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Existence of Bounded Solutions to Linear Differential Equations (I)

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Abstract

We deal with linear differential equations of the form $dx/dt = Ax(t) + f(t)$ in a Banach space $\mathbf{X}$, where A is the generator of a C_0-semigroup on $\mathbf{X}$ and f is a periodic function. In this paper, we give a method to show the existence of bounded solutions and a structure of them. As results, we can obtain criteria for the existence of quasi-periodic, periodic, asymptotically periodic solutions.

1 Introduction

Let $\mathbf{X}$ be a Banach space and $\mathbb{R}$ the real line. In this paper we investigate criteria on the existence of bounded solutions to the linear differential equation of the form

$$\frac{d}{dt}u(t) = Au(t) + f(t). \tag{1}$$

Throughout the present paper we make the following assumption.

Assumption : $A : \mathcal{D}(A) \subset \mathbf{X} \to \mathbf{X}$ is the generator of a C_0-semigroup $U(t)$, and $f : \mathbb{R} \to \mathbf{X}$ is a τ-periodic function.

If $x(t)$ is a continuous function which satisfies the following equation

$$x(t) = U(t)x(0) + \int_0^t U(t-s)f(s)ds, \quad t \in \mathbb{R}_+ := [0, \infty), \tag{2}$$

then it is called a (mild) solution to Equation (1).

The purpose of this paper is to give criteria for the existence of bounded solutions and a structure of bounded solutions to Equation (1). The relationship between the existence of bounded solutions and the existence of τ-periodic solutions is characterized by the Massera type theorem. To complete the Massera type theorem, it is practically and theoretically important to show the existence of bounded solutions.

2 The existence of bounded solutions

In this section, we give criteria on the existence of bounded solutions to Equation (1). For a solution $x(t)$ of Equation (1) such that $x(0) = x_0$, $x(n\tau)$ is expressed as

$$x(n\tau) = U(n\tau)x_0 + S_n(U(\tau))b_f,$$

where

$$S_n(U(\tau)) = \sum_{k=0}^{n-1} U(k\tau), \quad b_f = \int_0^\tau U(\tau - s)f(s)ds.$$

The solution $x(t)$ of Equation (1) is bounded on $\mathbb{R}_+$ if and only if $x(n\tau), n = 1, 2, \cdots$, are bounded. If we take $x_0 = b_f$(resp. $x_0 = 0$), then $x(n\tau) = S_{n+1}(U(\tau))b_f$ (resp. $x(n\tau) = S_n(U(\tau))b_f$). Hence a solution $x(t)$ of Equation (1) such that $x(0) = b_f$ (or $x(0) = 0$) is bounded on $\mathbb{R}_+$ if and only if

$$\limsup_{n\to\infty} \|S_n(U(\tau))b_f\| < \infty. \tag{3}$$

Based on this relation, we give criteria on the existence of bounded solutions to Equation (1). The following result can be found in [2].

Theorem 2.1 *Let* $\mathbf{Z}$ *be a subset of* $\mathbf{X}$. *Assume that for any* $x \in \mathbf{Z}$ *there exists a positive number* $\alpha_x > 0$ *such that* $\|U_{\mathbf{Z}}(n\tau)x\| \le \alpha_x$ *for all* $n \in \mathbb{N}$. *Then the following three statements are equivalent :*

1) *Every solution* $x(t)$ *of Equation* (1) *such that* $x(0) \in \mathbf{Z}$ *is bounded on* $\mathbb{R}_+$.
2) *Condition* (3) *holds.*
3)

$$\limsup_{t\to\infty} \| \int_0^t U(t-s)f(s)ds\| < \infty.$$

2.1 The case of finite dimension

We will check Condition (3) for the case where $\mathbf{X} = \mathbb{C}^m, A = (a_{ij})$, an $m \times m$ matrix. Let the characteristic polinomial of A be factorized as follows :

$$\Phi(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_\ell)^{m_\ell},$$

where $\lambda_1, \cdots, \lambda_\ell$ are the distinct roots of $\Phi(\lambda)$, and $m_1 + \cdots + m_\ell = m$. Put $\lambda_p =: a_p + ib_p, a_p, b_p \in \mathbb{R}$. Denote by $P_p : \mathbb{C}^m \to \mathbf{M}_p$ the projection corresponding to the direct sum decomposition $\mathbb{C}^m = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_\ell$, where $\mathbf{M}_p := \mathcal{N}((A - \lambda_p I)^{n_p})$ is the generalized eigenspace corresponding to λ_p.

Theorem 2.2 *For* $\tau > 0$, $b \in \mathbb{C}^m$, *the vector sequence* $\{S_n\}$, *given as*

$$S_n := S_n(e^{\tau A})b = \sum_{k=0}^{n-1} e^{k\tau A}b,$$

is bounded if and only if for every $p = 1, \cdots, \ell$, the following conditions hold :

(i) *If $a_p > 0$, then $P_p b = 0$.*

(ii) *The case where $a_p = 0$;*

(a) *if $\tau b_p \in 2\pi\mathbb{Z}$, then $P_p b = 0$.*

(b) *if $\tau b_p \notin 2\pi\mathbb{Z}$, then $P_p b \in \mathcal{N}(A - \lambda_p I)$.*

(iii) *If $a_p < 0$, then $P_p b$ is arbitrary.*

To prove the theorem, the following lemma is needed.

Lemma 2.3 *Let $Q(t)$ be a vector in $\mathbb{C}^n$, whose component is a polinomial of t, and $\lambda = a + ib \in \mathbb{C}, a, b \in \mathbb{R}$. The vector sequence $\{R_n\}$, given as*

$$R_n = \sum_{j=1}^{n} e^{j\lambda} Q(j),$$

is bounded if an only if the following conditions hold :

(i) *In the case where $a > 0$, $Q(t) \equiv 0$.*

(ii) *In the case where $a = 0$, if $b \in 2\pi\mathbb{Z}$, then $Q(t) \equiv 0$; if $b \notin 2\pi\mathbb{Z}$, then $Q(t) = c$ (a constant vector).*

(iii) *In the case where $a < 0$, $Q(t)$ is arbitrary.*

Proof Set $z = e^{\lambda}$. Then $R_n = \sum_{j=1}^{n} z^j Q(j)$. If $\{R_n\}$ is bounded, the sequence $\{R_n - R_{n-1}\}_{n=2}^{\infty}$ is also bounded and

$$\|R_n - R_{n-1}\| = \|z^n Q(n)\| = e^{na}\|Q(n)\|.$$

Hence in the case where $a > 0$, $Q(t) \equiv 0$ if and only if $\{R_n\}$ is bounded. If $a < 0$, then $Q(t)$ is arbitrary if and only if $\{R_n\}$ is bounded. So, we see the case where $a = 0$. We note that

$$\|R_n - R_{n-1}\| = \|Q(n)\|.$$

From the definition of $Q(t)$ it follows that $\{Q(n)\}$ is bounded if and only if $Q(t) = c$ (a constant vector). If $b \in 2\pi\mathbb{Z}$, then $z = 1$, and so, $R_n = nc$. Namely, $c = 0$ if and only if $\{R_n\}$ is bounded. If $b \notin 2\pi\mathbb{Z}$, then $z \neq 1$. Hence we have

$$\|R_n\| = \|\frac{1 - z^{n+1}}{1 - z} c\| \leq \frac{2}{1 - z}\|c\|,$$

which implies that $\{R_n\}$ is bounded. Therefore the proof of the lemma is finished. □

The proof of Theorem 2.2 $\mathbb{C}^m$ is decomposed as

$$\mathbb{C}^m = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_\ell.$$

Take a circle C_p centered at λ_p, whose radius is sufficiently smale and its disk does not contain the other points $\lambda_q, q \neq p$. Then the projection P_p is expressed as

$$P_p = \frac{1}{2\pi} \int_{C_p} (\lambda I - A)^{-1} d\lambda.$$

Then P_p is a bounded operator having the following properties :

$$P_p \mathbb{C}^m = \mathbf{M}_p, \quad AP_p = P_p A, \quad P_p P_q = 0 \ (p \neq q), \quad P_p^2 = P_p, \quad P_1 + P_2 + \cdots + P_\ell = I.$$

Furthermore, e^{tA} is decomposed as follows :

$$e^{tA} = \sum_{p=1}^{\ell} e^{\lambda_p t} Q_p(t) P_p, \quad Q_p(t) = \sum_{k=0}^{n_p - 1} \frac{t^k}{k!} (A - \lambda_p I)^k.$$

Using those facts, we have

$$S_n(e^{\tau A}) = \sum_{j=0}^{n-1} e^{j\tau A} = \sum_{j=0}^{n-1} \sum_{p=1}^{\ell} e^{j\tau\lambda_p} Q_p(j\tau) P_p = \sum_{p=1}^{\ell} \sum_{j=0}^{n-1} e^{j\tau\lambda_p} Q_p(j\tau) P_p.$$

Since $P_p P_q = 0 (p \neq q)$, $P_p^2 = P_p$, it follows that

$$P_p S_n(e^{\tau A}) b = \sum_{j=0}^{n-1} e^{j\tau\lambda_p} Q_p(j\tau) P_p b := R_n^p.$$

Hence, the sequence $\{S_n(e^{\tau A})b\}_{n \in \mathrm{N}}$ is bounded if and only if for every $p = 1, \cdots, \ell$, the sequence $\{R_n^p\}_{n \in \mathrm{N}}$ is bounded. Since

$$Q_p(j\tau) P_p b = \sum_{k=0}^{n_p - 1} j^k \frac{\tau^k}{k!} (A - \lambda_p I)^k P_p b, \quad p \in \{1, 2, \cdots, \ell\},$$

we have, by Lemma 2.3, the following facts.

i) If $a_p > 0$, then $Q_p(j\tau) P_p b \equiv 0$, from which we have $P_p b = 0$.

ii) If $a_p < 0$, then $Q_p(j\tau) P_p b$ is arbitrary ; that is, $P_p b$ is also arbitrary.

iii) Let $a_p = 0$. If $\tau b_p \in 2\pi\mathbb{Z}$, then $Q_p(j\tau) P_p b \equiv 0$; that is, $P_p b = 0$. If $\tau b_p \notin 2\pi\mathbb{Z}$, then $Q_p(j\tau) P_p b = c$(constant vector). Notice that $(A - \lambda_p I) P_p b = 0$ if and only if $Q_p(j\tau) P_p b = P_p b$. This means that $c = P_p b$; namely, $P_p b \in (A - \lambda_p I)$. Hence we obtain the conclusion of the theorem. □

2.2 The case of infinite demension

We consider the condition 2) in Theorem 2.1 from the point of view of the spectrum of A in Equation (1).

Suppose that $\omega_e(U) := \lim_{t\to\infty} t^{-1}\log\alpha(U(t)) < 0$. Then $\exp(t\omega_e(U)) < 1$, $(t > 0)$. This implies that there exists a $\gamma > 0$ such that $\sigma(U(t)) \cap \{z : |z| \geq e^{-\gamma t}\}$ and $\sigma(A) \cap \{z : \Re z \geq -\gamma\}$ consist of finite number of normal eigenvalues and that $\sigma(A) \cap \{z : -\gamma < \Re z < 0\} = \emptyset$. Hence $\sigma(A) \cap \{z : \Re z > -\gamma\}$ consists of finite number of normal eigenvalues $\lambda_j, j = 1, 2, \cdots, r$, with nonnegative real parts. Set $a_j = \Re\lambda_j, b_j = \Im\lambda_j$. Assume that $a_j = 0$ for $1 \leq j \leq q$ $(\leq r)$ and that $a_j > 0$ for $q < j \leq r$. Thus 1 is a normal eigenvalue of $U(\tau)$. Set $\sigma_0(A) = \{\lambda_j : 1 \leq j \leq q\}$ and $\sigma_+(A) = \{\lambda_j : q+1 \leq j \leq r\}$. We understand that $\sigma_+(A) = \emptyset$, provided $q = r$.

Let $\mathbf{M}_j$ be the generalized eigenspace of A corresponding to λ_j. Since λ_j is a normal eigenvalue of A, $n_j := \dim \mathbf{M}_j$ is finite and there exists a positive integer m_j such that $\mathbf{M}_j = \mathcal{N}((\lambda_j - A)^{m_j})$. The space $\mathbf{X}$ is decomposed as follows:

$$\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z}, \quad \mathbf{Z} = \mathbf{M}_0 \oplus \mathbf{M}_+, \quad \mathbf{Y} = \bigcup_{j=1}^{r} \mathcal{R}((\lambda_j I - A)^{m_j}),$$

$$\mathbf{M}_0 = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_q, \quad \mathbf{M}_+ = \mathbf{M}_{q+1} \oplus \cdots \oplus \mathbf{M}_r$$

The subspaces $\mathbf{Y}$ and $\mathbf{M}_j$ are closed in $\mathbf{X}$ and $\dim \mathbf{Z} = n_1 + n_2 + \cdots + n_r =: d$. If we define P_j as in Section 2.1, where $(\lambda I - A)^{-1}$ is understood as the resolvent operator $R(\lambda, A)$, then $P_j : \mathbf{X} \to \mathbf{M}_j$ are projections such that $P_j P_k = \delta_{jk} P_j$ and $AP_j x = P_j Ax$ for $x \in D(A)$. If we set

$$P = P_1 + P_2 + \cdots + P_r, P_0 = I - P, \tag{4}$$

then $P : \mathbf{X} \to \mathbf{Z}$ and $P_0 : \mathbf{X} \to \mathbf{Y}$ are projections. $\mathbf{Y}, \mathbf{M}_j$ and $\mathbf{Z}$ are invariant subspaces of $U(t)$.

Since $U(t)x = U(t)P_0 x + U(t)Px$, we have

$$\|S_n(U(\tau))x\| \leq \|S_n(U(\tau))P_0 x\| + \|S_n(U(\tau))Px\|.$$

It follows from Proposition 4.15 in [6] that there are an $\varepsilon_0 > 0$ and a constant $K \geq 1$ such that

$$\|U(t)P_0 x\| \leq K e^{-\varepsilon_0 t}\|P_0 x\| \quad \text{for all } \; x \in \mathbf{X}, \; t \geq 0.$$

Hence we have

$$\|S_n(U(\tau))P_0 x\| \leq K \sum_{k=0}^{n-1} e^{-\varepsilon_0 \tau}\|P_0 x\| \leq \frac{K}{1 - e^{-\varepsilon_0 \tau}}\|P_0 x\| < \infty.$$

As a result, $\|S_n(U(\tau))x\|, n = 1, 2, \cdots$, are bounded if and only if $\|S_n(U(\tau))Px\|, n = 1, 2, \cdots$, are bounded.

Since $d = \dim \mathbf{Z} < \infty$, $A_{\mathbf{Z}}$, the restriction of A to $\mathbf{Z}$, is regarded as a $d \times d$ matrix with eigenvalues $\lambda_j, 1 \leq j \leq r$, and $U(t)Px = \exp(tA_{\mathbf{Z}})Px$ for all $x \in \mathbf{X}$. Thus we have the following result from Theorem 2.2.

Theorem 2.4 *Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then $S_n(U(\tau))x$, $n = 1, 2, \cdots$, are bounded if and only if the following conditions hold:*

(i) *For $q < j \leq r$, $P_j x = 0$.*

(ii) *For $1 \leq j \leq q$,*

(a) *if $\tau b_j \in 2\pi\mathbb{Z}, P_j x = 0$;*

(b) *if $\tau b_j \notin 2\pi\mathbb{Z}, P_j x \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Corollary 2.5 *Assume that $\sigma(U(t))$ and $\sigma(A)$ are as in the above. Then the solution $x(t)$ of Equation (1) such that $x(0) = b_f$ is bounded on $\mathbb{R}_+$ if and only if the following conditions hold :*

(i) *For $q < j \leq r$, $P_j b_f = 0$.*

(ii) *For $1 \leq j \leq q$,*

(a) *if $\tau b_j \in 2\pi\mathbb{Z}, P_j b_f = 0$;*

(b) *if $\tau b_j \notin 2\pi\mathbb{Z}, P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Combining Theorem 2.4 with Theorem 2.1, we obtain the following result.

Corollary 2.6 *Suppose $U(t)$ is a bounded C_0-semigroup such that $\omega_e(U) < 0$. Then every solution of Equation (1) is bounded on $\mathbb{R}_+$ if and only if for $j = 1, \cdots, q$ the following conditions hold :*

(a) *If $\tau b_j \in 2\pi\mathbb{Z}$, then $P_j b_f = 0$;*

(b) *If $\tau b_j \notin 2\pi\mathbb{Z}$, then $P_j b_f \in \mathcal{N}(A_{\mathbf{Z}} - \lambda_j I)$.*

Using Corollary 2.5, we obtain the following result on the existence of a τ-periodic solution to Equation (1).

Theorem 2.7 *Assume that $\omega_e(U) < 0$ and that b_f satisfies the conditions* (i) *and* (ii) *in Corollary* 2.5. *Then Equation* (1) *has a τ-periodic solution.*

Proof Since $\{S_n(U(\tau))b_f\}_n$ is bounded, it follows from Corollary 2.5 that the solution $x(t)$ of Equation (1) such that $x(0) = b_f$ is bounded on $\mathbb{R}_+$. Since 1 is a normal point of $U(\tau)$, we see that $\mathcal{R}(I - U(\tau))$ is a closed subspace of $\mathbf{X}$. Therefore the fixed point theorem by Chow and Hale implies that Equation (1) has a τ-periodic solution. □

3 A structure of bounded solutions

In this section we will give a structure of bounded solutions obtained in Section 2. Throughout this section, we assume the following conditions :

1) $\omega_e(U) < 0$,

2) 1 is a normal eigenvalue of $U(\tau)$,

3)

$$\limsup_{n\to\infty} \|S_n(U(\tau))b_f\| < \infty.$$

We use the same notations for the points in $\sigma(A)\cap\{z : \Re z \geq 0\}$ as in Section 2. Denote by **SP** and $\mathbf{SP_X}$ the set of all τ-periodic solutions of Equation (1) and the set of all solutions of the equation $(I - U(\tau))x = b_f$, respectively. They are affine spaces. If we take an vector $x_0 \in \mathbf{SP_X}$, then $\mathbf{SP_X} = x_0 + \mathcal{N}(I - U(\tau))$. Set

$$\mathbf{N}_0 = \mathcal{N}(A - ib_1 I) \oplus \cdots \oplus \mathcal{N}(A - ib_q I).$$

Suppose that $\tau b_j \in 2\pi\mathbb{Z}$ for $1 \leq j \leq p (\leq q)$ and that $\tau b_j \notin 2\pi\mathbb{Z}$ for $p+1 \leq j \leq q$. Since 1 is a normal eigenvalue of $U(\tau)$, it follows that, $p \geq 1$ and

$$\mathcal{N}(I - U(\tau)) = \mathcal{N}(A - ib_1 I) \oplus \cdots \oplus \mathcal{N}(A - ib_p I) =: \mathbf{N}_{00} \subset \mathbf{N}_0, \tag{5}$$

cf. Proposition 4.13 in [6]. Hence $\mathbf{SP_X} = x_0 + \mathbf{N}_{00}$. Let P and P_0 be defined by (4).

Proposition 3.1 *The following results hold.*
1) *$U(t)x$ is bounded for $t \geq 0$ if and only if $x \in \mathbf{Y} \oplus \mathbf{N}_0$; that is,*
 (i) *$P_j x \in \mathcal{N}(A - \lambda_j I)$ for $1 \leq j \leq q$.*
 (ii) *$P_j x = 0$ for $q+1 \leq j \leq r$.*
2) *$U(t)x$ is τ-periodic if and only if $x \in \mathbf{N}_{00}$; that is,*
 (iii) *$P_0 x = 0$*
 (iv) *$P_j x \in \mathcal{N}(A - ib_j I)$ for $1 \leq j \leq p$.*
 (v) *$P_j x = 0$ for $p+1 \leq j \leq r$.*

Proof $U(t)x$ is bounded if and only if $P_0U(t)x = U(t)P_0x$ and $PU(t)x = U(t)Px$ are bounded. Since $P_0U(t)x$ is bounded for all $x \in \mathbf{X}$, it suffices to check the boundedness of $U(t)Px$. Since $P = P_1 + \cdots + P_r$, $\{U(n\tau)Px\}_n$ is bounded if and only if $\{U(n\tau)P_jx\}_n, j = 1, 2, \cdots, r$, are bounded. Since

$$\|U(t)P_jx\| = \|e^{(a_j+ib_j)t}\sum_{m=0}^{m_j-1}\frac{t^m}{m!}(A-\lambda_j I)^m P_jx\| = e^{a_jt}\|\sum_{m=0}^{m_j-1}\frac{t^m}{m!}(A-\lambda_j I)^m P_jx\|$$

for $1 \leq j \leq r$, the assertion 1) is easily derived from this relation.

Similarly $U(t)x$ is τ-periodic if and only if $P_0U(t)x = U(t)P_0x$ and $PU(t)x = U(t)Px$ are τ-periodic. If $U(t)P_0x$ is τ-periodic, we have $P_0x = U(n\tau)P_0x$ for all $n = 1, 2, \cdots$. Since $U(t)P_0x \to 0$ as $t \to \infty$, we have $P_0x = 0$.

If $U(t)Px$ is τ-periodic, $U(t)P_jx$ is τ-periodic for $1 \leq j \leq r$. It follows at first that $P_jx \in \mathcal{N}(A - ib_jI)$ for $1 \leq j \leq q$ and that $P_jx = 0$ for $q+1 \leq j \leq r$. If $p+1 \leq j \leq q$, and if $P_jx \neq 0$, then $U(t)P_jx = e^{ib_jt}P_jx$ is not τ-periodic. Consequently, $x \in \mathbf{N}_{00}$. Clearly, if $x \in \mathbf{N}_{00}$, $U(t)x$ is τ-periodic. □

Theorem 3.2 *A solution $x(t)$ of Equation* (1) *is bounded on $\mathbb{R}_+$ if and only if $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$.*

Proof The solution $x(t)$ is written as Equation (2) in Introduction. Notice that the integral in this equation is bounded if and only if Condition 3) holds. Hence $x(t)$ is bounded if and only if $U(t)x(0)$ is bounded. From Proposition 3.1 we have the result in the theorem. □

The following result follows from Condition 3) and Theorem 3.2.

Corollary 3.3 *The following assertions hold true.*

1) $\mathbf{SP_X} \neq \emptyset$, $\mathbf{SP_X} \subset \mathbf{Y} \oplus \mathbf{N}_0$.

2) $\mathbf{M}(b_f) \subset \mathbf{Y} \oplus \mathbf{N}_0$, *where* $\mathbf{M}(b_f)$ *is the linear space generated by* $\{U(n\tau)b_f\}_{n=0}^{\infty}$.

Theorem 3.4 *Take a τ-periodic solution $u_0(t)$ of Equation (1). Then the following statements are valid.*

1) *Any bounded solution $x(t)$ of Equation (1) on $\mathbb{R}_+$ is written as*

$$x(t) = u_0(t) + \sum_{j=1}^{p} e^{ib_jt}x_j + \sum_{j=p+1}^{q} e^{ib_jt}x_j + U_{\mathbf{Y}}(t)y_0,$$

with some vectors $x_j \in \mathcal{N}(A - ib_jI), 1 \leq j \leq q$, and $y_0 \in \mathbf{Y}$.

2) *Any τ-periodic solution $u(t)$ of Equation (1) is written as*

$$u(t) = u_0(t) + \sum_{j=1}^{p} e^{ib_jt}u_j,$$

with some vectors $u_j \in \mathcal{N}(A - ib_jI), 1 \leq j \leq p$.

Proof Since $u_0(t)$ is the τ-periodic solution of Equation (1), $u_0(0) \in \mathbf{SP_X} \subset \mathbf{Y} \oplus \mathbf{N}_0$. Let $x(t)$ be a bounded solution of Equation (1) on $\mathbb{R}_+$. Then it follows from Theorem 3.2 that $x(0) \in \mathbf{Y} \oplus \mathbf{N}_0$. Therefore $x(0) - u_0(0) \in \mathbf{Y} \oplus \mathbf{N}_0$; it is expressed as

$$x(0) - u_0(0) = \sum_{j=1}^{q} x_j + y_0,$$

where $x_j \in \mathcal{N}(ib_jI - A)$ and $y_0 \in \mathbf{Y}$. Since $x(t) - u_0(t)$ is a solution of the homogeneous equation, we have

$$\begin{aligned} x(t) - u_0(t) &= U(t)[x(0) - u_0(0)] \\ &= U(t)\sum_{j=1}^{p} x_j + U(t)\sum_{j=p+1}^{q} x_j + U_{\mathbf{Y}}(t)y_0 \\ &= \sum_{j=1}^{p} e^{ib_jt}x_j + \sum_{j=p+1}^{q} e^{ib_jt}x_j + U_{\mathbf{Y}}(t)y_0, \end{aligned}$$

as required. The remainder is obvious. Therefore the proof of the theorem is completed. □

Corollary 3.5 *The following statements are valid.*

1) *There is a τ-periodic solution to Equation* (1) ; $\dim \mathbf{SP} = p$.

2) *There is an asymptotically τ-periodic solution to Equation* (1).

3) *If $p < q$, there is an asymptotically quasi-periodic solution to Equation* (1).

4) *If $p = q$, every bounded solution is an asymptotically τ-periodic solution to Equation* (1).

5) *If $p = q = r$, and if $R(\lambda, A) := (\lambda I - A)^{-1}$ has a pole of order 1 at $\lambda = \lambda_j, 1 \leq j \leq p$, then all τ-periodic solutions of Equation* (1) *are stable.*

Proof Statements 1,2,3,4) are trivial. Assume that the conditions in 5) hold. Then we have $\mathbf{X} = \mathbf{Y} \oplus \mathbf{N}_0$. Hence there is a positive constant H such that $\|U(t)x\| \leq H\|x\|$ for $t \geq 0, x \in \mathbf{X}$. Let $u_0(t)$ be a τ-periodic solution of Equation (1). Then for every solution $u(t)$, $u(t) - u_0(t)$ is a solution of the homogeneous equation. Hence $u(t) - u_0(t) = U(t)(u(0) - u_0(0))$, which implies $\|u(t) - u_0(t)\| \leq H\|u(0) - u_0(0)\|$ for $t \geq 0$. Therefore the assertion 5) is valid. □

In a subsequent paper, we will consider the case where

$$\limsup_{n \to \infty} \|S_n(U(\tau))b_f\| = \infty.$$

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