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q-deformed Araki-Woods factors

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1 Construction of the q-deformed functor

Let $\mathcal{H}_{\mathbf{R}}$ be a separable real Hilbert space and U_t a strongly continuous one-parameter group of orthogonal transformations on $\mathcal{H}_{\mathbf{R}}$. By linearity U_t extends to a one-parameter unitary group on the complexified Hilbert space $\mathcal{H}_{\mathbf{C}} := \mathcal{H}_{\mathbf{R}} + i\mathcal{H}_{\mathbf{R}}$. Write $U_t = A^{it}$ with the generator A (a positive non-singular operator on $\mathcal{H}_{\mathbf{C}}$) and define an inner product $\langle\cdot,\cdot\rangle_U$ on $\mathcal{H}_{\mathbf{C}}$ by

$$\langle x, y\rangle_U = \langle 2A(1+A)^{-1}x, y\rangle, \qquad x, y \in \mathcal{H}_{\mathbf{C}}.$$

Let $\mathcal{H}$ be the complex Hilbert space obtained by completing $\mathcal{H}_{\mathbf{C}}$ with respect to $\langle\cdot,\cdot\rangle_U$.

For $-1 < q < 1$ the q-Fock space $\mathcal{F}_q(\mathcal{H})$ was introduced in [BS1, BKS] as follows. Let $\mathcal{F}^{\mathrm{finite}}(\mathcal{H})$ be the linear span of $f_1 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$ $(n = 0, 1, \dots)$ where $\mathcal{H}^{\otimes 0} = \mathbf{C}\Omega$ with vacuum Ω. The sesquilinear form $\langle\cdot,\cdot\rangle_q$ on $\mathcal{F}^{\mathrm{finite}}(\mathcal{H})$ is given by

$$\langle f_1 \otimes \cdots \otimes f_n\,,\, g_1 \otimes \cdots \otimes g_m\rangle_q = \delta_{nm} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)}\rangle_U \cdots \langle f_n, g_{\pi(n)}\rangle_U\,,$$

where $i(\pi)$ denotes the number of inversions of the permutation $\pi \in S_n$. For $-1 < q < 1$, $\langle\cdot,\cdot\rangle_q$ is strictly positive and the q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{\mathrm{finite}}(\mathcal{H})$ with respect to $\langle\cdot,\cdot\rangle_q$. Given $h \in \mathcal{H}$ the q-creation operator $a_q^*(h)$ and the q-annihilation operator $a_q(h)$ on $\mathcal{F}_q(\mathcal{H})$ are defined by

$$a_q^*(h)\Omega = h\,,$$

$$a_q^*(h)(f_1 \otimes \cdots \otimes f_n) = h \otimes f_1 \otimes \cdots \otimes f_n\,,$$

and

$$a_q(h)\Omega = 0\,,$$

$$a_q(h)(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^n q^{i-1} \langle h, f_i\rangle_U f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_n\,.$$

The operators $a_q^*(h)$ and $a_q(h)$ are bounded operators on $\mathcal{F}_q(\mathcal{H})$ and they are adjoins of each other (see [BKS, Remark 1.2]).

Following [Sh1] we consider the von Neumann algebra $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$, called a q-deformed Araki-Woods algebra, generated on $\mathcal{F}_q(\mathcal{H})$ by

$$s_q(h) := a_q^*(h) + a_q(h), \qquad h \in \mathcal{H}_{\mathbf{R}}\,.$$

The vacuum state $\varphi\ (= \varphi_{q,U}) := \langle\Omega, \cdot\,\Omega\rangle_q$ on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is called the q-quasi-free state.

Proposition 1.1 *Ω is cyclic and separating for $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$.*

One can canonically extend U_t on $\mathcal{H}$ to a one-parameter unitary group (the so-called second quantization) $\mathcal{F}_q(U_t)$ on $\mathcal{F}_q(\mathcal{H})$ by

$$\mathcal{F}_q(U_t)\Omega = \Omega\,,$$

$$\mathcal{F}_q(U_t)(f_1 \otimes \cdots \otimes f_n) = (U_t f_1) \otimes \cdots \otimes (U_t f_n)\,.$$

Notice $\mathcal{F}_q(U_t)a_q^*(h)\mathcal{F}_q(U_t)^* = a_q^*(U_t h)$ for $h \in \mathcal{H}$ so that

$$\mathcal{F}_q(U_t)s_q(h)\mathcal{F}_q(U_t)^* = s_q(U_t h)\,, \qquad h \in \mathcal{H}_{\mathbf{R}}\,.$$

Thus, $\alpha_t := \operatorname{Ad}\mathcal{F}_q(U_t)$ defines a strongly continuous one-parameter automorphism group on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$.

Proposition 1.2 *The q-quasi-free state φ on $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ satisfies the KMS condition with respect to α_t at $\beta = 1$.*

Let $(\mathcal{K}_{\mathbf{R}}, V_t)$ be another pair of a separable real Hilbert space and a one-parameter group V_t of orthogonal transformations on $\mathcal{K}_{\mathbf{R}}$. Let $T : \mathcal{H}_{\mathbf{R}} \to \mathcal{K}_{\mathbf{R}}$ be a contraction such that $TU_t = V_t T$ for all $t \in \mathbf{R}$. By linearity T extends to a contraction $T : \mathcal{H}_{\mathbf{C}} \to \mathcal{K}_{\mathbf{C}}$ and it satisfies $TU_t = V_t T$ on $\mathcal{H}_{\mathbf{C}}$. Let B be the generator of V_t so that $V_t = B^{it}$. Since

$$TA(\mathbf{1} + A)^{-1} = B(\mathbf{1} + B)^{-1}T\,,$$

T can further extend to a contraction from $(\mathcal{H}, \langle\cdot,\cdot\rangle_U)$ to $(\mathcal{K}, \langle\cdot,\cdot\rangle_V)$. Then:

Proposition 1.3 *There is a unique completely positive normal contraction $\Gamma_q(T)$: $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \to \Gamma_q(\mathcal{K}_{\mathbf{R}}, V_t)''$ such that*

$$(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega)\,, \qquad x \in \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'',$$

where $\mathcal{F}_q(T) : \mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{K})$ is given by

$$\mathcal{F}_q(T)(f_1 \otimes \cdots \otimes f_n) = (Tf_1) \otimes \cdots \otimes (Tf_n)\,.$$

In this way, we have presented a q-analogue of Shlyakhtenko's free CAR functor; namely, a von Neumann algebra with a specified state, $(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi)$, is associated to each real Hilbert space with a one-parameter group of orthogonal transformations, $(\mathcal{H}_{\mathbf{R}}, U_t)$, and a unital completely positive state-preserving map $\Gamma_q(T) : \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \to \Gamma_q(\mathcal{K}_{\mathbf{R}}, V_t)''$ to every contraction $T : (\mathcal{H}_{\mathbf{R}}, U_t) \to (\mathcal{K}_{\mathbf{R}}, V_t)$.

When $q = 0$, $\Gamma(\mathcal{H}_{\mathbf{R}}, U_t)'' \equiv \Gamma_0(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a free Araki-Woods factor (of type III) in [Sh1]. On the other hand, when $U_t = \mathrm{id}$ a trivial action, $\Gamma_q(\mathcal{H}_{\mathbf{R}})'' \equiv \Gamma_q(\mathcal{H}_{\mathbf{R}}, \mathrm{id})$ is a q-deformation of the free group factor in [BKS]; in particular, $\Gamma_0(\mathcal{H}_{\mathbf{R}})'' \cong L(\mathbb{F}_{\dim \mathcal{H}_{\mathbf{R}}})$ a free group factor.

2 Factoriality and non-injectivity of $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$

The following were proven in [BS2, BKS], but it is still open whether $\Gamma_q(\mathcal{H}_{\mathbf{R}})''$ is a non-injective type II_1 factor whenever $\dim \mathcal{H}_{\mathbf{R}} \geq 2$.

(i) If $-1 < q < 1$ and $\dim \mathcal{H}_{\mathbf{R}} > 16/(1-|q|)^2$, then $\Gamma_q(\mathcal{H}_{\mathbf{R}})''$ is not injective.

(ii) If $\dim \mathcal{H}_{\mathbf{R}} = \infty$, then $\Gamma_q(\mathcal{H}_{\mathbf{R}})$ is a factor (of type II_1) for all $-1 < q < 1$.

These results can be extended to $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ as follows.

Theorem 2.1 *If there is $T \in [1, \infty)$ such that*

$$\frac{\dim E_A([1,T])\mathcal{H}_{\mathbf{C}}}{T} > \frac{16}{(1-|q|)^2}$$

where E_A is the spectral measure of A, then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is not injective. In particular, $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is not injective if A has a continuous spectrum.

Theorem 2.2 *Assume that the almost periodic part of $(\mathcal{H}_{\mathbf{R}}, U_t)$ is infinite dimensional, that is, A has infinitely many mutually orthogonal eigenvectors. Then*

$$\left(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''\right)'_{\varphi} \cap \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' = \mathbf{C}1,$$

where $\left(\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''\right)_{\varphi}$ is the centralizer of $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ with respect to the vacuum state φ. In particular, $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a factor.

3 Type classification of $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$

As usual let S_φ be the closure of the operator given by

$$S_\varphi(x\Omega) = x^*\Omega, \qquad x \in \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'',$$

and let Δ_φ, J_φ be the associated modular operator and the modular conjugation. Then the following are seen as in [Sh1]: For $h_1, \dots, h_n \in \mathcal{H}_{\mathbf{R}}$,

$$S_\varphi(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = h_n \otimes h_{n-1} \otimes \cdots \otimes h_1,$$

and for $h_1, \dots, h_n \in \mathcal{H}_{\mathbf{R}} \cap \operatorname{dom} A^{-1}$,

$$\Delta_\varphi(h_1 \otimes \cdots \otimes h_n) = (A^{-1}h_1) \otimes \cdots \otimes (A^{-1}h_n).$$

Noting that $\mathcal{D} := \{h + ig : h, g \in \mathcal{H}_{\mathbf{R}} \cap \operatorname{dom} A^{-1}\}$ is a core of A^{-1} (on $\mathcal{H}$) such that $U_t\mathcal{D} = \mathcal{D}$ for all $t \in \mathbf{R}$, we see that

$$\Delta_\varphi^{it} = \mathcal{F}_q(A^{-it}) = \mathcal{F}_q(U_{-t}), \qquad t \in \mathbf{R}.$$

By this and Theorem 2.2 we obtain the following type classification result:

Theorem 3.1 *Assume that A has infinitely many mutually orthogonal eigenvectors. Let G be the closed multiplicative subgroup of $\mathbf{R}_+$ generated by the spectrum of A ($U_t = A^{it}$). Then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a non-injective factor of type II_1 or type III_λ ($0 < \lambda \leq 1$), and*

$$\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \text{ is } \begin{cases} \text{type } II_1 & \text{if } G = \{1\}, \\ \text{type } III_\lambda & \text{if } G = \{\lambda^n : n \in \mathbf{Z}\} \ (0 < \lambda < 1), \\ \text{type } III_1 & \text{if } G = \mathbf{R}_+. \end{cases}$$

This result for free Araki-Woods factors (in case of $q = 0$) was shown in [Sh1, Sh2] generally when $\dim \mathcal{H}_{\mathbf{R}} \geq 2$. Moreover, it was shown as a consequence of Barnett's theorem that free Araki-Woods factors are full whenever U_t is almost periodic (i.e. the eigenvectors of A span $\mathcal{H}$). The assumption of Theorems 2.2 and 3.1 is a bit too restrictive while the following opposite extreme case is easy to see:

Proposition 3.2 *If U_t has no eigenvectors, then $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ is a type III_1 factor.*

It is worthwhile to note that the type III_0 case does not appear in the above type classifications.

For example, let $(\mathcal{H}_{\mathbf{R}}, U_t) = \bigoplus_{k=1}^\infty (\mathbf{R}^2, V_t)$ where $V_t := \begin{bmatrix} \cos(t\log\lambda) & -\sin(t\log\lambda) \\ \sin(t\log\lambda) & \cos(t\log\lambda) \end{bmatrix}$, $0 < \lambda \leq 1$, and write $(T_{q,\lambda}, \varphi_{q,\lambda}) := (\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi)$ with two parameters $q \in (-1, 1)$ and $\lambda \in (0, 1]$. For $0 < \lambda < 1$, $T_{q,\lambda}$ is a type III_λ q-deformed Araki-Woods factor. In particular when $q = 0$, $(T_{0,\lambda}, \varphi_{0,\lambda})$ coincides with the type III_λ free Araki-Woods factor $(T_\lambda, \varphi_\lambda)$ discussed in [Ra, Sh1]. For $\lambda = 1$, $T_{q,1}$ is the q-deformed type II_1 factor treated in [BKS].

The C^*-algebra $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)$, $-1 < q < 1$, generated by $\{s_q(h) : h \in \mathcal{H}_{\mathbf{R}}\}$ on $\mathcal{F}_q(\mathcal{H})$ is considered as the q-analogue of the CAR algebra. From this point of view, the above $T_{q,\lambda}$ ($0 < \lambda < 1$) may be considered as the q-analogue of Powers' III_λ factor. In fact, we remark that, for $q = -1$, our construction of $T_{q,\lambda}$ provides Powers' III_λ factor. To be more precise, for given $(\mathcal{H}_{\mathbf{R}}, U_t)$, let $\Gamma_-(\mathcal{H}_{\mathbf{R}}, U_t)''$ denote the von Neumann algebra generated by $s_-(h) := a_-^*(h) + a_-(h)$ ($h \in \mathcal{H}_{\mathbf{R}}$) on the Fermion Fock space $\mathcal{F}_-(\mathcal{H})$, where $a_-^*(h)$ and $a_-(h)$ are the Fermion (CAR) creation and annihilation operators. If $(\mathcal{H}_{\mathbf{R}}, U_t) = \bigoplus_{k=1}^\infty (\mathcal{H}_{\mathbf{R}}^{(k)}, U_t^{(k)})$ where $\mathcal{H}_{\mathbf{R}}^{(k)} = \mathbf{R}^2$, $U_t^{(k)} = \begin{bmatrix} \cos(t\log\lambda_k) & -\sin(t\log\lambda_k) \\ \sin(t\log\lambda_k) & \cos(t\log\lambda_k) \end{bmatrix}$ with $\lambda_k \leq 1$, then $\big(\Gamma_-(\mathcal{H}_{\mathbf{R}}, U_t)'', \varphi := \langle\Omega, \cdot\,\Omega\rangle_-\big)$ becomes an Araki-Woods factor

$$\bigotimes_{k=1}^\infty \left(M_2(\mathbf{C}), \mathrm{Tr}\left(\cdot \begin{bmatrix} \frac{\lambda_k}{\lambda_k+1} & 0 \\ 0 & \frac{1}{\lambda_k+1} \end{bmatrix} \right) \right).$$

Upon these considerations we called $\Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)''$ a q-deformed Araki-Woods algebra.

4 Hypercontractivity of $\Gamma_q(T)$

When $T = e^{-t}\mathbf{1}_{\mathcal{H}_\mathbf{R}}$ $(t > 0)$, we obtain a semigroup $\Gamma_q(e^{-t})$ $(t > 0)$ of completely positive normal contractions on $\Gamma_q(\mathcal{H}_\mathbf{R}, U_t)''$. This is a non-tracial extension of q-Ornstein-Uhlenbeck semigroup discussed in [Bi, Bo]. In the tracial case (i.e. the case of U_t being trivial), the ultracontractivity for $\Gamma_q(e^{-t})$ was proven in [Bo] as follows:

$$\|\Gamma_q(e^{-t})x\| \le C_{|q|}^{3/2}\sqrt{\frac{1+e^{-2t}}{(1-e^{-2t})^3}}\,\|x\Omega\|\,, \qquad x \in \Gamma_q(\mathcal{H}_\mathbf{R})''$$

with $C_{|q|}$ given below. In the non-tracial type III case, we have the following hypercontractivity property. This reduces to the above ultracontractivity when $A = \mathbf{1}$ or $\gamma = 0$.

Theorem 4.1 *Assume that A is bounded (in particular, this is the case if $\dim\mathcal{H}_\mathbf{R} < +\infty$), and let $\gamma := \frac{1}{2}\log\|A\|$. If $-1 < q < 1$ and $t > \gamma$, then*

$$\|\Gamma_q(e^{-t})x\| \le C_{|q|}^{3/2}\sqrt{\frac{1+e^{-(2t-\gamma)}}{(1-e^{-2t})(1-e^{-(2t-\gamma)})(1-e^{-2(t-\gamma)})}}\,\|\Delta_\varphi^{\theta/2}x\Omega\|$$

for all $x \in \Gamma_q(\mathcal{H}_\mathbf{R}, U_t)''$ and $0 \le \theta \le 1$, where

$$C_{|q|} := \frac{1}{\prod_{m=1}^{\infty}(1-|q|^m)}\,.$$

It might be expected that the hypercontractivity given in the above theorem is valid for the whole $t > 0$. However, the next proposition says that it is impossible to remove the assumption $t > \gamma$, so Theorem 4.1 seems more or less best possible. Also, it says that the hypercontractivity in the sense that $\|\Gamma_q(e^{-t})x\| \le C\|x\Omega\|_q$ holds for some $t > 0$ and for all $x \in \Gamma_q(\mathcal{H}_\mathbf{R}, U_t)''$ is impossible when A is unbounded; for example, this is the case when $U_t f = f(\cdot + t)$ on $\mathcal{H}_\mathbf{R} = L^2(\mathbf{R};\mathbf{R})$.

Proposition 4.2 *Let $-1 < q < 1$, $0 \le \theta \le 1$ and $t > 0$. If there exists a constant $c > 0$ such that*

$$\|\Gamma_q(e^{-t})x\| \le c\|\Delta_\varphi^{\theta/2}x\Omega\|\,, \qquad x \in \Gamma_q(\mathcal{H}_\mathbf{R}, U_t)'',$$

then A is bounded and

$$\|A\| \le \exp\left(\frac{2t}{\max\{\theta, 1-\theta\}}\right).$$

It seems that it is convenient to consider the hypercontractivity of $\Gamma_q(T)$ in the setting of Kosaki's interpolated L^p-spaces. For a general von Neumann algebra $\mathcal{M}$ and $1 \le p \le \infty$ let $L^p(\mathcal{M})$ be Haagerup's L^p-space. Given a faithful normal state φ on $\mathcal{M}$ let h_φ denote the element of $L^1(\mathcal{M})$ $(\cong \mathcal{M}_*)$ corresponding to φ. For each $1 < p < \infty$

and $0 \le \theta \le 1$, Kosaki's L^p-space $L^p(\mathcal{M};\varphi)_\theta$ with respect to φ is introduced as the complex interpolation space

$$C_{1/p}(h_\varphi^\theta \mathcal{M} h_\varphi^{1-\theta}, L^1(\mathcal{M}))$$

equipped with the complex interpolation norm $\|\cdot\|_{p,\theta}\ (= \|\cdot\|_{C_{1/p}})$.

Let $T : \mathcal{H}_{\mathbf{R}} \to \mathcal{K}_{\mathbf{R}}$ be a contraction with $TU_t = V_tT$, $t \in \mathbf{R}$. The adjoint operator $T^* : \mathcal{K}_{\mathbf{R}} \to \mathcal{H}_{\mathbf{R}}$ is also a contraction satisfying $T^*V_t = U_tT^*$, $t \in \mathbf{R}$. For each $-1 < q < 1$ let

$$\begin{aligned} \mathcal{M} &:= \Gamma_q(\mathcal{H}_{\mathbf{R}}, U_t)'' \quad \text{with} \quad \varphi = \langle \Omega, \cdot\, \Omega \rangle_q, \\ \mathcal{N} &:= \Gamma_q(\mathcal{K}_{\mathbf{R}}, V_t)'' \quad \text{with} \quad \psi = \langle \Omega, \cdot\, \Omega \rangle_q, \end{aligned}$$

where the vacuums in $\mathcal{F}_q(\mathcal{H})$ and in $\mathcal{F}_q(\mathcal{K})$ are denoted by the same Ω. Then, by Proposition 1.3 the completely positive normal contractions

$$\Gamma_q(T) : \mathcal{M} \to \mathcal{N} \quad \text{and} \quad \Gamma_q(T^*) : \mathcal{N} \to \mathcal{M}$$

are determined by

$$\begin{aligned} (\Gamma_q(T)x)\Omega &= \mathcal{F}_q(T)(x\Omega), \qquad x \in \mathcal{M}, \\ (\Gamma_q(T^*)y)\Omega &= \mathcal{F}_q(T^*)(y\Omega), \qquad y \in \mathcal{N}. \end{aligned}$$

One can define the contraction $\omega \mapsto \omega \circ \Gamma_q(T^*)$ of $\mathcal{M}_*$ into $\mathcal{N}_*$. Via $\mathcal{M}_* \cong L^1(\mathcal{M})$ and $\mathcal{N}_* \cong L^1(\mathcal{N})$ this induces the contraction $\tilde{\Gamma}_q(T)$ of $L^1(\mathcal{M})$ into $L^1(\mathcal{N})$ as follows:

$$\tilde{\Gamma}_q(T)h_\omega = h_{\omega\circ\Gamma_q(T^*)}, \qquad \omega \in \mathcal{M}_*.$$

We see that for every $0 \le \theta \le 1$ and $x \in \mathcal{M}$,

$$\tilde{\Gamma}_q(T)(h_\varphi^\theta x h_\varphi^{1-\theta}) = h_\psi^\theta(\Gamma_q(T)x)h_\psi^{1-\theta},$$

so that $\tilde{\Gamma}_q(T) : L^1(\mathcal{M}) \to L^1(\mathcal{N})$ is the (unique) continuous extension of the linear mapping from $h_\varphi^\theta \mathcal{M} h_\varphi^{1-\theta}\ (\subset L^1(\mathcal{M}))$ into $h_\psi^\theta \mathcal{N} h_\psi^{1-\theta}\ (\subset L^1(\mathcal{N}))$ given by

$$h_\varphi^\theta x h_\varphi^{1-\theta} \mapsto h_\psi^\theta(\Gamma_q(T)x)h_\psi^{1-\theta}, \qquad x \in \mathcal{M}.$$

Moreover, the Riesz-Thorin theorem implies that for each $0 \le \theta \le 1$ and $1 \le p \le \infty$, $\tilde{\Gamma}_q(T)$ maps $L^p(\mathcal{M};\varphi)_\theta$ into $L^p(\mathcal{N};\psi)_\theta$ and

$$\|\tilde{\Gamma}_q(T)a\|_{p,\theta} \le \|a\|_{p,\theta}, \qquad a \in L^p(\mathcal{M};\varphi)_\theta.$$

The next theorem is shown by using Theorem 4.1.

Theorem 4.3 *Assume that either A ($U_t = A^{it}$) or B ($V_t = B^{it}$) is bounded, and let $\rho := \min\{\|A\|, \|B\|\}$. Let $T : \mathcal{H}_{\mathbf{R}} \to \mathcal{K}_{\mathbf{R}}$ be a bounded operator such that $TU_t = V_tT$ for all $t \in \mathbf{R}$ and $\|T\| < \rho^{-1}$. Then $\tilde{\Gamma}_q(T)$ maps $L^1(\mathcal{M})$ into $\bigcap_{0\le\theta\le1} h_\psi^\theta \mathcal{N} h_\psi^{1-\theta}$ and*

$$\|\tilde{\Gamma}_q(T)a\|_{\infty,\theta} \le C_{|q|}^3 \frac{1 + \rho^{1/2}\|T\|}{(1 - \|T\|)(1 - \rho^{1/2}\|T\|)(1 - \rho\|T\|)} \|a\|_1$$

for all $a \in L^1(\mathcal{M})$, $0 \le \theta \le 1$.

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