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Some Stability Criteria for a Class of Volterra Integro-differential Systems

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Abstract

We study the stability and boundedness of the solutions of a system of Volterra integro-differential equations of the form $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t,s)\mathbf{g}(\mathbf{x}(s))ds + \mathbf{h}(t)$. Our results extend some of the more well-known criteria.

1 Introduction

We consider the stability and boundedness of solutions of systems of Volterra integro-differential equations, with forcing functions, of the form

$$\frac{d}{dt}[\mathbf{x}(t)] = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t,s)\mathbf{g}(\mathbf{x}(s))ds + \mathbf{h}(t), \tag{1}$$

in which $\mathbf{A}(t)$ is an $n \times n$ matrix function continuous on $[0,\infty)$, $\mathbf{B}(t,s)$ is an $n \times n$ matrix continuous for $0 \leq s \leq t < \infty$, $\mathbf{f}$ and $\mathbf{g}$ are $n \times 1$ vector functions continuous on $(-\infty,\infty)$ and $\mathbf{h}$ is an $n \times 1$ vector function continuous on $[0,\infty)$.

The qualitative behaviour of the solutions of systems of Volterra integro-differential equations, especially the case where $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = \mathbf{x}$, has been thoroughly analyzed by many researchers. Among the contributions in the 1980s, those of Burton are worthy of mention. His work ([1], [2]) laid the foundation for a systematic treatment of the basic structure and stability properties of Volterra integro-differential equations, mainly, via the direct method of Lyapunov. This paper essentially looks into some of the many interesting results established by Burton and proposes ways of utilizing the form of the Lyapunov functionals proposed by Burton to construct new or similar ones for system (1).

Now, if $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{h}(t) = \mathbf{0}$, then system (1) reduces to

$$\frac{d}{dt}[\mathbf{x}(t)] = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t,s)\mathbf{g}(\mathbf{x}(s))ds, \tag{2}$$

so that $\mathbf{x}(t) \equiv \mathbf{0}$ is a solution of (2) called the *zero solution*. Hence, the stability analysis of (1) could be considered as the stability analysis of its solution $\mathbf{x}(t) \equiv \mathbf{0}$ given the forcing function or the external disturbance $\mathbf{h}(t)$. The initial conditions for integral equations such as (1) or (2) involve continuous *initial functions* on an *initial interval*, say, $\mathbf{x}(t) = \phi(t)$ for $0 \le t \le t_0$. Hence, $\mathbf{x}(t; t_0, \phi)$, $t \ge t_0 \ge 0$ denotes the solution of (1) or (2), with the initial function $\phi : [0, t_0] \to \mathbf{R}^n$ assumed to be bounded and continuous on $[0, t_0]$.

The definitions of the stability and the boundedness of solutions of (1) are given in Burton [1]. It is assumed that the functions in (1) are well-behaved, that continuous initial functions generate solutions, and that solutions which remain bounded can be continued.

2 The Scalar Equation

2.1 Unperturbed Case

Consider the scalar equation

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds\,. \tag{3}$$

We suppose that

$$A(t) \text{ is continuous for } 0 \le t < \infty; \tag{4}$$

$$B(t,s) \text{ is continuous for } 0 \le s \le t < \infty; \tag{5}$$

$$\int_0^t |B(u,s)|du \text{ is defined and continuous for } 0 \le s \le t < \infty; \tag{6}$$

$$f(x) \text{ and } g(x) \text{ are continuous on } (-\infty, \infty); \tag{7}$$

$$xf(x) > 0 \ \ \forall x \ne 0\,, \text{ and } f(0) = g(0) = 0\,. \tag{8}$$

For comparison sake, we first state Burton's theorem regarding the stability of the zero solution of (3).

Theorem 1 (Burton [7]). *Let (4)–(8) hold and suppose there are constants $m > 0$ and $M > 0$ such that $g^2(x) \le m^2 f^2(x)$ if $|x| \le M$. Let*

$$\beta(t,k) = A(t) + k\int_t^\infty |B(u,t)|du + \frac{1}{2}\int_0^t |B(t,s)|ds$$

If there exists $k > 0$ with $m^2 < 2k$ and $\beta(t,k) \le 0$ for $t \ge 0$, then the zero solution of (3) is stable.

We next state an extension of Theorem 1, which Burton proved via the Lyapunov functional

$$V_1(t, x(\cdot)) = \int_0^x f(s)ds + k\int_0^t \int_t^\infty |B(u,s)|du f^2(x(s))ds\,. \tag{9}$$

We are motivated here by the fact that a Lyapunov function for an asymptotically stable system governed by ordinary differential equations gives conservative estimates of the region of asymptotic stability. A superior Lyapunov function would be considered to be the one that gives better estimates of the exact region, a knowledge of which is a necessity in some engineering disciplines, such as power system engineering (see, for example, Pai [3]). Judging whether a Lyapunov function is superior is inherently numerical.

We intend to show via numerical examples that a Lyapunov functional could also provide a better picture of the stability of a Volterra equation. Hence, we propose another stability criterion proved by a new functional that is a combination of Burton's functional (9) and a generalized Lyapunov function proposed by Miyagi et. al for power systems [4] and single-machine systems [5].

Theorem 2. *Let (4)-(8) hold, with $A(t) < 0$, and suppose there are constants*

$$m > 0 \text{ and } M > 0 \text{ such that } g^2(x) \le m^2 f^2(x) \text{ if } |x| \le M\,, \tag{10}$$

$$\alpha > 4 \text{ and } N > 0 \text{ such that } 4x^2 \le (\alpha - 4) f^2(x) \text{ if } |x| \le N\,, \text{ and} \tag{11}$$

$$J \ge 1 \text{ such that } \quad -\frac{1}{4A(t)} \int_0^t |B(t,s)|ds < \frac{1}{J} \quad \text{for every} \quad t \ge 0\,. \tag{12}$$

Suppose there is some constant $k > 0$ such that

$$\frac{(1+\alpha)m^2}{J} < k\,, \tag{13}$$

and

$$A(t) + k\int_t^\infty |B(u,t)|du \le 0 \tag{14}$$

for $t \ge 0$. Then the zero solution of (3) is stable.

Proof. We use the Lyapunov functional

$$V_2(t, x(\cdot)) = \frac{1}{2}x^2 + \sqrt{\alpha}\int_0^x \sqrt{uf(u)}\,du + \frac{1}{2}\alpha\int_0^x f(u)du + k\int_0^t\int_t^\infty |B(u,s)|du f^2(x(s))ds\,.$$

to prove

$$V'_{2(3)}(t, x(\cdot)) \le \left[A(t) + k\int_t^\infty |B(u,t)|du\right] f^2(x) - \left[k - \frac{m^2(1+\alpha)}{J}\right]\int_0^t |B(t,s)|f^2(x(s))ds\,.$$

which will be negative semidefinite If equations (13) and (14) are satisfied, then $V'_{2(3)}(t, x(\cdot))$ is negative semidefinite. This implies the stability of zero solution of (3). □

Next we state a result which might be easier to use than Theorems 1 and 2.

Theorem 3. *Let (4)-(6) hold and assume that f and g are differentiable at $x = 0$. Let*

$$D(x) = \begin{cases} \dfrac{f(x)}{x}, & x \ne 0\,, \\ f'(0)\,, & x = 0\,, \end{cases} \qquad E(x) = \begin{cases} \dfrac{g(x)}{x}, & x \ne 0\,, \\ g'(0)\,, & x = 0\,, \end{cases}$$

and

$$\beta(t,k,x) = A(t)D(x) + k\int_t^\infty |B(u,t)|du|E(x)|\,.$$

Suppose there is some constant $k \ge 1$ such that $\beta(t,k,x) \le 0$ for all $t \ge 0$ and $x \in \mathbf{R}$. Then the zero solution of (3) is stable.

Theorem 3 is the special case of Theorem 7 for system (1) in Section 3. We can give several illustrative examples which show the differences of Theorems 1,2 and 3.

2.2 Perturbed Case

The next two results, which extend Theorem 1 and Theorem 2, give a class of forcing functions that maintains the boundedness of the solutions of the equation

$$x'(t) = A(t)f(x(t)) + \int_0^t B(t,s)g(x(s))ds + h(t), \tag{15}$$

where $h : [0,\infty) \to \mathbf{R}$ is defined almost everywhere on $[0,\infty)$.

Theorem 4. *Let (4)–(8) hold and suppose there is a constant $m > 0$ such that $g^2(x) \le m^2 f^2(x)$ for all $x \in \mathbf{R}$. Define*

$$\beta(t,k) = A(t) + k\int_t^\infty |B(u,t)|du + \frac{1}{2}\int_0^t |B(t,s)|ds$$

and let there be constants $\rho > 0$ and $k > 0$ such that $m^2 < 2k$ and $\beta(t,k) \le -\rho$ for $t \ge 0$. If $h(\cdot) \in L^2[0,\infty)$, then all solutions of (15) are bounded.

Proof. Let $\epsilon > 0$ and consider the functional

$$V_3(t,x(\cdot)) = V_1(t,x(\cdot)) + \frac{1}{4\epsilon}\int_t^\infty h^2(u)du.$$

Since $h(\cdot) \in L^2[0,\infty)$, we have

$$\frac{d}{dt}\left[\int_t^\infty h^2(u)du\right] = \frac{d}{dt}\left[\int_0^\infty h^2(u)du - \int_0^t h^2(u)du\right] = -h^2(t),$$

implying, therefore, the differentiability and hence the existence on $[0,\infty)$ of the second term of the functional V_3. Thus, we have

$$\begin{aligned} V_3{}'_{(15)} &\le \beta(t,k)f^2(x) + f(x)h(t) - \frac{1}{4\epsilon}h^2(t) \le -\rho f^2(x) + \epsilon f^2(x) + \frac{1}{4\epsilon}h^2(t) - \frac{1}{4\epsilon}h^2(t) \\ &= -(\rho - \epsilon)f^2(x). \end{aligned}$$

This completes the proof of Theorem 4 since we can always find some $\epsilon > 0$ small enough such that $(\rho - \epsilon) > 0$. □

In the same fashion, we prove the following extension of Theorem 2 similarly as in the proof of Theorem 4

Theorem 5. *Let (4)–(8) hold, with $A(t) < 0$, and suppose there are constants*

$m > 0$ *such that* $g^2(x) \le m^2 f^2(x)$ *for all* $x \in \mathbf{R}$,

$\alpha > 4$ *such that* $4x^2 \le (\alpha - 4)\, f^2(x)$ *for all* $x \in \mathbf{R}$, *and*

$J \ge 1$ *such that* $\quad -\dfrac{1}{4A(t)}\displaystyle\int_0^t |B(t,s)|ds < \frac{1}{J}$ *for every* $t \ge 0$.

Further, suppose there are constants $k > 0$ and $\rho > 0$ such that

$$\frac{(1+\alpha)m^2}{J} < k, \qquad A(t) + k\int_t^\infty |B(u,t)|du \le -\rho,$$

for all $t \ge 0$. If $h(\cdot) \in L^2[0,\infty)$, then all solutions of (15) are bounded.

3 The Vector Equation

In this section we shall give the stability and boundedness results for the vector equations without proofs because of the limitation of pages.

3.1 Unperturbed Case

Let us first look at the linear system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \int_0^t \mathbf{B}(t,s)\mathbf{x}(t)\,ds\,. \tag{16}$$

Let $\mathbf{x}^T = (x_1, \ldots, x_n)$, $\mathbf{A}(t) = [a_{ij}(t)]_{n\times n}$, and $\mathbf{B}(t,s) = [b_{ij}(t,s)]_{n\times n}$. One of the more effective results so far, in terms of ease of use, was proposed recently by Elaydi [6].

Theorem 6 (Elaydi [6]). *Suppose that for* $1 \le i \le n$, $t \ge 0$,

$$a_{ii}(t) + \sum_{\substack{j=1\\ j\ne i}}^{n} |a_{ji}(t)| + \sum_{j=1}^{n} \int_t^{\infty} |b_{ij}(u,t)|du \le 0\,.$$

Then the zero solution of system (16) is stable.

To have a generalization of Theorem 6, we consider the more general system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{f}(\mathbf{x}(t)) + \int_0^t \mathbf{B}(t,s)\mathbf{g}(\mathbf{x}(s))\,ds\,. \tag{17}$$

If we suppose that $\mathbf{f},\mathbf{g} \in C^1[\mathbf{R}^n, \mathbf{R}^n]$, then we can define

$$\mathbf{D}(\mathbf{x}) = [d_{ij}(\mathbf{x})]_{n\times n} \quad \text{with} \quad d_{ij}(\mathbf{x}) = \begin{cases} \displaystyle\int_0^1 \frac{\partial f_i(u\mathbf{x})}{\partial(ux_j)}du, & x_j \ne 0\,, \\[2ex] \displaystyle\frac{\partial}{\partial x_j}[f_i(x_1,\ldots,x_j=0,\ldots,x_n)]\,, & x_j = 0\,, \end{cases} \tag{18}$$

and

$$\mathbf{E}(\mathbf{x}) = [e_{ij}(\mathbf{x})]_{n\times n} \quad \text{with} \quad e_{ij}(\mathbf{x}) = \begin{cases} \displaystyle\int_0^1 \frac{\partial g_i(u\mathbf{x})}{\partial(ux_j)}du, & x_j \ne 0\,, \\[2ex] \displaystyle\frac{\partial}{\partial x_j}[g_i(x_1,\ldots,x_j=0,\ldots,x_n)]\,, & x_j = 0\,. \end{cases} \tag{19}$$

Then we have

$$\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{0}) = \mathbf{D}(\mathbf{x})\mathbf{x}\,, \quad \text{and} \quad \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{0}) = \mathbf{E}(\mathbf{x})\mathbf{x}\,.$$

Hence, assuming $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$, system (17) can be written as

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{D}(\mathbf{x}(t))\mathbf{x}(t) + \int_0^t \mathbf{B}(t,s)\mathbf{E}(\mathbf{x}(s))\mathbf{x}(s)\,ds\,, \tag{20}$$

the i-th component of which is

$$\begin{aligned}
x_i'(t) &= a_{ii}(t)\left[d_{ii}(\mathbf{x}(t))x_i(t)+\sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}(\mathbf{x}(t))x_j(t)\right] \\
&+\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\left[d_{ji}(\mathbf{x}(t))x_i(t)+\sum_{\substack{k=1\\ k\neq i}}^{n} d_{jk}(\mathbf{x}(t))x_k(t)\right] \\
&+\sum_{k=1}^{n}\int_0^t\left[b_{ii}(t,s)e_{ik}(\mathbf{x}(s))+\sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}(t,s)e_{jk}(\mathbf{x}(s))\right]x_k(s)ds.
\end{aligned} \tag{21}$$

The next result is new.

Theorem 7. *Assume that*

$$\mathbf{f},\mathbf{g}\in C^1[\mathbf{R}^n,\mathbf{R}^n] \quad \textit{and} \quad \mathbf{f}(\mathbf{0})=\mathbf{g}(\mathbf{0})=\mathbf{0}\,. \tag{22}$$

Let

$$\begin{aligned}
\beta_i(t,\kappa_i,\mathbf{x}) &= \Bigg\{a_{ii}(t)d_{ii}(\mathbf{x})+\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)d_{ji}(\mathbf{x}) \\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[|a_{jj}(t)d_{ji}(\mathbf{x})|+|a_{ji}(t)d_{ii}(\mathbf{x})|+\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}|a_{kj}(t)d_{ji}(\mathbf{x})|\right] \\
&+\sum_{k=1}^{n}\kappa_i\int_t^\infty\left[|b_{kk}(u,t)e_{ki}(\mathbf{x})|+\sum_{\substack{j=1\\ j\neq i}}^{n}|b_{kj}(u,t)e_{ji}(\mathbf{x})|\right]du\Bigg\}\,.
\end{aligned} \tag{23}$$

Suppose there is some $\kappa_i \geq 1$, $1 \leq i \leq n$, such that $\beta_i(t,\kappa_i,\mathbf{x}) \leq 0$ for all $t \geq 0$ and $\mathbf{x} \in \mathbf{R}^n$. Then the zero solution of system (17) is stable.

In proving Theorem 7 we utilize the functional

$$\begin{aligned}
V_5(t,\mathbf{x}(\cdot)) &= \sum_{i=1}^{n}|x_i(t)| \\
&+\sum_{i=1}^{n}\sum_{k=1}^{n}\kappa_i\int_0^t\int_t^\infty\left[|b_{ii}(u,s)e_{ik}(\mathbf{x}(s))|+\sum_{\substack{j=1\\ j\neq i}}^{n}|b_{ij}(u,s)e_{jk}(\mathbf{x}(s))|\right]du|x_k(s)|ds\,.
\end{aligned}$$

Remark 1. If $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = \mathbf{x}$, then it is clear that Theorem 6 gives us back Elaydi's Theo-

Remark 2. In Burton's Theorem 5 [7], due to the type of the Lyapunov functional used, a term that can be derived from

$$\sum_{k=1}^{n}\int_0^t \left[|b_{ii}(t,s)e_{ik}(\mathbf{x}(s))| + \sum_{\substack{j=1\\ j\neq i}}^{n} |b_{ij}(t,s)e_{jk}(\mathbf{x}(s))| \right] ds\,,$$

which appears at the end of (??), is added to the last term in (23) of Theorem 7. In this sense, Theorem 7 improves Burton's Theorem 5 [7] by having one term less.

Remark 3. If $i = 1$, then Theorem 7 gives Theorem 3, its scalar version, proven by the Lyapunov functional

$$V(t, x(\cdot)) = |x| + k\int_0^t \int_t^{\infty} \big|B(u,s)E(x(s))\big| du |x(s)| ds\,,$$

the time-derivative of which is taken with respect to a trajectory of the scalar equation (3) rewritten as

$$x' = A(t)D(x)x + \int_0^t B(t,s)E(x(s))x(s)ds\,,$$

where D and E are defined in Theorem 3.

3.2 Perturbed Case

Define $\mathbf{h}^{\mathbf{T}}(t) = (h_1(t), \dots, h_n(t))$ and $[d_{ij}]_{n\times n}$ and $[e_{ij}]_{n\times n}$ as in (18) and (19), respectively. Then the i-th component of system (1) is

$$\begin{aligned} x_i'(t) &= a_{ii}(t)\left[d_{ii}(\mathbf{x}(t))x_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} d_{ij}(\mathbf{x}(t))x_j(t) \right] \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)\left[d_{ji}(\mathbf{x}(t))x_i(t) + \sum_{\substack{k=1\\ k\neq i}}^{n} d_{jk}(\mathbf{x}(t))x_k(t) \right] \\ &\quad + \sum_{k=1}^{n}\int_0^t \left[b_{ii}(t,s)e_{ik}(\mathbf{x}(s)) + \sum_{\substack{j=1\\ j\neq i}}^{n} b_{ij}(t,s)e_{jk}(\mathbf{x}(s)) \right] x_k(s)ds + h_i(t)\,. \end{aligned} \tag{24}$$

The next result simply establishes the existence of a functional from which boundedness of solutions of system (1) can be deduced.

Theorem 8. *Assume that* $\mathbf{f}, \mathbf{g} \in C^1[\mathbf{R}^n, \mathbf{R}^n]$, *and* $\mathbf{f}(\mathbf{0}) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$. *Let* $\alpha_i \in C[[0,\infty), \mathbf{R}]$, $i = 1, \dots, n$, *and*

$$\begin{aligned} \beta_i(t, \mathbf{x}) &= \Bigg\{ \alpha_i(t) + a_{ii}(t)d_{ii}(\mathbf{x}) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)d_{ji}(\mathbf{x}) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{n} \left[|a_{jj}(t)d_{ji}(\mathbf{x})| + |a_{ji}(t)d_{ii}(\mathbf{x})| + \sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n} |a_{kj}(t)d_{ji}(\mathbf{x})| \right] \Bigg\}\,. \end{aligned}$$

Suppose there is some $c_i > 0$, $1 \le i \le n$, such that $\beta_i(t,\mathbf{x}) \le -c_i$ for all $t \ge 0$ and $\mathbf{x} \in \mathbf{R}^n$. Let $c = \min\{c_1, \dots, c_n\}$. Then, along a solution of system (1), the functional

$$\begin{aligned} W(t,\mathbf{x}(\cdot)) &= \sum_{i=1}^n |x_i(t)| + \sum_{i=1}^n \int_0^t \alpha_i(s) e^{-c(t-s)} |x_i(s)|\, ds \\ &\quad - \sum_{i=1}^n \sum_{k=1}^n \int_0^t \int_s^t e^{-c(t-u)} \left[|b_{ii}(u,s) e_{ik}(\mathbf{x}(s))| + \sum_{\substack{j=1\\ j\neq i}}^n |b_{ij}(u,s) e_{jk}(\mathbf{x}(s))| \right] du |x_k(s)| ds \end{aligned}$$

satisfies

$$W'_{(1)} \le -cW(t,\mathbf{x}(\cdot)) + \sum_{i=1}^n |h_i(t)|,$$

so that

$$W(t,\mathbf{x}(\cdot)) \le W(t_0, \phi(\cdot)) e^{-c(t-t_0)} + \sum_{i=1}^n \int_{t_0}^t e^{-c(t-s)} |h_i(s)| ds.$$

Remark 4. Theorem 8 is a generalization of Theorem 7.2.1, Burton [1], page 205. Then Corollary 1, Corollary 2 and Corollary 3 in Burton [1], pages 205-207, can be used to conclude ultimate boundedness of solutions of system (1) for some specific cases. For example, we shall apply Burton's corollaries to the case where

$$g_i(\mathbf{x}) = x_1 + x_2 + \ldots + x_n, \tag{25}$$

for $1 \le i \le n$. The assumption (25) implies that $\mathbf{E}(\mathbf{x}) = \mathbf{1}$, an $n \times n$ matrix with all entries being 1.

Corollary 1. *Let the conditions of Theorem 8 hold, with*

$$g_i(\mathbf{x}) = x_1 + x_2 + \ldots + x_n,$$

for $1 \le i \le n$. Further, suppose there is a constant P_i and a continuous scalar function $\Phi_i(t,s) \ge 0$ such that

$$\alpha_i(s) e^{-c(t-s)} - \sum_{k=1}^n \int_s^t e^{-c(t-u)} \left[|b_{ii}(u,s)| + \sum_{\substack{j=1\\ j\neq i}}^n |b_{ij}(u,s)| \right] du \ge -\Phi_i(t,s),$$

and

$$0 \le \int_0^t \Phi_i(t,s) ds \le P_i < 1,$$

for $1 \le i \le n$ and $0 \le s \le t < \infty$. Let $p_i = 1 - P_i$. Then each solution $x_i(t)$ of (1) on an interval $[t_0, T]$ having $|x_i(T)|$ as the absolute maximum of $|x_i(t)|$ on $[0,T]$ satisfies

$$|x_i(T)| \le \frac{1}{p_i} \left[W_i(t_0, \phi(\cdot)) e^{-c(T-t_0)} + \int_{t_0}^T e^{-c(T-s)} |h_i(s)| ds \right]. \tag{26}$$

Corollary 2. *Let the conditions of Corollary 1 hold. Further, suppose there are constants* $M_i > 0$ *and* $K_i > 0$ *such that*

$$\alpha_i(t) < M_i\,,$$

and

$$\int_0^t e^{-c(t-s)}|h_i(s)|\,ds \le K_i\,,$$

for $1 \le i \le n$ *and* $t \ge 0$. *Then all solutions of (1) are uniform, ultimate bounded.*

4 Conclusion

The main contribution of this paper is Theorem 7, in which the ith component of system (1) is presented in such a way that enables the utilization of the form of a well-known Lyapunov functional that can guarantee stability. The ith component, given in (21), is shown to be also useful in obtaining the boundedness of the solutions of system (1).

Other noteworthy results include Theorem 2 and Theorem 3, which give new stability criteria for the scalar case (3), and Theorem 4 and Theorem 5, which give new boundedness results for the perturbed scalar case (15).

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