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# A Construction of Type III Factors from Boundary Actions

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## 1 Introduction

One of our purposes in this note is to determine the types of quasi-free KMS states on Cuntz-Krieger algebras. The Cuntz-Krieger algebra $\mathcal{O}_A$ [CK], associated with a 0-1 $N\times N$-matrix $A$, is the universal $C^*$-algebra generated by the family of partial isometries $\{S_i\}_{i=1}^N$ satisfying:

$$S_i^*S_i=\sum_{j=1}^N A(i,j)S_jS_j^*,$$

and

$$1=\sum_{j=1}^N S_jS_j^*.$$

The universal property of $\mathcal{O}_A$ allows us to define the gauge action $\alpha$ on $\mathcal{O}_A$ by

$$\alpha_t(S_i)=e^{\sqrt{-1}t}S_i$$

for $t\in\mathbb{R}$. The KMS states for the gauge actions on the Cuntz algebra $\mathcal{O}_n$ and the Cuntz-Krieger algebra $\mathcal{O}_A$ were obtained by D. Olesen and G. K. Pedersen [OP] and M. Enomoto, M. Fujii and Y. Watatani [EFW], respectively. More generally, D. E. Evans [Eva] determined the KMS states on $\mathcal{O}_n$ for the quasi-free actions. In order to construct examples of subfactors, M. Izumi [Izu] determined the types of factors obtained by the GNS-representations of the quasi-free KMS states. We will generalize these results to Cuntz-Krieger algebras. However the existence and the uniqueness of the quasi-free KMS states on Cuntz-Krieger algebras were proved by R. Exel and M. Laca [EL].

Therefore it suffices to compute the Connes spectrum of the modular automorphism group.

The other purpose is to show that there is one-to-one correspondence between quasi-free KMS states on some Cuntz-Krieger algebras and some class of random walks on groups. Namely, J. Spielberg [Spi] proved that some Cuntz-Krieger algebras can be obtained by the crossed product construction of the boundary action $(\Omega, \Gamma)$, where $\Gamma$ is the free product of cyclic groups and $\Omega$ is some compact space, on which $\Gamma$ acts by homeomorphisms. This construction was generalized to amalgamated free product groups in [O1]. By identifying $\Omega$ with the Poisson boundary, harmonic measures on $\Omega$ induce quasi-free KMS states.

By combining the above two results, we can construct type III factors from boundary actions and harmonic measures on the boundary, which generalizes J. Ramagge and G. Robertson's result in [RR].

## 2 Quasi-Free KMS States on Cuntz-Krieger Algebras

We first introduce some notations and known results. Let $I = \{1, \ldots, N\}$ be the index set. For $i \in I$, we denote $S_i S_i^* = P_i$. We put the set of all admissible word by

$$\mathcal{W}_A = \{\xi = (\xi_1, \ldots, \xi_n) \mid n \in \mathbb{N}, \xi_k \in I, A(\xi_k, \xi_{k+1}) = 1\}.$$

For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{W}_A$, we define two maps $s$ and $r$ by $s(\xi) = \xi_1$ and $r(\xi) = \xi_n$. Let us say that $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{W}_A$ is a *loop* if $A(\xi_n, \xi_1) = 1$. Moreover, we say that a loop $\xi$ is a *circle* if $\xi_k \neq \xi_l$ for any $1 \leq k, l \leq n$, $(k \neq l)$.

For $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}_+^N$, we define the action $\alpha^\omega$ of $\mathbb{R}$ on $\mathcal{O}_A$ by

$$\alpha_t^\omega(S_i) = e^{\sqrt{-1}\omega_i t} S_i$$

for $t \in \mathbb{R}$ and $i \in I$. Note that if $\omega = (1, \ldots, 1)$, then $\alpha^\omega$ is the gauge action. We define two word-length functions. For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{W}_A$, we denote the canonical one by $|\xi| = n$ and the other by $\omega_\xi = \omega_{\xi_1} + \cdots + \omega_{\xi_n}$. Note that there is the faithful conditional expectation $\Phi$ from $\mathcal{O}_A$ onto $\overline{\mathrm{span}}\{S_\xi S_\xi^* \mid \xi \in \mathcal{W}_A\} \simeq C(\Omega_A)$, where

$$\Omega_A = \{(a_k)_{i=k}^\infty \mid A(a_k, a_{k+1}) = 1\}$$

is the set of all one-sided infinite admissible words.

We assume that there is $\beta \in \mathbb{R}_+$ and $x_i > 0$ that satisfy:

$$x_i = \sum_{j=1}^{N} e^{-\beta\omega_i} A(i,j) x_j,$$

and

$$1 = x_1 + \cdots + x_N.$$

Then we can define a probability measure $\nu$ on $\Omega_A$ by

$$\nu(\Omega_A(\xi_1, \ldots, \xi_{n-1}, \xi_n)) = e^{-\beta\omega_{\xi_1}} \cdots e^{-\beta\omega_{\xi_{n-1}}} x_{\xi_n},$$

where $\Omega_A(\xi_1, \ldots, \xi_n)$ is the cylinder set

$$\{(a_k)_{k=1}^{\infty} \in \Omega_A \mid a_1 = \xi_1, \ldots, a_n = \xi_n\}.$$

This probability measure induces a $\beta$-KMS state for $\alpha^\omega$ on $\mathcal{O}_A$ by $\phi^\omega = \nu \circ \Phi$.

**Remark 2.1** If we set $A_\omega(i,j) = e^{-\beta\omega_i} A(i,j)$, then the vector $x = {}^T(x_1, \ldots, x_N)$ is the right Perron eigenvector of the matrix $A_\omega$ with respect to the Perron eigenvalue 1.

R. Exel and M. Laca [EL], in fact, showed the existence of such $\beta \in \mathbb{R}_+$ and $x_i > 0$.

**Theorem 2.2 ([EL, Theorem 18.5])** *If $A$ is irreducible, then there exists the unique $\beta$-KMS state $\phi^\omega$ of the Cuntz-Krieger algebra $\mathcal{O}_A$ for the action $\alpha^\omega$ and the inverse temperature $\beta$ is also unique.*

Throughout this note, we assume that $A$ is irreducible and not a permutation matrix. Let $(\pi_{\phi^\omega}, H_{\phi^\omega}, \xi_{\phi^\omega})$ be the GNS-triple of $\phi^\omega$. The above theorem, in particular, says that the von Neumann algebra $M = \pi_{\phi^\omega}(\mathcal{O}_A)''$ is a factor.

In order to compule the Connes spectrum of the modular automorphism of $\phi^\omega$, we investigate the weak-closure of the fixed-point algebra $\mathcal{O}_A^{\alpha^\omega}$ under $\alpha^\omega$. To do this, we need a technical lemma. Let $p$ be the period of $A$, where the period of the matrix $A$ means that

$$p(i) = \text{g.c.d.}\{m \in \mathbb{N} \mid A^m(i,i) \neq 0\}$$

for $i \in I$. If $A$ is irreducible, then this is independent on the choice of $i \in I$, and hence it is well-defined. For $m \in \mathbb{N}$, $i \in I$, we define partial isometries for $m \in \mathbb{N}$, $i \in I$ by

$$\theta_m^{(i)} = \sum_{\xi,\eta \in L_i(mp)} S_\xi S_\eta P_i S_\xi^* S_\eta^*,$$

where $L_i(n) = \{\xi \in \mathcal{W}_A \mid s(\xi) = i, A(r(\xi), i) = 1, |\xi| = n\}$ is the set of all loops of $i$ with the canonical length $n$. Note that $\theta_m^{(i)}$ is self-adjoint. We define the tracial state by $\psi^\omega = \phi^\omega|_{\mathcal{O}_A^{\alpha^\omega}}$ on $\mathcal{O}_A^{\alpha^\omega}$, and use the same symbol $\psi^\omega$ for its normal extension to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ for simplicity.

**Lemma 2.3 ([O2, Lemma 3.3])** *Let $f \in \pi_{\psi^\omega}(C(\Omega_A))''$ and $a \in \pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. Then for any $i \in I$,*

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) = \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2,$$

*where $y = (y_1, \ldots, y_N)$ is the left Perron eigenvector of $A_\omega$ with $\sum_{i\in I} x_i y_i = p$.*

*Proof.* It follows from the so-called Perron-Frobenius theorem below. □

**Theorem 2.4 ([Kit, Theorem 1.3.8])** *Let $A$ be an irreducible matrix with non-negative entries and $p$ the period of $A$. If $x = {}^T(x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are the right and left Perron eigenvectors of the Perron eigenvalue $\alpha$ such that $\sum_{i=1}^N x_i y_i = p$, then*

$$\lim_{n\to\infty} A^{pn}(i,j)/\alpha^{pn} = x_i y_j,$$

*for any $i, j = 1, \ldots, N$.*

Using the above lemma, we can completely determine the center of $\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$

**Definition 2.5** We say that $i$ is equivalent to $j$ if there are $\xi, \eta \in \mathcal{W}_A$ such that $s(\xi) = i, s(\eta) = j, r(\xi) = r(\eta)$ and $\omega_\xi = \omega_\eta$. Then we obtain the corresponding disjoint union $I = I_1^\omega \cup \cdots \cup I_{n_\omega}^\omega$. Set $P_{I_k^\omega} = \sum_{i \in I_k^\omega} P_i$.

**Proposition 2.6 ([O2, Lemma 3.1])**

$$Z(\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'' \cap \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})' = \bigoplus_{k=1}^{n_\omega} \mathbb{C} P_{I_k^\omega}.$$

*Proof.* It is easy to show that $P_{I_k^\omega} \in Z(\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'')$ for $k = 1, \ldots, n_\omega$. Note that $\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ is isomorphic to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. It therefore suffices to show that $Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \bigoplus_{k=1}^{n_\omega} \mathbb{C}P_{I_k^\omega}$ and use Lemma 2.3. □

Now we have the necessary ingredient for the proof of the main theorem.

**Theorem 2.7 ([O2, Theorem 4.2])** (1) *If $\omega_\xi/\omega_\eta \in \mathbb{Q}$ for any circles $\xi, \eta$, then $M = \pi_{\psi^\omega}(\mathcal{O}_A)''$ is the AFD type* $\mathrm{III}_\lambda$ *factor for some* $0 < \lambda < 1$.

(2) *If $\omega_\xi/\omega_\eta \notin \mathbb{Q}$ for some circles $\xi, \eta$, then $M = \pi_{\psi^\omega}(\mathcal{O}_A)''$ is the AFD type* $\mathrm{III}_1$ *factor.*

*Proof.* Since $\phi^\omega$ is $\alpha^\omega$-invariant, $\alpha^\omega$ can be extended to an action on $M$. We use the same symbol $\phi^\omega$ for its normal extension. Let $\sigma^{\phi^\omega}$ be the modular automorphism group for $\phi^\omega$ , which satisfies $\sigma_t^{\phi^\omega} = \alpha^\omega_{-\beta t}$ for $t \in \mathbb{R}$. We remark that $M^\sigma = \pi_{\phi^\omega}(\mathcal{O}_A^\alpha)''$. Therefore it follows from Proposition 2.6 that $\Gamma(\sigma^{\phi^\omega})$ is the additive subgroup of $\mathbb{R}$ generated by $\beta\omega_\xi$ for all circles $\xi$. □

# 3 Quasi-Free KMS States and Random Walks

In this section, we introduce some results in [O1] by using the following simple example.

**Example 3.1 ([Spi])** Let $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ be the free group with generators $a$ and $b$, and $S = \{a, b, a^{-1}, b^{-1}\}$ a generating set. We define the compact space

$$\Omega = \{\omega = (z_k)_{k=1}^\infty \mid z_k \in S, z_k \neq z_{k+1}^{-1}\} \subseteq \prod_{k=1}^\infty S.$$

Left multiplications of $\mathbb{F}_2$ on $\Omega$ induce an action of $\mathbb{F}_2$ on $C(\Omega)$:

$$(tf)(\omega) = f(t^{-1}\omega),$$

for $f \in C(\Omega), t \in \Gamma$ and $\omega \in \Omega$. Let $\Omega(x)$ be the set of infinite words with beginning $x \in S$. Consider the crossed product

$$C(\Omega) \rtimes \mathbb{F}_2 = C^*(f, u_x \mid f \in C(\Omega), x \in S),$$

where $u_x$ is the inplementing unitary of $x \in S$. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$S \times S$-matrix and $\mathcal{O}_A = C^*(S_x \mid x \in S)$ be the Cuntz-Krieger algebra associated to $A$. We denote by $\chi_{\Omega(x)}$, the characteristic function on $\Omega(x)$. Then we have the following identification:

$$\begin{array}{ccc} C(\Omega) \rtimes \mathbb{F}_2 & \simeq & \mathcal{O}_A \\ \chi_{\Omega(x)} & \leftrightarrow & P_x \\ u_x & \leftrightarrow & S_x + S^*_{x^{-1}} \\ u_x \chi_{\Omega \backslash \Omega(x^{-1})} & \leftrightarrow & S_x \end{array}$$

We use the symbol $\mathcal{O}_{\mathbb{F}_2}$ instead of $\mathcal{O}_A$.

For $\omega = (\omega_x)_{x \in S} \in \mathbb{R}^4_+$, we consider the action $\alpha^\omega$ of $\mathbb{R}$ on $\mathcal{O}_{\mathbb{F}_2}$, given by

$$\alpha^\omega_t(S_x) = e^{\sqrt{-1}\omega_x t} S_x.$$

By Theorem 2.1, we have the unique $\beta$-KMS state $\phi$ for $\alpha^\omega$, which has the form $\nu \circ \Phi$, where $\nu$ is a probability measure $\Omega$ and $\Phi$ is the canonical conditional expectation from $\mathcal{O}_{\mathbb{F}_2}$ onto $C(\Omega)$. Our purpose is to construct the above probability measure $\nu$ from a random walk on $\mathbb{F}_2$. The reader is referred to [W2] for a good book of random walks. We use the following result (e.g. see [W1]).

**Proposition 3.2** *Let $\mu$ be a probability measure on $\mathbb{F}_2$ such that* $\operatorname{supp}(\mu)$ *is finite and*

$$\bigcup_{n \geq 1} (\operatorname{supp}(\mu))^n = \mathbb{F}_2.$$

*Then there exists the unique probability measure $\nu$ on $\Omega$ such that*

(1) $(\Omega, \nu)$ *is the Poisson boundary,*

(2) $\nu = \mu * \nu$,

*where* $\mu * \nu$ *is defined by*

$$\int_\Omega f(\omega)d\mu * \nu(\omega) = \int_\Omega \int_{\mathrm{supp}\mu} f(t\omega)d\mu(t)d\nu(\omega)$$

*for* $f \in C(\Omega)$.

Using the above, we can prove the following.

**Theorem 3.3 ([O1, Theorem 8.1])** *For any* $\omega = (\omega_x) \in \mathbb{R}_+^4$, *there exists the unique probability measure* $\mu$ *on* $\mathbb{F}_2$ *such that*

(1) $\mathrm{supp}(\mu) = S$,

(2) $\phi = \nu \circ \Phi$ *is the KMS state for* $\alpha^\omega$,

*where* $\nu$ *is the corresponding probability measure on* $\Omega$ *in Proposition 3.2.*

We next discuss the converse. Let $\mu$ be a probability measure on $\mathbb{F}_2$ with $\mathrm{supp}(\mu) = S$. By Proposition 3.2, there is the unique probability measure $\nu$ on $\Omega$ such that $\mu * \nu = \nu$. Let $\phi = \nu \circ \Phi$ be a state on $\mathcal{O}_{\mathbb{F}_2}$. Let $(\pi_\phi, H_\phi, \xi_\phi)$ be the GNS-triple of $\phi$. We also denote by $\phi$ its normal extension on $\pi_\phi(\mathcal{O}_{\mathbb{F}_2})''$. Let $\sigma^\phi$ be the modular automorphism group of $\phi$. Then we have the following.

**Theorem 3.4 ([O1, Theorem 8.4])** *There is some* $\omega = (\omega_x)_{x \in S} \in \mathbb{R}_+^4$ *such that*

$$\sigma_t^\phi(\pi_\phi(S_x)) = e^{\sqrt{-1}\omega_x t}\pi_\phi(S_x)$$

*for* $x \in S$ *and* $t \in \mathbb{R}$.

Now we can apply Theorem 2.6 to $\mathcal{O}_{\mathbb{F}_2}$.

**Corollary 3.5** *Let* $\nu$ *be the probability measure on* $\Omega$ *that gives the quasi-free KMS state* $\nu \circ \Phi$. *Then*

(1) *If* $\omega_x/\omega_y \in \mathbb{Q}$ *for any* $x, y \in S$, *then* $L^\infty(\Omega, \nu) \rtimes \mathbb{F}_2$ *is the AFD type* $\mathrm{III}_\lambda$ *factor for some* $0 < \lambda < 1$.

(2) *If* $\omega_x/\omega_y \notin \mathbb{Q}$ *for some* $x, y \in S$, *then* $L^\infty(\Omega, \nu) \rtimes \mathbb{F}_2$ *is the AFD type* $\mathrm{III}_1$ *factor.*

**Remark 3.6** It was shown in [O2] that we can apply Theorem 2.7 to the boundary actions arising from some amalgamated free product group $\Gamma$ if $\Gamma$ satisfies some conditions. These generalize results of [RR].

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