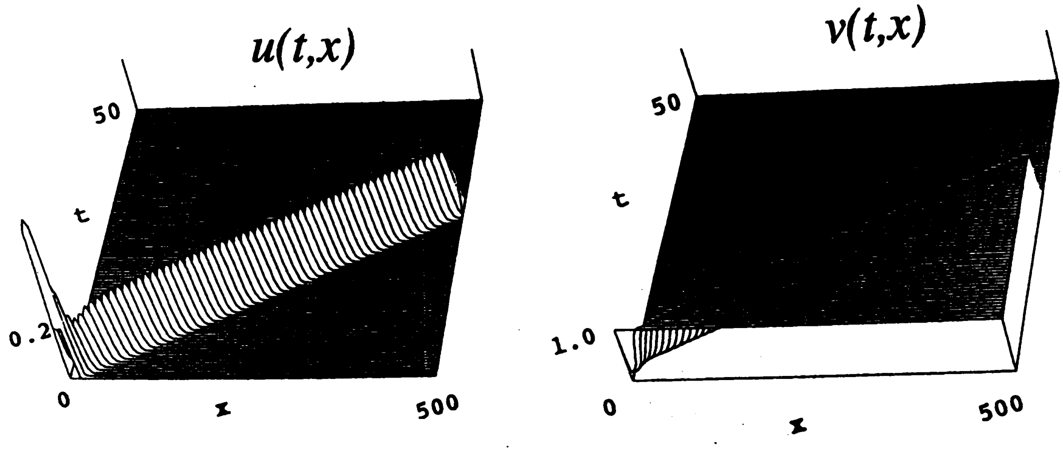

Fig. 1-2 (a) Time evolution of the solution (u,v) of the one dimensional problem (1.1)-(1.3) with $m = n = 1$ where $d_u = 1.0$, $d_v = 11.5$, $k = 1$ and $v_0 = 1.0$

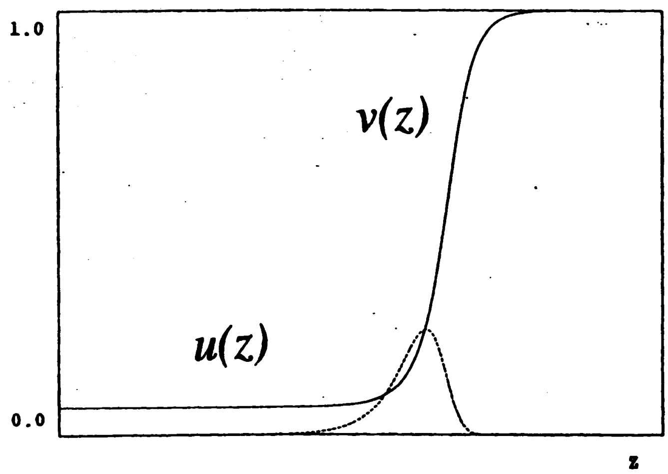

Fig. 1-2 (b) Spatial profiles of traveling wave solutions (u,v) of (1.1)-(1.3) where the parameters are same as those in (a).