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Representation of Choquet Integral

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1 Introduction

Non-additive set functions on measurable space is used in economics, decision theory and artificial intelligence, called by various name, such as cooperative game , capacity or fuzzy measure. In this paper, according to Denneberg [4] we call them non-additive measures. The Choquet integral with respect to a non-additive measure is a basic tool for multicriteria decision making, image processing and recognition [8, 9]. We consider the space $\mathcal{FM}^+$ of non-additive measures with topology introduced by Choquet integral.

The subspace $\mathcal{FM}_1^+$ of non-additive measures μ satisfying $\mu(X) = 1$ where X is the universal set is compact. The space $\mathcal{FM}^+$ of non-additive measures is a locally convex space. Applying the facts mentioned above, we obtain the additive representation of Choquet integral, that is, the Choquet integral with respect to a non-additive measure is represented by the classical integral with respect to the classical measure.

The similar theorems are shown in various contexts. In [11, 12], Murofushi and Sugeno show the additive representation theorem and propose an interpretation that non-additive measures express with their non additivity interaction among subset. Denneberg [5] shows the additive representation theorem, that is a generalization in various fields of papers, such as Gilboa and Schmeidler [6, 7] and Marinacci [10].

We compare these representation theorems, and show the equivalent points and the difference among them. We show that the domain of the representing classical measure and the representing integrand of classical integral are equivalent, but the classical measures which represent the non-additive measures are different.

2 Non-additive measure and Choquet integral

In this subsection, we present basic definitions and theorems about non-additive measures and the Choquet integral.

Definition 2.1. Let $(X, \mathcal{X})$ be a measurable space. A *non-additive measure* μ is an real valued set function, $\mu : \mathcal{X} \longrightarrow R^+$ with the following properties: (i) $\mu(\emptyset) = 0$ and (ii) $\mu(A) \leq \mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{X}$, where $R^+ = [0, \infty)$ is the set of extended nonnegative real numbers. We define *the conjugate* μ^c *of* μ by $\mu^c(A) = \mu(X) - \mu(A^c)$ for $A \in \mathcal{X}$.

The class of bounded measurable functions is denoted by $\mathcal{L}^\infty$ and the class of bounded non negative measurable functions by $\mathcal{L}^{\infty+}$.

Definition 2.2. [1, 11] Let μ be a non-additive measure on $(X, \mathcal{X})$.

(1) *The Choquet integral* of $f \in \mathcal{L}^{\infty+}$ with respect to μ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu_f(r) dr,$$

where $\mu_f(r) = \mu(\{x | f(x) \geq r\})$.

(2) Suppose $\mu(X) < \infty$. The Choquet integral of $f \in \mathcal{L}^\infty$ with respect to μ is defined by

$$(C)\int f d\mu = (C)\int f^+ d\mu - (C)\int f^- d\mu^c,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.

Definition 2.3. A non monotonic non-additive measure μ is k-monotone ($k \geq 2$) if for $A_1, \cdots A_k \in \mathcal{X}$

$$\mu(\bigcup_{i=1}^{k} A_i) + \sum_{I \subset 1, \cdots, k, I \neq \emptyset} (-1)^{|I|} \mu(\bigcap_{i \in I} A_i) \geq 0.$$

We say that μ is totally monotone if it is monotone and k-monotone for all $k \geq 2$.

If $\mu(X) = 1$ and μ is totally monotone, μ is a belief function.

Let $\mathcal{FM}^+$ be the class of (monotone) non-additive measures. We define $\mathcal{FM} := \{\mu - \nu | \mu, \nu \in \mathcal{FM}^+\}$. and $\mathcal{FM}^1 := \{\mu \in \mathcal{FM}^+ | \mu(X) = 1\}$.

Let f be a nonnegative measurable function. We define the map $C_f : \mathcal{FM} \to R$ by $C_f(\mu) := (C)\int f d\mu$. We define $C_A = C_{1_A}$ for $X \in \mathcal{X}$. We denote the set of bounded nonnegative measurable functions by B^+. It is obvious that C_f is a linear map on $\mathcal{FM}$ for all $f \in B^+$.

Definition 2.4. We shall say that the coarsest topology for which every C_A is continuous for $A \in \mathcal{X}$ is $\mathcal{X}$ - topology for $\mathcal{FM}$, and that the coarsest topology for which every C_f is continuous for $f \in B^+$ is B^+ - topology for $\mathcal{FM}$.

Definition 2.5. Let E be a vector space and $A \subset E$.

We define the convex hull c(A) by

$$c(A) = \cap\{Y | A \subset Y, Y \text{is a convex set}\}.$$

We say that $x \in X$ is an extreme point of X if $x = \lambda x_1 + (1-\lambda)x_2; x_1, x_2 \in X, 0 \leq \lambda \leq 1$ implies $x_1 = x_2 = x$. We denote the set of extreme points of A by $\mathcal{E}(A)$.

Definition 2.6. We say that $\mu \in \mathcal{FM}^1$ is $0-1$ non-additive measure if $\mu(A) = 0$ or $\mu(A) = 1$ for all $A \in \mathcal{X}$. We denote the set of $0-1$ non-additive measures by $\mathcal{FM}_0^1$. That is,

$$\mathcal{FM}_0^1 = \{\mu | \mu \in \mathcal{FM}^+, \mu : \mathcal{X} \longrightarrow \{0,1\}\}.$$

3 Integral representations

In this section, we show three integral representation theorem.

3.1 Representation by Choquet theorem

In this subsection, we show the representation theorem of Choquet integral by topological approach. The details of the proofs are in [13].

Let E be a Hausdorff locally convex space and $Y \subset E$ be compact and convex. The set of continuous convex function on Y is denoted by $S(Y)$. Define $A(Y) := S(Y) \bigcap (-S(Y))$.

Let $K(E,R)$ be the class of continuous functions $f : E \to R$ with compact support. On $K(E,R)$ we put the order defined by $f \geq 0$ if and only if $f(x) \geq 0$ for all $x \in E$. A radon measure on E is a linear map $\mu : K(E,R) \to R$ such that for any $Y \subset E$ compact there exists a number M_Y such that $f \in K(E,R)$ and $supp(f) \subset Y$ implies $\mu(f) \leq M_Y ||f||$, where $supp(f)$ is a support of f and the norm $||\cdot||$ is the sup norm. The collection of Radon measures on E is denoted by $\mathcal{R}(E)$ and the set of positive Radon measures with the order defined by $\mu \geq 0$ if and only if $\mu(f) \geq 0$ for all $f \geq 0, f \in K(E,R)$ by $\mathcal{R}^+(E)$. We define the order $\prec$ in $\mathcal{R}^+(Y)$ by $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for all $f \in S(Y)$. There exists a maximal element $m \in \mathcal{R}^+(Y)$ with respect to $\prec$. We say that the maximal element $m \in \mathcal{R}^+(Y)$ is a maximal measure. For $\mu \in \mathcal{R}(E)$, we define $||\mu|| := \sup\{|\mu(f)| | f \in K(E,R), ||f|| \leq 1\}$. We define $\mathcal{R}^1 := \{\mu \in \mathcal{R}^+(E) | ||\mu|| = 1\}$. Let f be a bounded real-valued function. Define $\hat{f}(Y) := \inf\{g(x) | g \in (-S(Y)), g \geq f\}$. We say that Y_f is the bordering set of f if $Y_f = \{x \in Y | f(x) = \hat{f}(x)\}$

The space $\mathcal{FM}$ with B^+ - topology is a Hausdorff locally convex space, and $\mathcal{FM}^1$ is compact convex. Define $h_f : \mathcal{FM}^1 \to R$ by $h_f(\mu) = C_f(\mu)$ for $f \in B^+$. Then h_f is linear and continuous. Applying Choquet theorem [3, 2] there exists a maximal radon measure $m \in \mathcal{R}(\mathcal{FM}^1)$ such that $h_f = m(h_f)$ and $m(\mathcal{FM}^1 \setminus G_A) = 0$ for $G_A := \{\mu \in \mathcal{FM}^1 | \mu(A) = 0 \text{ or } \mu(A) = 1\}$, $A \in \mathcal{X}$.

Applying Riesz's Representation theorem we have the next theorem.

Theorem 3.1. *For every $\mu \in \mathcal{FM}^1$, there exists a maximal Radon measure $m \in \mathcal{R}^1$ such that*

$$(C)\int f d\mu = \int h_f dm,$$

for all $f \in B^+$ and $m(\mathcal{FM}^1 \setminus G_A) = 0$ for every $A \in \mathcal{X}$. Especially $m(\mathcal{FM}^1 \setminus \mathcal{FM}^1_0) = 0$ if $\mathcal{FM}^1$ is metrizable.

We say that (h_f, m) is Choquet representation for (f, μ).

As to the metrizability of $\mathcal{FM}^1$ we have the next proposition.

Proposition 3.2. *If B^+ is separable, the $\mathcal{FM}^1$ is separable and metrizable.*

Next we consider the uniqueness of Choquet representation. First we define the Choquet simplex. Let $Y \subset E$ be convex and compact. Denote $\tilde{Y} = \{(\lambda x, \lambda)/x \in Y, \lambda > 0\}$ and $\hat{Y} = \tilde{Y} - \tilde{Y}$. We say that Y is Choquet simplex if there exists $\sup(x_1, x_2)$ for

$x_1, x_2 \in \hat{Y}$, where the order $\prec$ is defined by $x_1 \prec x_2$ if and only if $x_2 - x_1 \in \tilde{Y}$. It follows Choquet theorem [3] that the Choquet representation is unique if and only if $\mathcal{FM}^1$ is Choquet simplex. But $\mathcal{FM}^1$ is not always Choquet simplex. Therefore the Choquet representation is not always unique.

3.2 Interpreter representation

In this subsection, according to Murofushi and Sugeno we present the interpreter representation theorem. All proofs are shown in [11, 12].

Definition 3.3. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces.

(1) A mapping $H : \mathcal{X} \longrightarrow \mathcal{Y}$ is called an interpreter from $\mathcal{X}$ to $\mathcal{Y}$ if H satisfies (a) $H(\emptyset) = \emptyset$, (b) $H(A) \subset H(B)$ whenever $A \subset B$.

A triplet $(Y, \mathcal{Y}, H)$ is called a frame of $(X, \mathcal{X})$ if H is an interpreter from $\mathcal{X}$ to $\mathcal{Y}$.

Let $(X, \mathcal{X}, \mu)$ be a non-additive measure space. A quadruplet $(Y, \mathcal{Y}, m, H)$ is called an interpreter representation of μ if H is an interpreter from $\mathcal{X}$ to $\mathcal{Y}$, m is a classical measure on $(Y, \mathcal{Y})$ and $\mu = m \circ H$.

A semifilter θ in a measurable space $(X, \mathcal{X})$ is a non empty subclass of $\mathcal{X}$ with the properties ; (1) $\emptyset \notin \theta$, (2) if $A \in \theta$ and $A \subset B \in \mathcal{X}$ then $B \in \theta$.

Denote the set of all semifilters in $(X, \mathcal{X})$ by S_X, and define a mapping $H_X : \mathcal{X} \longrightarrow 2^{S_X}$ by $H_X(A) := \{\theta \in S_X | A \in \theta\}$.

$\mathcal{S}_X$ denotes the σ-algebra generated by $\{H_X(A) | A \in \mathcal{X}\}$.

The triplet $(S_X, \mathcal{S}_X, H_X)$ is called the universal frame of $(X, \mathcal{X})$ for representation.

Theorem 3.4. *For every non-additive measure μ on $(X, \mathcal{X})$ there exists a classical measure m on $\mathcal{S}_X$ such that $(S_X, \mathcal{S}_X, m, H_X)$ is a representation of μ.*

Definition 3.5. Let $(Y, \mathcal{Y}, m, H)$ be an interpreter representation of a non-additive measure space $(X, \mathcal{X}, \mu)$.

For a non negative measurable function f on X we define a function i_f on Y by $i_f(y) := \sup\{r | y \in H(\{f \geq r\})\}$. We call i_f an interpreter for a measurable function f induced by H.

Theorem 3.6. *Let $(Y, \mathcal{Y}, m, H)$ be an interpreter representation of a non-additive measure space $(X, \mathcal{X}, \mu)$ and i be an interpreter induced by H. We have*

$$(C) \int f d\mu = \int i_f dm.$$

for $f \in \mathcal{L}^{\infty+}$.

We have the next theorem from Theorem 3.4 and Theorem 3.6 .

Theorem 3.7. (Interpreter representation theorem) *Let $(X, \mathcal{X}, \mu)$ be a non-additive measure space. There exists a classical measure m on $\mathcal{S}_X$ such that*

$$(C) \int f d\mu = \int i_f dm$$

for $f \in \mathcal{L}^{\infty+}$.

3.3 Representation with Möbius transform

In this subsection, we present the representation with Möbius transform by Denneberg. The essence of the proofs are shown in [5].

$\mathcal{FM}_0^1$ denotes the set of $0 - 1$ non-additive measures. We define the tilde operator which assigns to a measurable function $f : X \to [0, \infty]$ the function

$$\widetilde{f}(\eta) := (C) \int f d\eta, \eta \in \mathcal{FM}_0^1.$$

If $A \in \mathcal{X}$, $\widetilde{A}$ is defined by $\widetilde{A} := \{\eta \in \mathcal{FM}_0^1 | \eta(A) = 1\}$. We use the notation $\widetilde{\mathcal{T}} := \{\widetilde{A} | A \in \mathcal{T}\}$, for a class $\mathcal{T} \subset 2^X$.

Definition 3.8. A non-additive measure $u_A \in \mathcal{FM}_0^1$ defined by

$$u_A(B) = \begin{cases} 1 & A \subset B \\ 0 & o.w. \end{cases}$$

is called a unanimity game for coalition A.

In some literature, it is called a $0-1$ necessity measure.

We write the set of all unanimity games on $\mathcal{X}$ by $\mathcal{FM}_{0_u}^1$. $\mathcal{FM}_{0_s}^1$ denotes the set of all supermodular $0-1$ non-additive measures, that is,

$$\eta \in \mathcal{FM}_{0_s}^1 \Leftrightarrow \eta(A \cup B) + \eta(A \cap B) \geq \eta(A) + \eta(B)$$

for $A, B \in \mathcal{X}$

It is obvious that $\mathcal{FM}_{0_u}^1 \subset \mathcal{FM}_{0_s}^1 \subset \mathcal{FM}_0^1$.

Let $f :\to R$ be a measurable function. We denote by $\mathcal{M}_f$ the class of upper level sets $\{x \in X | f(x) \geq \alpha\}$.

We denote by $\mathcal{D} \subset 2^{\mathcal{FM}_0^1}$, $\mathcal{D}_u \subset 2^{\mathcal{FM}_{0u}^1}$ and $\mathcal{D}_s \subset 2^{\mathcal{FM}_{0s}^1}$ the algebra generated by $\widetilde{\mathcal{X}}$ in $2^{\mathcal{FM}_0^1}$,$2^{\mathcal{FM}_{0u}^1}$ and $2^{\mathcal{FM}_{0s}^1}$ respectively.

We use $\mathcal{FM}_{0_\bullet}^1$ as a variable for one of the sets $\mathcal{FM}_0^1, \mathcal{FM}_{0_u}^1$ or $\mathcal{FM}_{0_s}^1$, and $\mathcal{D}_\bullet$ as $\mathcal{D}, \mathcal{D}_u$ or $\mathcal{D}_s$.

Definition 3.9. A kernel function κ for $\mathcal{X}$ is a function $\kappa : \mathcal{FM}_{0_\bullet}^1 \times \mathcal{X} \to [0, b]$; $(\eta, A) \mapsto \kappa_\eta(A)$ such that

(1) κ_η is a non-additive measure on $\mathcal{X}$, $\eta \in \mathcal{FM}_{0_\bullet}^1$.

(2) For fixed $A \in \mathcal{X}$, the real function $\kappa_\bullet(A)$ on $\mathcal{FM}_{0_\bullet}^1$ is $\mathcal{D}_\bullet$-measurable.

Next we define κ-extension.

Definition 3.10. Let $f \in \mathcal{L}^\infty$ and κ be a kernel function. κ-extension f^κ of f defined by

$$f^\kappa(\eta) := (C)\int f(x)d\kappa_\eta(x).$$

Let ν be a non-additive measure on $\mathcal{D}_\bullet$. We define the κ-transform μ on $\mathcal{X}$ of ν by

$$\mu(A) := \int \kappa_\eta(A)d\nu(\eta), A \in \mathcal{X}.$$

If $\kappa_\eta(A) = \eta(A)$ for $(\eta, A) \in \mathcal{FM}^1_{0\bullet} \times \mathcal{D}$, then κ is a kernel function for $\mathcal{X}$. The kernel function $\kappa_\eta = \eta$ on $\mathcal{FM}^1_{0u}$ is called the zeta function for $\mathcal{X}$. The corresponding transform of ν is called the zeta transform of ν.

The next proposition is Example 4.1 in Denneberg [5].

Proposition 3.11. *Let κ be a zeta function. Then we have $f^\kappa = \widetilde{f}$ for $f \in \mathcal{L}^\infty$.*

The next theorems are the main theorems in this subsection.

Theorem 3.12. *Let κ_i be kernel functions for $\mathcal{X}$ and ν_i monotone and additive set functions on $\mathcal{D}_\bullet, i = 1, 2$ respectively . Let μ_{ij} the k_i transform of ν_j for $i, j = 1, 2$. Define $\kappa := \kappa_1 - \kappa_2$, $\nu := \nu_1 - \nu_2$ and $\mu := (\mu_{11} - \mu_{12}) - (\mu_{21} - \mu_{22})$. If $f \in \mathcal{L}^\infty$, then*

$$f^\kappa = f^{\kappa_1} - f^{\kappa_2} \in \mathcal{L}^\infty$$

and

$$(C)\int f d\mu = \int f^\kappa d\nu.$$

ν, μ and κ_η are non monotonic non-additive measures of bounded variation for $\mathcal{X}$.

Corollary 3.13. *Let k be a zeta function for $\mathcal{X}$ and ν an additive set function on $\mathcal{D}_\bullet$. Let μ be the zeta transform of ν. If $f \in \mathcal{L}^\infty$, then*

$$(C)\int f d\mu = \int \widetilde{f} d\nu.$$

Theorem 3.14. *For any non-additive measure μ on $\mathcal{X}$ there exists a unique additive set function ν on $\mathcal{D}_u$ (or $\mathcal{D}_s$) so that $\mu(A) = \nu(\widetilde{A})$ for every $A \in \mathcal{X}$. Furthermore, μ is the (signed) zeta transform of ν.*

Corollary 3.15. *For any non-additive measure μ on $\mathcal{X}$ there exists a unique additive set function ν on $\mathcal{D}_u$ (or $\mathcal{D}_s$) so that*

$$(C)\int f d\mu = \int \tilde{f} d\nu$$

for every measurable function f.

We shall call ν the Möbius transform of μ on $\mathcal{D}_p$ (or $\mathcal{D}_u$) and denote it ν^μ.

The Möbius transform is not always monotone. The next proposition shows the necessary and sufficient condition for the Möbius transform to be monotone.

Proposition 3.16. *A non-additive measure μ on $\mathcal{X}$ is totally monotone if and only if its Möbius transformation ν^μ is monotone.*

Since Möbius transformation is additive, the monotonicity is equivalent to the positiveness of it.

The next two theorems show that the Möbius transform can be extended to a σ-additive (signed) measure.

Theorem 3.17. *Any additive set function ν on $\mathcal{D}_s$ is $\sigma-$ additive. If $\nu \in \mathcal{FM}$, then it has a unique σ-additive extension to the σ- algebra $\sigma(\widetilde{\mathcal{X}})$ generated by $\widetilde{\mathcal{X}}$ in $2^{\mathcal{FM}_0^1}$.*

Theorem 3.18. *Let $(X, \mathcal{X}, \mu)$ be a non-additive measure space. There exists an additive set function ν^μ on $\sigma(\widetilde{\mathcal{X}})$ generated by $\widetilde{\mathcal{X}}$ in $2^{\mathcal{FM}_0^1}$.*

4 Comparison

In this section we show that there exists a bijection from the class of $0-1$ non-additive measures to the class of semifilters of X.

Theorem 4.1. *Let $\mathcal{FM}_0^1$ be the class of $0-1$ non-additive measures, S_X be the class of semifilters of X. There exists a bijection $\varphi : \mathcal{FM}_0^1 \to S_X$.*

We call the bijection φ in previous theorem a mediator for representation.

The next theorem follows from the definition of h_f.

Theorem 4.2. *Let $(X, \mathcal{X}, \mu)$ be a non-additive measure space, $(S_X, \mathcal{S}_X, m, H_X)$ is an interpreter representation, and (h_f, m) be the Choquet integral representation for $(f, \mu), f \in B^+$. Then we have*

$$h_f(\nu) := \sup\{r | \nu \in H_X(\{x | f(x) \geq r\})\},$$

that is, $h_f = i_f$.

As to the interpreter representation and representation with the Möbius transform, we have the next theorem.

Theorem 4.3. *Let i be a interpreter from $\mathcal{X}$ to $\mathcal{S}_X$, τ be a tilde operator from $\mathcal{X}$ to $\widetilde{\mathcal{X}}$, φ be the mediator for representation. Define a mapping $H_X : \mathcal{X} \longrightarrow 2^{S_X}$ by $H_X(A) := \{\theta \in S_X | A \in \theta\}$.*

(1) $H_X(A) = \varphi(\widetilde{A})$ *for* $A \in \mathcal{X}$.

(2) $i_f \circ \varphi = \tau(f)$ *for* $f \in \mathcal{L}^{\infty+}$.

As we show above, the representing integrand are equivalent. On the other hand concerning the representing measure, in the interpreter representation the measure is monotone and the uniqueness is not always true. In the representation with Möbius transform, the measure is not always monotone and the existence is unique.

It is not proved in [5] that there exists an additive set function on $\mathcal{D}$ so that it represents a non-additive measure and the Choquet integral. Using the interpreter representation theorem, we can show the existence.

Theorem 4.4. *For any non-additive measure μ on $\mathcal{X}$ there exists a measure on $\mathcal{D}$ so that*

$$(C)\int f d\mu = \int \tilde{f} d\nu$$

for every $f \in \mathcal{L}^{\infty+}$.

Concerning belief function on the finite X, the two theorems are perfectly the same.

We conclude this paper by showing the next table about the properties of three representation theorems.

Table 1: Three representation theorem

	Topology		Int.	Möbius	
Space	$\mathcal{FM}^1$	$\mathcal{FM}^1_{0r}$	S_X	$\mathcal{FM}^1_0$	$\mathcal{FM}^1_{0s}$
Monotone	○	○	○	×	×
Unique	×	×	×	×	○

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