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REACTION–DIFFUSION: FROM SYSTEMS TO NONLOCAL EQUATIONS IN A CLASS OF FREE BOUNDARY PROBLEMS

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We consider a class of reaction–diffusion systems where the diffusivity of the second equation tends to infinity and we illustrate in model problems the use of energy estimates for basic existence and convergence results of the solutions.

We consider also free boundary problems of obstacle type as a special class of partial differential equations with discontinuous nonlinearities, following the plan:

1. Elliptic problems
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Although most results of this paper can be found in previous works, namely in a joint work with D. Hilhorst [HR] and in the references quoted there, some new extensions to the obstacle problem, whose general references can be found in the books [L], [F] or [R2], are taken from [R4] and [RS]. In this last work an application to the diffusion of the oxygen with a nonlocal diffusion coefficient is considered. Other motivations for considering these type of mathematical problems arise in the study of dynamics of the mechanism of basic pattern formation (see, for instance, [N], [LS], [HS] or [K]), in excitable media (see [OMK] and its references), in combustion problems (see, for instance, [FT], [FN] or [BRS]) or in some phase transitions models (see [CHL] and its references).

1 – Elliptic problems

1. A model nonlocal equation

Consider in a bounded open subset $\Omega \subset \mathbb{R}^n$, an arbitrary $f \in L^2(\Omega)$ and a given measurable function $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$, continuous in the second variable, i.e., $a(x,\cdot) \in C^0(\mathbb{R})$ for a.e. $x \in \Omega$ and, such that, for some constants $\underline{\alpha}, \overline{\alpha}$:

$$0 < \underline{\alpha} \le a(x,\rho) \le \overline{\alpha}, \qquad \forall \rho \in \mathbb{R}, \quad \text{a.e. } x \in \Omega\,. \tag{1.1}$$

For $\sigma > 0$, we consider the homogeneous Dirichlet–Neumann problem for the reaction–diffusion system (∂_n denotes the normal derivative $\partial/\partial n$):

$$-\nabla\cdot\Big(a(v_\sigma)\,\nabla u_\sigma\Big) = f \;\text{ in } \Omega\,, \qquad u_\sigma = 0 \;\text{ on } \partial\Omega\,, \tag{1.2}$$

$$-\sigma\,\Delta v_\sigma = u_\sigma - v_\sigma \;\text{ in } \Omega\,, \qquad \partial_n v_\sigma = 0 \;\text{ on } \partial\Omega\,. \tag{1.3}$$

Proposition 1.1. *There exist solutions (u_σ, v_σ) to (1.2),(1.3) such that*

$$u_\sigma \to u \;\text{ in } H^1_0(\Omega)\,, \qquad v_\sigma \to \fint_\Omega u \;\text{ in } H^1(\Omega) \qquad \text{as } \sigma \to \infty\,,$$

where $\fint_\Omega u$ is the average of u in Ω and u solves the nonlocal problem

$$-\nabla\cdot\Big(a\Big(\fint_\Omega u\Big)\,\nabla u\Big) = f \;\text{ in } \Omega\,, \qquad u = 0 \;\text{ on } \partial\Omega\,. \tag{1.4}$$

Proof: We write (1.2) and (1.3) in variational form

$$u_\sigma \in H^1_0(\Omega)\colon \int_\Omega a(v_\sigma)\,\nabla u_\sigma\cdot\nabla\varphi = \int_\Omega f\,\varphi\,, \qquad \forall \varphi \in H^1_0(\Omega)\,, \tag{1.5}$$

$$v_\sigma \in H^1(\Omega)\colon \;\sigma \int_\Omega \nabla v_\sigma\cdot\nabla\zeta = \int_\Omega (u_\sigma - v_\sigma)\,\zeta\,, \qquad \forall \zeta \in H^1(\Omega)\,. \tag{1.6}$$

For any given $v_\sigma \in L^2(\Omega)$ in (1.5), with $\varphi = u_\sigma$ we obtain the *a priori* estimate

$$c_0 \int_\Omega u_\sigma^2 \le \int_\Omega |\nabla u_\sigma|^2 \le C\,, \tag{1.7}$$

where C depends only on $\underline{\alpha}$, f and the constant c_0 of Poincaré inequality, and therefore it is independent of v_σ and $\sigma > 0$.

Letting $\zeta = v_\sigma$ in (1.6) we immediately obtain also

$$\int_\Omega v_\sigma^2 \le \int_\Omega u_\sigma^2 \le C' = \frac{C}{c_0} \quad \text{and} \quad \int_\Omega |\nabla v_\sigma|^2 \le \frac{C'}{\sigma} . \tag{1.8}$$

Since (1.6) is a linear problem in v_σ for fixed $u_\sigma \in L^2(\Omega)$, we easily construct a nonlinear operator S from the ball B of radius $\sqrt{C'}$ in $L^2(\Omega)$, by solving (1.5) with those solutions of (1.6). By (1.7), its image $S(B) \subset B$ and S is compact by the compactness of $H_0^1(\Omega) \subset L^2(\Omega)$. By the Schauder fixed point theorem, there exist solutions (u_σ, v_σ) to (1.5),(1.6). By the estimates (1.7) and (1.8), for subsequences, we have as $\sigma \to \infty$

$$u_\sigma \rightharpoonup u \ \text{ in } H_0^1(\Omega)\text{-weak} \quad \text{and} \quad v_\sigma \to v = \text{const. in } H^1(\Omega) .$$

Letting $\zeta = 1$ in (1.6) we have $\int_\Omega v_\sigma = \int_\Omega u_\sigma$ and since $⨍_\Omega v_\sigma \to ⨍_\Omega v$ and $⨍_\Omega u_\sigma \to ⨍_\Omega u$ as $\sigma \to \infty$, we find $v = ⨍_\Omega v = ⨍_\Omega u$.

Taking this limit in (1.5), we obtain

$$u \in H_0^1(\Omega): \int_\Omega a\Big(⨍_\Omega u\Big) \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi, \quad \forall \varphi \in H_0^1(\Omega) , \tag{1.9}$$

which is the variational formulation of (1.4). Finally comparing (1.9) with (1.5), and observing that $a(v_\sigma) \to a(⨍_\Omega u)$ in $L^p(\Omega)$, $\forall p < \infty$ and a.e. in Ω, we easily conclude the strong convergence $u_\sigma \to u$ in $H_0^1(\Omega)$. ∎

In general we cannot expect uniqueness of solutions in (1.2),(1.3) nor in (1.4) even in the case when a is independent of x, as it was observed in [CR]. Indeed, we remark that u is a solution of

$$-a\Big(⨍_\Omega u\Big) \Delta u = f \ \text{ in } \Omega , \qquad u = 0 \ \text{ on } \partial\Omega , \tag{1.10}$$

if and only if $u = u_1/a(⨍_\Omega u)$, where u_1 is the unique solution of (1.10) with $a \equiv 1$. hence by integrating in Ω, we see that $\rho = ⨍_\Omega u$ solves the equation in $\mathbb{R}$

$$a(\rho) = ⨍_\Omega u_1/\rho . \tag{1.11}$$

Reciprocally, if ρ solves (1.11), then $u = \rho u_1 / ⨍_\Omega u_1$ solves (1.10).

Since the equation (1.11) may have, in general, more than one real root (it may have even a continuum of solutions) the same may occur for (1.10). However, this cannot happen if $a(x,\rho)$ is Lipschitz continuous in ρ, with small oscillation, i.e., if there exists a sufficiently small $\alpha' > 0$ such that

$$\Big|a(x,\rho) - a(x,\tau)\Big| \le \alpha' |\rho - \tau|, \qquad \text{a.e. } x \in \Omega . \tag{1.12}$$

Proposition 2. *There exists $\delta > 0$ such that, if (1.12) for $\alpha' < \delta$ then (1.4) admits at most one solution. The same conclusion holds for the system (1.2),(1.3), if, in addition, a is continuous in $x \in \overline{\Omega}$, $f \in L^p(\Omega)$ for $p > n$ and $\partial\Omega$ is of class C^1.*

Proof: If u and $\widehat{u}$ are two solutions to (1.4) (or (1.9)) then we may write for their difference $w = u - \widehat{u}$ (using (1.1), (1.7) and (1.12)):

$$\underline{\alpha} \int_\Omega |\nabla w|^2 \le \int_\Omega a\Big(\fint_\Omega u\Big) |\nabla w|^2 \le \int_\Omega \Big[a\Big(\fint_\Omega \widehat{u}\Big) - a\Big(\fint_\Omega u\Big)\Big] \nabla\widehat{u} \cdot \nabla w$$
$$\le \alpha' \Big|\fint_\Omega w\Big| \Big(\int_\Omega |\nabla\widehat{u}|^2\Big)^{1/2} \Big(\int_\Omega |\nabla w|^2\Big)^{1/2} \le \alpha' \sqrt{\frac{C}{c_0\,|\Omega|}} \int_\Omega |\nabla w|^2 .$$

Therefore if $\alpha' < \underline{\alpha}\sqrt{c_0\,|\Omega|/C}$, we must have $w = 0$, i.e. $u = \widehat{u}$.

For the system (1.2),(1.3) we need to use some elliptic regularity theory (see [R2], for references). If $f \in L^p(\Omega)$, $p > n$, we have $\widehat{u}_\sigma \in C^0(\overline{\Omega})$ and then also $\widehat{v}_\sigma \in C^0(\overline{\Omega})$; hence $a(\widehat{v}_\sigma) \in C^0(\overline{\Omega})$ and also $\nabla\widehat{u}_\sigma \in L^p(\Omega)$ for $p > n$. We observe $u_\sigma - \widehat{u}_\sigma$ solves the equation

$$\nabla \cdot \Big(a(v_\sigma)\, \nabla(u_\sigma - \widehat{u}_\sigma)\Big) = \nabla \cdot \Big\{\Big[a(v_\sigma) - a(\widehat{v}_\sigma)\Big]\, \nabla\widehat{u}_\sigma\Big\} \quad \text{in } \Omega .$$

Hence, using the generalized maximum principle in this equation, we have

$$\|u_\sigma - \widehat{u}_\sigma\|_{L^\infty(\Omega)} \le C \Big\|\Big[a(v_\sigma) - a(\widehat{v}_\sigma)\Big]\, \nabla\widehat{u}_\sigma\Big\|_{L^p(\Omega)}$$
$$\le \alpha' \widehat{C}\, \|v_\sigma - \widehat{v}_\sigma\|_{L^\infty(\Omega)} \le \alpha' \widehat{C}\, \|u_\sigma - \widehat{u}_\sigma\|_{L^\infty(\Omega)} .$$

The last inequality is also a consequence of the maximum principle applied to (1.3). Again, we see that if $\alpha' < 1/\widehat{C}$ we must have $u_\sigma = \widehat{u}_\sigma$ and the uniqueness follows for the system (1.2),(1.3). ∎

1.2. Discontinuous reaction terms

We can extend the framework of the preceding section to more general reaction terms in the right hand side of (1.2). We may suppose $f = f(x, u, v)$, under appropriate growth conditions on (u, v), and allow this dependence to have certain discontinuities. However, the notion of solution must be extended as the following counter-example shows.

If h denotes the Heaviside function ($h(s) = 1$ if $s > 0$, and $h(s) = 0$ if $s \le 0$), consider the Dirichlet problem

$$-\Delta u = h\Big(\mu - \fint_\Omega u\Big) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega , \tag{1.13}$$

where $0 < \mu < \fint_\Omega u_1$. Here u_1 denotes the solution of (1.13) with h replaced by 1 and we have $\fint_\Omega u_1 > 0$. Since $0 \le h \le 1$, by the maximum principle, if u solves (1.13) we have

$0 \leq u \leq u_1$ in Ω and we obtain the absurds: if $\fint_\Omega u \geq \mu > 0$ then $h \equiv 0$ and $u = 0$; if $\fint_\Omega u < \mu < \fint_\Omega u_1$ then $h \equiv 1$ and $u = u_1$. Therefore it cannot exists a classical solution to (1.13). However, using the method of "filling in the jumps" and introducing the maximal monotone graph H associated with h by setting $H(s) = h(s)$ if $s \neq 0$ and $H(0) = [0,1]$, we replace (1.13) by

$$-\Delta u \in H\Big(\mu - \fint_\Omega u\Big) \text{ a.e. in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.14}$$

Then we may obtain solutions to (1.14) provided $\fint_\Omega u = \mu \in [0, \fint_\Omega u_1]$. Indeed if $u_\lambda \in H_0^1(\Omega)$ denotes the solution in Ω of $-\Delta u = \lambda \in [0,1]$, we may construct the linear mapping $[0,1] \ni \lambda \mapsto \fint_\Omega u_\lambda \in [0, \fint_\Omega u_1]$. Hence, for each $\mu \in [0, \fint_\Omega u_1]$ there exist one $\lambda \in [0,1]$ such that u_λ is a solution to (1.14).

In general, we have nonuniqueness for (1.14). For instance, for any function $g \in L^2(\Omega)$, $0 \leq g \leq 1$, the solution $u_g \in H_0^1(\Omega)$ of $-\Delta u = g$ in Ω, clearly also solves (1.14) for $\mu = \fint_\Omega u_g$.

We consider now more general discontinuities with a given measurable function $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that,

$$|f(x,u,v)| \leq f_0(x) \quad \text{a.e. } x \in \Omega, \ \forall u, v \in \mathbb{R}, \tag{1.15}$$

where $f_0 \in L^p(\Omega)$, with $p \geq 2n/(n+2)$ if $n \geq 3$ or $p > 1$ if $n = 2$, is such that $f_0 \in H^{-1}(\Omega)$, by Sobolev imbedding. More generally we could also admit a certain growth in u and v under suitable conditions.

As in [C] and [HR], we construct the multivalued function F: $(x,u,v) \mapsto [\underline{f}(x,u,v), \overline{f}(x,u,v)]$, where $\underline{f}$ and $\overline{f}$ are, respectively, lower and upper semicontinuous functions in (u,v) defined by

$$\underline{f}(x,u,v) = \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|z-u|+|w-v| \leq \delta} f(x,z,w)$$

and

$$\overline{f}(x,u,v) = \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|z-u|+|w-v| \leq \delta} f(x,z,w), \qquad \text{for a.e. } x \in \Omega .$$

Of course, if f is continuous in (u,v) we have $f(u,v) = \overline{f}(u,v) = \underline{f}(u,v)$.

We replace (1.2) by the extended reaction–diffusion system

$$-\nabla \cdot (a(v_\sigma)\nabla u_\sigma) \in F(u_\sigma, v_\sigma) \text{ in } \Omega, \qquad u_\sigma = 0 \text{ on } \partial\Omega, \tag{1.16}$$

$$-\sigma \Delta v_\sigma = u_\sigma - v_\sigma \text{ in } \Omega, \qquad \partial_n v_\sigma = 0 \text{ on } \partial\Omega. \tag{1.17}$$

Proposition 1.3. *Under the assumptions (1.1)–(1.15), there exist solutions (u_σ, v_σ) to (1.16)–(1.17) such that, as $\sigma \to \infty$, they converge to $(u, \fint_\Omega u)$ in $H_0^1(\Omega) \times H^1(\Omega)$, which is a solution to*

$$-\nabla \cdot \Big(a\Big(\fint_\Omega u\Big)\nabla u\Big) \in F\Big(u, \fint_\Omega u\Big) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.18}$$

Proof: First we regularize f by mollification in (u, v) and, arguing as in [HR] (see also [Ra]), we suppose initially f is continuous in those variables, being the general case obtained by approximation and a passage to the limit as in the Theorem 5.1 of [HR].

The existence to (1.16),(1.17) is then reduced to a Schauder fixed point argument, provided we obtain the equivalent to the a priori estimates (1.7) and (1.8). Now we use Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$ ($q \leq 2n/(n-2)$ if $n \geq 3$, or any $q < \infty$ if $n = 2$) and we reobtain the estimate (1.7) from

$$C_q \|u_\sigma\|_{L^q}^2 \leq \underline{\alpha} \int_\Omega |\nabla u_\sigma|^2 \leq \|f_0\|_{L^p} \|u_\sigma\|_{L^q}$$

where $q = p/(p-1)$. Hence (1.8) still holds, with constants independent of σ and independent of the mollification parameter.

In case of a continuous $f(x, \cdot)$ the passage to the limit is done without difficulty since, by compactness, we may also assume $u_\sigma \to u$ in $L^2(\Omega)$.

For F discontinuous but defined in terms of $\underline{f}$ and $\overline{f}$ as above, the passage to the limit $\sigma \to \infty$ is performed by using the following Lemma. ∎

Lemma 1.1. *Let $\varphi_\sigma \in F(u_\sigma, v_\sigma)$ a.e. in Ω, $\varphi_\sigma \rightharpoonup \varphi$ in $L^1(\Omega)$-weak. If $u_\sigma \to u$ and $v_\sigma \to v$ in $L^1(\Omega)$-strong, then $\varphi \in F(u, v)$ a.e. in Ω.*

Proof: We use an argument of [Ra] as in Theorem 5.3 of [HR]. For any $\eta > 0$, we may consider that $(u_\sigma, v_\sigma) \to (u, v)$ uniformly in $\Omega_\eta = \Omega \backslash \mathcal{O}$ with $\text{meas}(\mathcal{O}) < \eta$. Since $\varphi_\sigma \in F(u_\sigma, v_\sigma)$ is equivalent to

$$\underline{f}(x, u_\sigma(x), v_\sigma(x)) \leq \varphi_\sigma(x) \leq \overline{f}(x, u_\sigma(x), v_\sigma(x)) \quad \text{a.e. } x \in \Omega\,,$$

for any $g \in L^\infty(\Omega)$, $g \geq 0$, we have

$$\begin{aligned}\int_{\Omega_\eta} g\varphi &= \lim_\sigma \int_{\Omega_\eta} g\varphi_\sigma \geq \liminf_{\sigma\to\infty} \int_{\Omega_\eta} g\,\underline{f}(u_\sigma, v_\sigma) \\ &\geq \int_{\Omega_\eta} g \liminf_{\sigma\to\infty} \underline{f}(u_\sigma, v_\sigma) \geq \int_{\Omega_\eta} g\,\underline{f}(u, v)\end{aligned}$$

by Fatou's Lemma, semicontinuity and boundedness of $\underline{f}$ in Ω_η. Similarly we obtain $\varphi \leq \overline{f}(u, v)$ in Ω_η and, since η is arbitrary, we conclude that $\varphi \in F(u, v)$ a.e. in Ω. ∎

Remark 1.1. We may solve directly the nonlocal equation (1.18) by applying the fixed point Theorem of Schauder to the mollified problem with f_ε continuous and "approaching" F. Similarly to Lemma 1.1, $u_\varepsilon \to u$ in $L^1(\Omega)$ and $f_\varepsilon(u_\varepsilon, \fint_\Omega u_\varepsilon) \rightharpoonup \varphi$ in $L^1(\Omega)$-weak, implies $\varphi \in F(u, \fint_\Omega u)$ a.e. in Ω and we then obtain directly a solution to (1.18). See [HR] for the extension to the parabolic nonlocal problem.

1.3. Obstacle problems

In the equation (1.2) or (1.4), by the maximum principle, if $f \geq 0$ we have $u \geq 0$. But if f may change sign, i.e., $f = f^+ - f^-$ with $f^+ = \max(f, 0) \equiv 0$ and $f^- = (-f)^+ \not\equiv 0$, we cannot guarantee that u is nonnegative. If we impose then the unilateral constraint $u \geq 0$ in Ω, we have instead of (1.2) an obstacle problem, and we should look for u in the convex set

$$\mathbb{K} = \left\{ v \in H_0^1(\Omega) : \; v \geq 0 \;\text{ a.e. in } \Omega \right\} . \tag{1.19}$$

The variational formulation takes now the form

$$u_\sigma \in \mathbb{K} : \quad \int_\Omega a(v_\sigma) \, \nabla u_\sigma \cdot \nabla(\varphi - u_\sigma) \geq \int_\Omega f(\varphi - u_\sigma) , \qquad \forall \varphi \in \mathbb{K} , \tag{1.20}$$

where v_σ is given by (1.17) and $f = f(x)$ is given in $L^p(\Omega)$, with $p > 1$ if $n = 2$ or $p \geq 2n/(n+2)$ if $n \geq 3$. Taking $\varphi = 0$ in (1.20) we still have the estimate (1.7) and hence also (1.8). Using well-known properties of the obstacle problem (see [R2]), we can directly show that Propositions 1.1 and 1.2 hold for the problem (1.20),(1.3), being the corresponding nonlocal obstacle problem given by

$$u \in \mathbb{K} : \quad \int_\Omega a\Big(\fint_\Omega u \Big) \nabla u \cdot \nabla(\varphi - u) \geq \int_\Omega f(\varphi - u) , \qquad \forall \varphi \in \mathbb{K} . \tag{1.21}$$

We can regard the obstacle problem as a problem with the particular nonlinear discontinuity envolving the Heaviside graph:

$$F(x, u, v) = f^+(x) - f^-(x) \, H(u) . \tag{1.22}$$

In fact, if u denotes a solution to (1.16) (resp. to (1.18)), then, there exists a function $h = h(x) \in H(u(x))$ a.e. $x \in \Omega$, such that, with $a = a(v_\sigma)$ (resp. $a = a(\fint_\Omega u)$):

$$-\nabla \cdot (a \, \nabla u) = f^+ - f^- h \quad \text{a.e. in } \Omega . \tag{1.22}$$

Multiplying (1.22) by $-u^-$ and, integrating by parts, we obtain

$$\underline{\alpha} \int_\Omega |\nabla u^-|^2 \leq \int_\Omega a \, \nabla u \cdot \nabla(-u^-) = -\int_\Omega f^+ u^- + \int_\Omega f^- h u^- = -\int_\Omega f^+ u^- \leq 0 ,$$

since $h u^- = 0$. Then $u^- = 0$ and we have $u \geq 0$ in Ω, i.e. $u \in \mathbb{K}$.

Remarking that $(h - 1) \, u = 0$, for any $v \in \mathbb{K}$ we have a.e. in Ω

$$(f^+ - f^- h) \, (v - u) = \Big[f + f^- (1 - h) \Big] \, (v - u) \geq f(v - u)$$

and integrating (1.22) by parts in Ω, we conclude that we have as a special case of Proposition 3 the following conclusion.

Corollary 1.1. *With the choice (1.22), the solutions (u_σ, v_σ) to (1.16),(1.17) also solve (1.20),(1.17), and their cluster point $(u, \fint_\Omega u)$ as $\sigma \to \infty$ solves (1.18) and (1.21). In addition, under the assumptions of Proposition 1.2, the uniqueness of solutions holds and the whole sequence $(u_\sigma, v_\sigma) \to (u, \fint_\Omega u)$ converges in $H_0^1(\Omega) \times H^1(\Omega)$ as $\sigma \to \infty$.* ∎

Under additional conditions, in fact, the problem (1.16) (resp. (1.18)) with F given by (1.22) is equivalent to (1.20) (resp. (1.21)) as it was observed in [C] (see also [R2], page 146). Indeed, if u solves (1.20) (or (1.21)), it also satisfies the Lewy-Stampacchia's inequalities (see [R2], §5.3):

$$f \leq -\nabla \cdot (a \nabla u) \leq f^+ \quad \text{a.e. in } \Omega . \tag{1.23}$$

On the other hand, since $u \geq 0$ in Ω, we may consider two regions $\{u > 0\} = \{x \in \Omega : u(x) > 0\}$ and its complement $\{u = 0\}$ which is called the coincidence set. As it is well-known

$$-\nabla \cdot (a \nabla u) = f \quad \text{a.e. in } \{u > 0\} , \tag{1.24}$$

and, from (1.23), one should expect $\{u = 0\} \subset \{f \leq 0\}$ at least formally.

Assuming now more regularity, for instance, $f \in L^p(\Omega)$, $p > n/2$ (which yields $v_\sigma \in C^0(\overline{\Omega})$), and the coefficient a Lipschitz continuous in $x \in \Omega$ and in $\rho \in \mathbb{R}$,

$$|a_\rho'(x, \rho)| + |\nabla_x a(x, \rho)| \leq C, \quad \text{a.e. } x \in \Omega, \quad \rho \in \mathbb{R} , \tag{1.25}$$

by standard regularity in the obstacle problem (see [R2], §5.3 and its references) we have u_σ and u are in $W^{2,p}(\Omega)$ and satisfy

$$-\nabla \cdot (a \nabla u) = f + f^- \chi_{\{u=0\}} \quad \text{a.e. in } \Omega . \tag{1.26}$$

Here $\chi_{\{u=0\}}$ denotes the characteristic function of the coincidence set $\{u = 0\}$. Comparing (1.26) with (1.22), we easily see that we may choose $h = 1 - \chi_{\{u=0\}}$ and clearly $h \in H(u)$ a.e. in Ω, and u_σ and u satisfy also (1.16) and (1.18) with (1.22), respectively.

Using the equation (1.26) it is possible to show the continuous dependence of the coincidence set $\{u = 0\}$, through its characteristic function $\chi_{\{u=0\}}$, under the nondegeneracy assumption

$$f \neq 0 \quad \text{a.e. in } \Omega . \tag{1.27}$$

For instance, under the assumption (1.25), if u_j denote the solution to (1.20) corresponding to $v_j \to v$ in $C^0(\overline{\Omega})$, which $|\nabla v_j|$ are uniformly bounded in $L^\infty(\Omega)$, then not only $u_j \to u$ in $W^{2,p}(\Omega)$, where u is the solution to (1.20) corresponding to v, but also $\chi_{\{u_j=0\}} \to \chi_{\{u=0\}}$ in $L^q(\Omega)$, $\forall\, q < \infty$, provided (1.27) holds (see, for instance, Theor. 5:4.5 and Theor. 6:6.1 of [R2], respectively).

As in [R4], we can not only consider (1.20) associated with

$$v_\sigma \in H^1(\Omega): \quad \sigma \int_\Omega \nabla v_\sigma \cdot \nabla \zeta + \int_\Omega v_\sigma \zeta = \int_\Omega \chi_{\{u_\sigma=0\}} \zeta , \quad \forall \zeta \in H^1(\Omega) \tag{1.28}$$

instead of (1.17) or (1.6), but also consider the limit problem $\sigma \to \infty$ where the nonlocal obstacle problem (1.21) is replaced by

$$u \in \mathbb{K}: \quad \int_\Omega a(\langle u = 0\rangle)\, \nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u)\,, \qquad \forall v \in \mathbb{K}\,. \tag{1.29}$$

Here we have introduced the "fraction" of the coincidence set $\{u = 0\}$ with respect to the whole domain Ω:

$$\langle u = 0\rangle \;=\; \fint_\Omega \chi_{\{u=0\}} \;=\; \mathrm{meas}\{u=0\}/\,\mathrm{meas}(\Omega)\,. \tag{1.30}$$

Theorem 1.1. *Under the previous assumptions, namely (1.1), (1.25) and (1.27) with $f \in L^p(\Omega)$, $p > n/2$, and $\partial\Omega \in C^{1,1}$, there exist solutions $(u_\sigma, v_\sigma) \in [\mathbb{K} \cap W^{2,p}(\Omega)] \times W^{2,q}(\Omega)$, $\forall\, q < \infty$, to the coupled problem (1.20),(1.28), such that*

$$u_\sigma \to u \;\; \text{in } H_0^1(\Omega) \quad \text{and} \quad v_\sigma \to \langle u = 0\rangle \;\; \text{in } H^1(\Omega), \qquad \text{as } \sigma \to \infty\,,$$

where u is a solution to (1.29).

Proof: Remarking that by elliptic theory $v_\sigma \in W^{2,q}(\Omega) \cap D$, where $D = \{v \in C^1(\overline{\Omega}):\ 0 \leq v \leq 1\}$, the existence of solution for (1.20),(1.28) can be found as a Schauder fixed point in Ω for the mapping $w \mapsto z \mapsto \chi_{\{z=0\}} \mapsto w_\sigma$, where z solves uniquely (1.20) with v_σ replaced by $w \in D$ and w_σ solves uniquely (1.28) with $\chi_{\{z=0\}}$ in the second hand term (see [R4], for details).

For the passage to the limit $\sigma \to \infty$, as in Proposition 1, we know that $v_\sigma \to V = \text{const.}$ in $H^1(\Omega)$ and also

$$\langle u_\sigma = 0\rangle \;=\; \fint_\Omega v_\sigma \to V\,. \tag{1.31}$$

Then, we may pass to the limit in (1.20) and show that $u_\sigma \to u$ first in $H_0^1(\Omega)$-weak and afterwards also strongly, where u solves (uniquely) (1.20) for V in place of v_σ. By regularity, u also solves a.e. in Ω the equation (1.26) with $a = a(V)$. By Theorem 6:6.1 of [R1], we have then $\chi_{\{u_\sigma=0\}} \to \chi_{\{u=0\}}$ in $L^q(\Omega)$, $\forall\, q < \infty$, due to assumption (1.27). But then, using (1.31) we find $V = \fint_\Omega \chi_{\{u=0\}} = \langle u = 0\rangle$ and u solves (1.29). ∎

2 – Parabolic problems

2.1. Nonlocalization via the shadow system

We consider now the natural extension of the model nonlocal equation of Section 1.1 to an evolution problem in a cylindrical domain $Q_T = \Omega \times]0, T[$, $T > 0$, with $\Omega \subset \mathbb{R}^n$ an open bounded subset and with a prescribed $f = f(x,t) \in L^2(Q_T)$. We give $a\colon Q_T \times \mathbb{R} \to \mathbb{R}$, $a(x,t,\cdot) \in C^0(\mathbb{R})$, satisfying (1.1) for a.e. $(x,t) \in Q_T$ and initial conditions

$$u_0, v_0 \in L^2(\Omega) \ .$$

The for each $\sigma, \tau > 0$ the corresponding parabolic reaction-diffusion system reads ($\partial_t = \partial/\partial t$ and $\Sigma_T = \partial\Omega \times]0,T[$):

$$\partial_t u_{\tau\sigma} - \nabla \cdot \Big(a(v_{\tau\sigma})\, \nabla u_{\tau\sigma}\Big) = f \quad \text{in } Q_T \tag{2.1}$$

$$u_{\tau\sigma} = 0 \ \text{ on } \Sigma_T\,, \quad u_{\tau\sigma}(0) = u_0 \ \text{ in } \Omega \tag{2.2}$$

$$\tau\, \partial_t v_{\tau\sigma} - \sigma\, \Delta v_{\tau\sigma} + v_{\tau\sigma} = u_{\tau\sigma} \quad \text{in } Q_T \tag{2.3}$$

$$\partial_n v_{\tau\sigma} = 0 \ \text{ on } \Sigma_T\,, \quad v_{\tau\sigma}(0) = v_0 \ \text{ in } \Omega\ . \tag{2.4}$$

The passage to a nonlocal equation may be performed in two steps by letting first $\sigma \to \infty$ with fixed $\tau > 0$ and afterwards $\tau \to 0$. The intermediate shadow system is given by the Cauchy–Dirichlet (2.2) problem for ($\dot{\xi} = d\xi/dt$)

$$\partial_t u_\tau - \nabla \cdot \Big(a(\xi_\tau)\, \nabla u_\tau\Big) = f \quad \text{in } Q_T\,, \tag{2.5}$$

$$\tau\, \dot{\xi}_\tau + \xi_\tau = \fint_\Omega u_\tau \quad \text{in }]0,T[,\ \ \xi_\tau(0) = \fint_\Omega v_0\,, \tag{2.6}$$

and the nonlocal parabolic equation in the limit case $\tau = 0$ is now

$$\partial_t u - \nabla \cdot \Big(a\Big(\fint_\Omega u\Big)\, \nabla u\Big) = f \quad \text{in } Q_T\,, \tag{2.7}$$

with the conditions (2.2).

The standard energy estimates can be obtained by integration in $Q_t = \Omega \times]0,t[$, using only (1.1) and Poincaré inequality, yielding

$$\sup_{0<t<T} \int_\Omega |u_{\tau\sigma}(t)|^2 + \underline{\alpha} \int_{Q_T} |\nabla u_{\tau\sigma}|^2 \le \int_\Omega u_0^2 + \frac{1}{\underline{\alpha}\, c_0} \int_{Q_T} f^2 = C_0\,, \tag{2.8}$$

$$\tau \sup_{0<t<T} \int_\Omega |v_{\tau\sigma}(t)|^2 + \sigma \int_{Q_T} |\nabla v_{\tau\sigma}|^2 + \int_{Q_T} |v_{\tau\sigma}|^2 \le \tau \int_\Omega v_0^2 + C_0 T\ . \tag{2.9}$$

They are sufficient to obtain the existence of weak solutions to (2.1)–(2.4). It is also standard to multiply (2.3) by $t\, \partial_t v_{\tau\sigma}$ to obtain

$$\tau \int_\delta^t \int_\Omega |\partial_t v_{\tau\sigma}|^2 + \sigma \int_\Omega |\nabla v_{\tau\sigma}(t)|^2 \le \frac{C_\tau}{\delta}\,, \qquad 0 < \delta < t \le T\,,$$

where C_τ is independent of σ, but $C_\tau \to +\infty$ as $\tau \to 0$.

By (2.9), as $\sigma \to \infty$, there exists $\xi_\tau = \xi_\tau(t)$ and $v_{\tau\sigma} \to \xi_\tau$ in $L^2(0,T;H^1(\Omega))$ and in $C^0([\delta,T];L^2(\Omega))$ strongly, for each $\delta > 0$ by compactness.

Integrating (2.3) in $\Omega\times]\delta,t[$, we have

$$\tau\int_\Omega\Big[v_{\tau\sigma}(t)-v_{\tau\sigma}(\delta)\Big]=\int_\delta^t\int_\Omega(u_{\tau\sigma}-v_{\tau\sigma})\,.$$

Letting $\sigma\to\infty$ and then $\delta\to 0$, we obtain the weak form of (2.6)

$$\tau\,\xi_\tau(t)-\tau\fint_\Omega v_0=\int_0^t\fint_\Omega u_\tau-\int_0^t\xi_\tau$$

since ξ_τ does not depend on $x\in\Omega$, and u_τ is a limit of a subsequence $u_{\tau\sigma}$ in $L^2(0,T;H_0^1(\Omega))$-weak $\cap\,L^2(Q_T)$-strong. It is then easy to conclude the following special case of Theorem 2.1 of [HR].

Proposition 2.1. *There exist solutions $(u_{\tau\sigma},v_{\tau\sigma})$ to (2.1)–(2.4) in the class $L^2(0,T;H_0^1(\Omega))\times L^2(0,T;H^1(\Omega))\cap C^0([0,T];L^2(\Omega))^2$, such that, as $\sigma\to\infty$*

$$u_{\tau\sigma}\to u_\tau\quad\text{in }\ L^2(0,T;H_0^1(\Omega))\text{-strong}\,,\tag{2.10}$$

$$v_{\tau\sigma}\to \xi_\tau\quad\text{in }\ L^2(0,T;H^1(\Omega))\text{-strong}\,,\tag{2.11}$$

where $(u_\tau,\xi_\tau)\in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\times C^1[0,T]$ are solutions, in the generalized sense, of (2.5), (2.2) and (2.6).

Proof: Since, in particular, $v_{\tau\sigma}\to\xi_\tau$ in $L^2(Q_T)$ and a.e. in Q_T, also $a(v_{\tau\sigma})\to a(\xi_\tau)$ a.e. in Q_T and in $L^q(Q_T)$, $\forall\,q<\infty$. First we take the limit in the variational form

$$\int_{Q_T}\partial_t u_{\tau\sigma}\,\varphi+\int_{Q_T}a(v_{\tau\sigma})\,\nabla u_{\tau\sigma}\cdot\nabla\varphi=\int_{Q_T}f\,\varphi\,,\qquad\forall\,\varphi\in L^2(0,T;H_0^1(\Omega))\,,\tag{2.12}$$

where the first integral is understood in duality sense with $\partial_t u_{\tau\sigma}\in L^2(0,T;H^{-1}(\Omega))$, by considering a subsequence $\sigma\to\infty$, such that $u_{\tau\sigma}\rightharpoonup u_\tau$ in $L^2(0,T;H_0^1(\Omega))$-weak. Then u_τ solves (2.12) with $a(v_{\tau\sigma})$ replaced by $a(\xi_\tau)$. Finally taking the difference of the two corresponding variational formulations for $u_{\tau\sigma}$ and u_τ we obtain the strong convergence (2.10) for $w=u_{\tau\sigma}-u_\tau$:

$$\underline{a}\int_{Q_T}|\nabla w|^2\le\int_{Q_T}a(v_{\tau\sigma})\,|\nabla w|^2\le\int_{Q_T}\Big[a(\xi_\tau)-a(v_{\tau\sigma})\Big]\,\nabla u_\tau\cdot\nabla w\underset{\sigma\to\infty}{\longrightarrow}0\,.\ \blacksquare$$

The results of [HR] were obtained for the Neumann problem for u instead the Dirichlet condition (2.2), but there is no essential difference except in the next step $\tau\to\infty$. In fact, now we cannot obtain the estimate $\frac{d}{dt}\fint_\Omega u_\tau$ in $L^1(0,T)$, uniformly in τ, just by taking $\varphi=1$ in (2.12), what would be possible in the Neumann problem. However, we may use a different and more general argument to prove the next result, which is new.

Proposition 2.2. *There exists at least a solution* $u \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ *of the problem (2.5),(2.2), which can be obtained as the limit*

$$u_\tau \to u \quad \text{in } L^2(0,T;H_0^1(\Omega)) \ \text{ as } \tau \to 0\ , \tag{2.13}$$

$$\xi_\tau \to \fint_\Omega u \quad \text{in } L^q(0,T),\ \ \forall q < \infty\ , \tag{2.14}$$

where (u_τ, ξ_τ) *are weak solutions of (2.5), (2.2) and (2.6).*

Proof: The estimate (2.8) allows us to consider subsequences $u_\tau \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$-weak, $L^\infty(0,T;L^2(\Omega))$-weak* and also $L^2(Q_T)$-strongly, since the equation (2.5) also yields then $\partial_t u_\tau$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$.

Consequently, we may assume in (2.6)

$$\fint_\Omega u_\tau \to \fint_\Omega u \quad \text{in } L^2(0,T) \ \text{ as } \tau \to 0\ ,$$

and, by Lemma 2.1 below applied to $\zeta_\tau = \xi_\tau - \fint_\Omega v_0$, this implies

$$\xi_\tau \to \fint_\Omega u \quad \text{in } L^2(0,T), \ \text{ as } \tau \to 0\ .$$

By Proposition 3.2 of [HR] we have

$$\|\xi_\tau\|_{L^\infty(0,T)} \le \left|\fint_\Omega v_0\right| + \left\|\fint_\Omega u_\tau\right\|_{L^\infty(0,T)}\ ,$$

and the conclusion (2.14) follows. Then the conclusion (2.13) holds as in Proposition 2.1. ∎

Lemma 2.1. *Let* $\tau > 0$ *and consider for* $\eta_\tau \in L^2(0,T)$ *and* $\omega_\tau \in \mathbb{R}$

$$\tau\,\dot\zeta_\tau + \zeta_\tau = \eta_\tau \quad \text{in }]0,T[,\ \ \zeta_\tau(0) = \omega_\tau\ .$$

Then if $\eta_\tau \to \eta$ *in* $L^2(0,T)$ *and* $\omega_\tau \to \omega$ *we have*

$$\zeta_\tau \to \eta \quad \text{in } L^2(0,T) \ \text{ as } \tau \to 0\ .$$

Proof: We remark that d/dt is a maximal monotone operator in the Hilbert space $H = L^2(0,T)$ with domain

$$D\Big(\frac{d}{dt}\Big) = \Big\{\nu \in L^2(0,T):\ \dot\nu = \frac{d\nu}{dt} \in L^2(0,T),\ \nu(0) = 0\Big\}\ .$$

Indeed, we have

$$\int_0^T \dot\nu\,\nu\,dt = \frac{1}{2}\,|\nu(T)|^2 \ge 0, \qquad \forall \nu \in D\Big(\frac{d}{dt}\Big)$$

and $\forall \eta \in L^2(0,T)$, $\exists \nu \in D(\frac{d}{dt})$: $\dot{\nu} + \nu = \eta$. Hence, for each $\tau > 0$, its resolvent $J_\tau = (I + \tau \frac{d}{dt})^{-1}$ is a linear operator in $H = L^2(0,T)$ with norm $\|J_\tau\|_{\mathcal{L}} \leq 1$ and $J_\tau \nu \to \nu$, $\forall \nu \in H$, as $\tau \to 0$.

Now applying J_τ to $g_\tau = \eta_\tau - \omega_\tau \to \eta - \omega = g$ in $L^2(0,T)$, we conclude

$$\|J_\tau g_\tau - g\|_{L^2(0,T)} \leq \|J_\tau\|_{\mathcal{L}} \|g_\tau - g\|_{L^2(0,T)} + \|J_\tau g - g\|_{L^2(0,T)} \to 0, \quad \text{as } \tau \to 0\,. \blacksquare$$

2.2. Discontinuous nonlinearities

Following [HR] we allow in this section the reaction term f in the equations (2.1), (2.5) or (2.7) to be given by a nonlinear discontinuous function

$$f\colon Q_T \times \mathbb{R}^2 \to \mathbb{R}, \qquad (u,v) \mapsto f(x,t,u,v) \in L^\infty_{\text{loc}}(\mathbb{R}), \quad \text{a.e. } (x,t) \in Q_T\,,$$

under the assumptions that for $g_0 \in L^1(Q_T)$, $g_0 \geq 0$ and a constant $C_0 > 0$

$$u\, f(x,t,u,v) \leq g_0 + C_0(u^2 + v^2)\,, \qquad \forall u, v \in \mathbb{R}, \ \text{a.e. } (x,t) \in Q_T\,, \tag{2.16}$$

and, for any large $M > 0$, there are $g_M \in L^1(Q_T)$, $g_M \geq 0$ and a constant $C_M > 0$, such that for some δ ($\delta \leq 2$)

$$\sup_{|u| \leq M} |f(x,t,u,v)| \leq g_M(x,t) + C_M\,|v|^{2-\delta}\,, \qquad \forall v \in \mathbb{R}, \ \text{a.e. } (x,t) \in Q_T\,. \tag{2.17}$$

As in Section 1.2 we define for a.e. $(x,t) \in Q_T$ the multivalued function $F(x,t,u,v)$ in the same way. We may now consider the reaction–diffusion system $\{S_{\tau\sigma}\}$ consisting of (2.1)–(2.4) with f replaced by $F(u,v)$ in the following sense

$$f = f_{\tau\sigma} \in L^1(Q_T) \ \text{ and } \ f_{\tau\sigma} \in F(u_{\tau\sigma}, v_{\tau\sigma}) \quad \text{a.e. } (x,t) \in Q_T\,, \tag{2.18}$$

as well as the corresponding shadow system $\{S_\tau\}$ consisting of (2.5), (2.2), (2.6) and the limit nonlocal problem $\{S\}$ given by (2.7),(2.2), where we define

$$f = f_\tau \in L^1(Q_T) \ \text{ and } \ f_\tau \in F(u_\tau, \xi_\tau) \quad \text{a.e. } (x,t) \in Q_T\,, \tag{219}$$

$$f \in L^1(Q_T) \ \text{ with } f \in F(u, \fint_\Omega u) \quad \text{a.e. } (x,t) \in Q_T\,, \tag{2.20}$$

respectively, in the system $\{S_\tau\}$ and in $\{S\}$.

As it was shown in [HR], the assumptions (2.16),(2.17) are sufficiently to prove there exists at least a generalized solution $\{u_{\tau\sigma}, v_\sigma\}$ to the system $\{S_{\tau\sigma}\}$, as well as, $\{u_\tau, v_\tau\}$ and u respectively solutions to $\{S_\tau\}$ and to $\{S\}$, now in the class

$$u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))\,, \qquad \partial_t u \in L^2(0,T; H^{-1}(\Omega)) + L^1(Q_T)\,.$$

Using Lemma 1.1 we can also extend to these cases the previous asymptotic convergence results of Propositions 2.1 and 2.2, but now in a weaker sense. Indeed, since we only obtain

$$f_{\tau\sigma} \underset{\sigma\to\infty}{\rightharpoonup} f_\tau \text{ and } f_\tau \underset{\tau\to 0}{\rightharpoonup} f \quad \text{in } L^1(Q_T)$$

we can only show that $u_{\tau\sigma} \longrightarrow_{\sigma\to\infty} u_\tau$ and $u_\tau \longrightarrow_{\tau\to 0} u$ in $L^2(0,T;H_0^1(\Omega)$-weak, in $L^2(Q_T)$-strong and a.e. in Q_T. As in [HR], using now Lemma 2.1, we can illustrate these results in the following proposition, where we consider the simultaneous limit in σ and τ.

Proposition 2.3. *Under the previous assumptions we can obtain a solution u to the nonlocal problem (2.7),(2.2),(2.20) as limits when $(\tau,\sigma)\to(0,\infty)$*

$$u_{\tau\sigma} \to u \;\text{ in } L^2(0,T;H_0^1(\Omega))\text{-weak}, \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T\,, \tag{2.21}$$

$$v_{\tau\sigma} \to \fint_\Omega u \quad \text{in } L^2(0,T;H^1(\Omega))\text{-strong}\,, \tag{2.22}$$

where $u_{\tau\sigma}, v_{\tau\sigma}$ are solutions of $\{S_{\tau\sigma}\}$, i.e., (2.1)–(2.4) with (2.18).

As in Section 2.3, we may consider the parabolic obstacle problem in this form, by choosing

$$F(x,t,u,v) \;=\; g^+(x,t) - g^-(x,t)\,H(u)\;, \tag{2.23}$$

where H is the Heaviside function and we prescribe, for instance,

$$g \in L^2(Q_T) \quad \text{and} \quad u_0 \in H_0^1(\Omega),\;\; u_0 \geq 0\,. \tag{2.24}$$

Similarly, to the elliptic problem, the weak maximum principle implies that $u \geq 0$ a.e. in Q_T in all the three problems. In fact, if u solves (2.1), (2.5) or (2.7) with $f \in F(u)$ given by (2.23), we find $u^- = 0$ by integrating in $Q_t = \Omega\times]0,t[$ the respective equation multiplied by $-u^-$, from

$$\frac{1}{2}\int_\Omega |u^-(t)|^2 + \underline{\alpha}\int_{Q_t} |\nabla u^-|^2 \;\leq\; \int_{Q_t} \partial_t u^- \cdot u^- + \int_{Q_t} a\nabla u\cdot\nabla(-u)^- \;\leq\; 0\,,$$

since $u^-(0) = u_0^- = 0$, $H(u)\,u^- = 0$ and $g^+(-u^-) \leq 0$. Here we have also denoted a as the coefficient $a(v_{\tau\sigma})$, $a(v_\tau)$ or $a(\fint_\Omega u)$ corresponding to each one of the three cases.

Since $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ and this space contains $L^2(Q_T)$ we may now conclude that, if $(u_{\tau\sigma},v_{\tau\sigma})$ (resp. (u_τ,ξ_τ) or u) solve the system $\{S_{\tau\sigma}\}$ (resp. $\{S_\tau\}$ or $\{S\}$), then $u_{\tau\sigma}$ (resp. u_τ, u) also satisfies the parabolic variational inequality

$$u \in L^2(0,T;H_0^1(\Omega))\cap C^0([0,T];L^2(\Omega))\,, \quad u(t)\in\mathbb{K} \text{ a.e. } t\in]0,T[,\;\; u(0)=u_0\,, \tag{2.25}$$

$$\int_\Omega \partial_t u\,(\varphi - u) + \int_\Omega a\nabla u\cdot\nabla(\varphi-u) \geq \int_\Omega g(\varphi-u)\,, \quad \forall\varphi\in\mathbb{K},\;\text{ a.e. } t\in]0,T[\,, \tag{2.26}$$

where the first integral is understood in the sense of duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Actually, it also holds in L^2 under additional regularity assumptions on the coefficient a and on the data g, u_0.

Therefore, the existence results and the asymptotic convergences, such as the one in Proposition 2.3, also hold for the evolutionary obstacle problem.

With respect to uniqueness results, as observed in [HR] for the nonlocal problem $\{S\}$ it is sufficient to assume the Lipschitz condition (1.22) on the coefficient a, now without restriction on the constant α', and also a Lipschitz property on the nonlinearities $f(u,v)$. It is easy to extend this result to the case of monotone discontinuities in u, as in the case of the obstacle problem:

$$\Big|f(x,t,u,v) - f(x,t,u,w)\Big| \le \Big(g_2(x,t) + C_2\,|u|\Big)\,|v-w|\,, \tag{2.27}$$

$$\Big[f(x,t,u,v) - f(x,t,z,v)\Big]\,(u-z) \le 0\,, \tag{2.28}$$

for a.e. $(x,t) \in Q_T$, $u,v,w,z \in \mathbb{R}$, where $C_2 > 0$ is a constant and $g_2 \in L^2(Q_T)$, $g_2 \ge 0$.

Proposition 2.4. *Under the additional assumptions (1.12), (1.27) and (2.28) there exists at most one solution u to the nonlocal problem (2.7),(2.2) with (2.20), in particular, also to the variational inequality (2.26) with $a = a(\fint_\Omega u)$.*

Proof: We remark first that $\fint_\Omega u \in L^\infty(0,T)$ and then also $f \in F(u, \fint_\Omega u)$ is in $L^2(Q_T)$. Now if $\widehat{u}$ is another solution with $\widehat{f} \in F(\widehat{u}, \fint_\Omega \widehat{u})$, for $\widehat{g} \in L^2(Q_T)$ such that $\widehat{g} \in F(u, \fint_\Omega \widehat{u})$, we obtain, using the assumptions (2.28) and (2.27)

$$(f-\widehat{f})\,(u-\widehat{u}) \le (f-\widehat{g})\,(u-\widehat{u}) \le \Big(g_2 + C_2\,|u|\Big)\left|\fint_\Omega u - \fint_\Omega \widehat{u}\right|\,|u-\widehat{u}|\,.$$

Then, integrating the difference of the equations (2.7) for u and $\widehat{u}$, multiplied by their difference $\overline{u}$:

$$\int_\Omega \partial_t \overline{u}\,\overline{u} + \int_\Omega a\,|\nabla\overline{u}|^2 \le \alpha'\left|\fint_\Omega \overline{u}\right|\left|\int_\Omega \nabla\widehat{u}\cdot\nabla\overline{u}\right| + \left|\fint_\Omega \overline{u}\right|\int_\Omega\Big(g_2 + C_2\,|u|\Big)\,|\overline{u}|\,,$$

where $a = a(\fint_\Omega u)$ and we have used (1.12). Then, recalling the Poincaré inequality and that $|\fint_\Omega \overline{u}| \le |\Omega|^{1/2}(\int_\Omega u^2)^{1/2}$, we easily conclude the uniqueness with a standard application of Gronwall inequality. ∎

Remark 2.1. For the shadow system $\{S_\tau\}$ this uniqueness results still hold exactly under the same assumptions, since the ξ_τ, being independent of x and solving (2.6), allow the same proof as in Proposition 2.4. However, for the initial reaction–diffusion system $\{S_{\tau\sigma}\}$ additional assumptions on the regularity of u are required. For instance, if $\nabla u \in L^\infty(Q_T)$ the same Gronwall type argument still applies.

Remark 2.2. As in the elliptic case, for general discontinuous nonlinearities, the parabolic problem may also exhibit multiplicity of solutions, as a counter example of [HR] shows for the nonlocal Neumann problem.

Remark 2.3. An interesting problem, only partly treated in special cases (see [CL] and [CM]) is the asymptotic behaviour of the evolutionary case when $t \to \infty$.

2.3. Extension to a unilateral problem

We consider now a nonlocal parabolic obstacle problem, where the diffusion coefficient $a = a(\rho)$ is a continuous strictly positive function, i.e. it satisfies (1.1) but it is supposed independent of x and t.

As in Section 1.3, we start with the obstacle problem (2.25),(2.26). Now we let a depend on a second variable $v_{\tau\sigma}$ or ξ_τ as in Section 2.1 with (2.3) replaced by

$$\tau\, \partial_t v_{\tau\sigma} - \sigma\, \Delta v_{\tau\sigma} + v_{\tau\sigma} = \chi_{(\{u_{\tau\sigma}=0\})} \quad \text{in } Q_T \tag{2.29}$$

or (2.6) replaced by

$$\tau\, \dot{\xi}_\tau + \xi_\tau = \fint_\Omega \chi_{\{u_\tau=0\}} = \langle u_\tau(t) = 0\rangle \quad \text{in }]0,T[\tag{2.30}$$

respectively, with $u_{\tau\sigma}$ solving (2.25),(2.26) for $a = a(v_{\tau\sigma})$ and u_τ solving (2.25),(2.26) for $a = a(\xi_\tau)$, together with the boundary conditions (2.4) or (2.6).

It is then natural to study the asymptotic limits $\sigma \to \infty$ and $\tau \to 0$ and, in the second case, obtain the parabolic nonlocal version of (1.29). This limit problem, for any

$$g = g(x,t) \in L^2(Q_T) \quad \text{and} \quad u_0 \in H_0^1(\Omega),\ u_0 \geq 0 \ \text{ in } \Omega\,, \tag{2.31}$$

corresponds to the nonlocal obstacle problem for $u = u(x,t) \geq 0$ satisfying (2.25) and

$$\int_\Omega \partial_t u\,(\varphi - u) + \int_\Omega a(\langle u = 0\rangle)\, \nabla u \cdot \nabla(\varphi - u) \geq \int_\Omega g(\varphi - u)\,, \qquad \forall\, \varphi \in \mathbb{K}, \ \text{a.e. } t \in]0,T[\,. \tag{2.32}$$

Indeed, it is still possible to extend the previous results to this new problem (see [RS] for the details) but the arguments are more delicate than in the elliptic problem. The regularity C^2 of the boundary $\partial\Omega$ and the nondegeneracy assumption

$$g \neq 0 \quad \text{a.e. in } Q_T \tag{2.33}$$

are also required in the following result of [RS]:

Theorem 2.1. *Under the previous assumptions, namely (2.31) and (2.33) there exist solutions* $(u_{\tau\sigma}, v_{\tau\sigma}) \in W_2^{2,1}(Q_T) \times W_q^{2,1}(Q_T)$, $\forall\, q < \infty$, *to the coupled problem (2.25),(2.26) (with* $a = a(v_{\tau\sigma})$*), (2.29),(2.4), such that as* $\sigma \to \infty$

$$u_{\tau\sigma} \to u_\tau \quad \text{in } L^2(0,T; H_0^1(\Omega)) ,$$
$$v_{\tau\sigma} \to \xi_\tau \quad \text{in } L^2(0,T; H^1(\Omega)) ,$$

where $(u_\tau, \xi_\tau) \in W_2^{2,1}(Q_T) \times W^{1,\infty}(0,T)$ *solve the coupled problem (2.25),(2.26) (with* $a = a(\xi_\tau)$*) and (2.30) with the initial condition of (2.6). Moreover, there exists at least a solution* $u \in W_2^{2,1}(Q_T)$ *to the nonlocal obstacle problem (2.25),(2.32), which can be obtained as the limit as* $\tau \to 0$ *of solutions* (u_τ, ξ_τ)*, i.e. such that*

$$u_\tau \to u \quad \text{in } L^2(0,T; H_0^1(\Omega)) \cap W_2^{2,1}(Q_T) ,$$
$$\xi_\tau \to \langle u = 0 \rangle = \fint_\Omega \chi_{\{u=0\}} \quad \text{in } L^q(0,T), \quad \forall\, q < \infty . \blacksquare$$

Remark 2.4. Here $W_p^{2,1}(Q_T) = L^p(0,T; W^{2,p}(\Omega)) \cap W^{1,p}(0,T; L^p(\Omega))$, $1 < p < \infty$, and this result uses the regularity for the obstacle problem and the "a priori" L^p-estimates for the linear parabolic problems of second order (see [LSU]), as well as the extension of the continuous dependence of the characteristic function $\chi_{\{u=0\}}$ of the coincidence set to the evolutionary obstacle problem (see [R1]).

Remark 2.5. The extension of Theorem 2.1 to the case of a nonlinear coupling $g = g(v_{\tau\sigma})$ can be done easily up to the convergence $\sigma \to \infty$ but presents a non obvious difficulty in the second passage $\tau \to 0$ (see [RS]). Therefore, the corresponding nonlocal problem (2.32) with a nonlinearity of the type $g = g(\langle u = 0 \rangle)$ seems to be an open problem.

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