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Aharonov–Bohm Effect in Scattering by a Chain of Point–like Magnetic Fields

Hiroshi T. Ito (伊藤 宏) and Hideo Tamura (田村 英男)

Department of Computer Science, Ehime University
Matsuyama 790–8577, Japan
and
Department of Mathematics, Okayama University
Okayama 700–8530, Japan

Abstract 量子力学に従う粒子が直接磁場に触れていなくても，ベクトル・ポテンシャルからの影響をうけ，離れた場所の磁場を感じる現象は Aharonov–Bohm 効果として知られている ([3]). 複数個の 2 次元 δ 型磁場 (magnetic vortex [8]) による散乱を考え，磁場の中心を大きく離したときの散乱振幅の漸近公式を導く．各磁場による散乱がポテンシャル（長距離型摂動となる）を通していかなる相互作用を及ぼしあうかを解析するのが目的である．結果には入射方向・散乱方向や各磁場のフラックスのみならず磁場のお互いの位置関係が反映する．特に，入射方向または散乱方向に沿って複数の δ 型磁場が並んでいる場合にはそれ以外の場合とは異なった興味深い結果が得られた．証明については [6] を参照してください．

Introduction and Results

We study the magnetic scattering by several point–like fields at large separation in two dimensions. The aim is to derive the asymptotic formula for scattering amplitudes as the distances between centers of fields go to infinity. Even if a magnetic field is compactly supported, the corresponding vector potential does not necessarily fall off rapidly, and in general, it has the long–range property at infinity. We discuss from a mathematical point of view how the scattering by separate fields interacts with one another through long-range magnetic potentials. A special emphasis is placed on the case of scattering by fields with centers on an even line. The obtained result depends on fluxes of fields and on ratios of distances between adjacent centers. It is known as the Aharonov–Bohm effect ([3]) that magnetic potential has a direct significance to the motion of quantum particles. An extensive list of physical literatures on the Aharonov–Bohm scattering can be found in the book [2]. We refer to the recent article [8] for the Aharonov–Bohm effect in many point–like fields, where the term *magnetic vortex* is used for point–like magnetic field.

Throughout the whole exposition, we work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$. We write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2}(-i\partial_j - a_j)^2, \qquad \partial_j = \partial/\partial x_j,$$

for the magnetic Schrödinger operator with vector potential $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b(x)$ is defined as

$$b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1$$

and the quantity $\alpha = (2\pi)^{-1} \int b(x)\,dx$ is called the total flux of field b, where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

We first consider the case of a single point–like field. The Hamiltonian with such a field is regarded as one of solvable models and the explicit representation for scattering amplitude has been already obtained by [1, 2, 9]. Let $2\pi\alpha\delta(x)$ be the magnetic field with flux α and center at the origin. Then the magnetic potential $A_\alpha(x)$ associated with the field is given by

$$A_\alpha(x) = \alpha\,(-\partial_2 \log|x|, \partial_1 \log|x|) = \alpha\left(-x_2/|x|^2, x_1/|x|^2\right). \tag{1}$$

In fact, we can easily see that

$$\nabla \times A_\alpha = \alpha\,\Delta \log|x| = 2\pi\alpha\delta(x).$$

We should note that $A_\alpha(x)$ does not fall off rapidly at infinity and it has the long–range property. We write $H_\alpha = H(A_\alpha)$. The potential $A_\alpha(x)$ has a strong singularity at the origin, so that H_α is not necessarily essentially self–adjoint in $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$ ([1, 4]). We have to impose some boundary conditions at the origin to define H_α as a self–adjoint extension in $L^2 = L^2(\boldsymbol{R}^2)$. We denote by the same notation H_α the operator with domain

$$\mathcal{D}(H_\alpha) = \{u \in L^2 : H(A_\alpha)u \in L^2,\ \lim_{|x|\to 0} |u(x)| < \infty\},$$

where $H(A_\alpha)u$ is understood in $\mathcal{D}'$ (distribution sense). Then H_α is known to be self–adjoint in L^2 and this operator is called the Aharonov–Bohm Hamiltonian. If, in particular, $\alpha \notin \boldsymbol{Z}$ is not an integer, $u \in \mathcal{D}(H_\alpha)$ is convergent to zero as $|x| \to 0$. As stated above, the amplitude $f_\alpha(\omega \to \tilde{\omega}; E)$ for the scattering from initial direction $\omega \in S^1$ to final one $\tilde{\omega}$ at energy $E > 0$ has been already calculated. If we identify the coordinates over the unit circle S^1 with the azimuth angles from the positive x_1 axis, then $f_\alpha(\omega \to \tilde{\omega}; E)$ is explicitly represented as

$$f_\alpha = c(E)\Big((\cos\alpha\pi - 1)\delta(\tilde{\omega} - \omega) - (i/\pi)\sin\alpha\pi\, e^{i[\alpha](\tilde{\omega}-\omega)} F_0(\tilde{\omega} - \omega)\Big) \tag{2}$$

with $c(E) = (2\pi/i\sqrt{E})^{1/2}$, where the Gauss notation $[\alpha]$ denotes the maximal integer not exceeding α, and $F_0(\theta)$ is defined by $F_0 = \text{v.p.}\, e^{i\theta}/(e^{i\theta} - 1)$.

We move to the scattering by point–like field supported on N points $d_j \in \boldsymbol{R}^2$, $1 \le j \le N$. We make large the distance $|d_{jk}| = |d_k - d_j|$ between centers d_j and d_k under the assumption that

$$\text{the direction } \widehat{d}_{jk} = d_{jk}/|d_{jk}| \text{ remains fixed} \tag{3}$$

for all pairs (j,k) with $j \ne k$, $1 \le j,k \le N$. We further assume that

$$\max |d_{jk}| \le c \min |d_{jk}| \tag{4}$$

for some $c > 1$. These two assumptions on the location of centers are always assumed to be fulfilled. By translation, we may assume d_1 to remain fixed, so that all the centers are in a disk $\{|x| < c\,d\}$ with another $c > 1$, where $d = \min |d_{jk}|$. We write

$$H_d = H(A_d), \qquad A_d(x) = \sum_{j=1}^{N} A_j(x) = \sum_{j=1}^{N} A_{\alpha_j}(x - d_j), \tag{5}$$

for the Schrödinger operator with field $\sum_{j=1}^{N} 2\pi\alpha_j \delta(x - d_j)$, where $A_\alpha(x)$ is defined by (1). According to the results in [5], H_d becomes a self–adjoint operator with domain

$$\mathcal{D}(H_d) = \{u \in L^2 : H(A_d)u \in L^2, \ \lim_{|x-d_j|\to 0} |u(x)| < \infty, \ 1 \le j \le N\}.$$

We also know from [5] that the wave operators

$$W_\pm(H_d, H_0) = s - \lim_{t\to\pm\infty} \exp(itH_d)\exp(-itH_0)$$

exist and are asymptotically complete, where $H_0 = -\Delta$ is the free Hamiltonian. We denote by $f_d(\omega \to \tilde{\omega}; E)$ the scattering amplitude of pair (H_d, H_0). The aim is to analyze the asymptotic behavior as $d \to \infty$ of $f_d(\omega \to \tilde{\omega}; E)$.

We fix the notation to state the obtained results. We denote by $\gamma(\hat{x}; \omega)$, $\hat{x} = x/|x|$, the azimuth angle from direction $\omega \in S^1$. Let $A_j(x)$, $1 \le j \le N$, be as in (5) and set

$$H_j = H(A_j), \qquad 1 \le j \le N. \tag{6}$$

The operator H_j admits a self–adjoint realization under the boundary condition $\lim_{|x-d_j|\to 0} |u(x)| < \infty$ at center $x = d_j$ and the scattering amplitude of pair (H_j, H_0) is given by

$$f_j(\omega \to \tilde{\omega}; E) = \exp(-i\sqrt{E}d_j \cdot (\tilde{\omega} - \omega)) f_{\alpha_j}(\omega \to \tilde{\omega}; E),$$

where f_α is defined by (2). The first main theorem is formulated as follows.

Theorem 1 *Let the notation be as above. Assume* (3) *and* (4). *If* $\omega \neq \hat{d}_{jk}$ *and* $\tilde{\omega} \neq \hat{d}_{jk}$ *for all pairs* (j,k) *with* $j \neq k$, $1 \leq j,\ k \leq N$, *and* $\omega \neq \tilde{\omega}$, *then* $f_d(\omega \to \tilde{\omega}; E)$ *obeys*

$$f_d(\omega \to \tilde{\omega}; E) = \sum_{j=1}^{N} \exp\left(i(\tau_j - \tilde{\tau}_j)\right) f_j(\omega \to \tilde{\omega}; E) + o(1), \quad d \to \infty,$$

where $\tau_j = \sum_{k=1,\ k\neq j}^{N} \alpha_k \gamma(\hat{d}_{kj}; \omega)$ *and* $\tilde{\tau}_j = \sum_{k=1,\ k\neq j}^{N} \alpha_k \gamma(\hat{d}_{kj}; -\tilde{\omega})$.

We can find the Aharonov–Bohm effect in the theorem above. As is seen from the asymptotic formula, the scattering by field $2\pi\alpha_j\delta(x-d_j)$ is influenced by other fields through the coefficient $\exp(i\tau_j)$, although the centers of fields are far away from one another. This means that vector potentials have a direct significance to quantum particles moving in magnetic fields. The magnetic effect is more strongly reflected in the case when $\omega = \hat{d}_{jk}$ or $\tilde{\omega} = \hat{d}_{jk}$. We add the new notation. We interpret $\exp(i\alpha\gamma(\omega;\omega))$ as

$$\exp(i\alpha\gamma(\omega;\omega)) := (1 + \exp(i2\alpha\pi))/2 = \cos\alpha\pi \exp(i\alpha\pi).$$

Then the same asymptotic formula as in Theorem 1 can be shown to remain true even for $\omega = \hat{d}_{kj}$ or $\tilde{\omega} = \hat{d}_{kj}$ under the assumption that

$$\text{there is no other center on } l_{jk} \text{ for all pairs } (j,k), \tag{7}$$

where l_{jk} is the line joining the two centers d_j and d_k. We do not intend to prove this result here. We have studied the case $N = 2$ in [5]. If $N = 2$, (7) is automatically satisfied. For example, we have obtained that the backward scattering amplitude obey

$$f_d(\omega \to -\omega; E) = f_1(\omega \to -\omega; E) + (\cos\alpha_1\pi)^2 f_2(\omega \to -\omega; E) + o(1) \tag{8}$$

for $\omega = \hat{d}_{12}$.

Our emphasis is placed on the case without (7). As a typical case, we study the scattering by point–like fields with centers on an even line. For brevity, we confine ourselves to the simple case $N = 3$. The argument extends to the general case $N \geq 4$. What is interesting is that the asymptotic formula depends not only on fluxes of fields but also on ratios of distances between adjacent centers. We assume that three centers are along the direction $\omega_1 = (1,0)$ in the order of d_1, d_2 and d_3. We further assume that

$$\text{the ratio } |d_{23}|/|d_{12}| = \delta_0 \text{ remains fixed} \tag{9}$$

for some $\delta_0 > 0$. This assumption can be weakened as $\lim_{d\to\infty} |d_{23}|/|d_{12}| = \delta_0$. We define $\theta_{\pm}\pi$ as

$$\theta_{\pm}\pi = \text{ angle between two vectors } (0,\pm 1) \text{ and } (1, -\delta_0^{1/2}). \tag{10}$$

It is obvious that $\theta_+ + \theta_- = 1$ and $0 < \theta_- < \theta_+ < 1$. If, for example, three centers are at even intervals, then $\delta_0 = 1$, so that $\theta_\pm$ are determined as $\theta_+ = 3/4$ and $\theta_- = 1/4$. We are now in a position to state the second main theorem.

Theorem 2 *Let the notation be as above. Assume that* (9) *is satisfied. If $\tilde{\omega} \neq \pm\omega_1$ for the incident direction $\omega_1 = (1,0)$, then $f_d = f_d(\omega_1 \to \tilde{\omega}; E)$ behaves like*

$$\begin{aligned} f_d &= e^{i\alpha_2(\pi-\gamma(-\omega_1;-\tilde{\omega}))} e^{i\alpha_3(\pi-\gamma(-\omega_1;-\tilde{\omega}))} f_1 \\ &+ (\cos\alpha_1\pi)\, e^{i\alpha_1(\pi-\gamma(\omega_1;-\tilde{\omega}))} e^{i\alpha_3(\pi-\gamma(-\omega_1;-\tilde{\omega}))} f_2 \\ &+ (\theta_+\cos(\alpha_1+\alpha_2)\pi + \theta_-\cos(\alpha_1-\alpha_2)\pi)\, e^{i\alpha_1(\pi-\gamma(\omega_1;-\tilde{\omega}))} e^{i\alpha_2(\pi-\gamma(\omega_1;-\tilde{\omega}))} f_3 + o(1) \end{aligned}$$

as $d \to \infty$, where $f_j = f_j(\omega_1 \to \tilde{\omega}; E)$ for $1 \le j \le 3$. Moreover the backward scattering amplitude $f_d(\omega_1 \to -\omega_1; E)$ obeys

$$f_d = f_1 + (\cos\alpha_1\pi)^2 f_2 + (\theta_+\cos(\alpha_1+\alpha_2)\pi + \theta_-\cos(\alpha_1-\alpha_2)\pi)^2 f_3 + o(1)$$

with $f_j = f_j(\omega_1 \to -\omega_1; E)$.

We make several comments on the two theorems above.

Remark 1 The quantity $|f_d(\omega_1 \to \tilde{\omega}; E)|^2$ is called the differential cross section. We figure the approximate values of cross sections obtained from the asymptotic formula on the right side in the appendix and we see how the pattern of interferences changes with three flux parameters α_1, α_2 and α_3.

Remark 2 The idea in the proof of Theorem 2, in principle, enables us to prove Theorem 1 without assuming that $\omega \neq \hat{d}_{kj}$ and $\tilde{\omega} \neq \hat{d}_{kj}$. For example, it is possible to extend Theorem 2 to the case of scattering by several chains of point–like fields. However the formula takes a rather complicated form and we do not have yet obtained a unified form of representation.

Remark 3 If we make a change of variables $x \to dy$, then Theorems 1 and 2 can be easily seen to yield the asymptotic behavior at high energy of scattering amplitudes when the distances between centers of fields remain fixed.

We write $R(z; H) = (H - z)^{-1} : L^2 \to L^2$, $\operatorname{Im} z \neq 0$, for the resolvent of self–adjoint operator H. We know ([5, Propositions 7.2 and 7.3]) that H_d has no bound states and that the boundary values to the positive axis

$$R(E \pm i0; H_d) = \lim_{\varepsilon \downarrow 0} R(E \pm i\varepsilon; H_d) : L^2_s(\boldsymbol{R}^2) \to L^2_{-s}(\boldsymbol{R}^2) \tag{11}$$

exist as a bounded operator from the weighted L^2 space $L^2_s(\boldsymbol{R}^2) = L^2(\boldsymbol{R}^2; \langle x\rangle^{2s}\, dx)$ into $L^2_{-s}(\boldsymbol{R}^2)$ for $s > 1/2$, where $\langle x\rangle = (1 + |x|^2)^{1/2}$. We take $0 < \sigma \ll 1$ small enough and denote by $s_j(x)$ the characteristic function of set

$$S_j = \{x \in \boldsymbol{R}^2 : |x - d_j| < Cd^\sigma\}, \quad 1 \le j \le N, \tag{12}$$

with $C > 1$. The proof of the main theorems is based on the resolvent estimate

$$\|s_j R(E \pm i0; H_d) s_k\| = O(d^{-1/2+c\sigma}) \tag{13}$$

for $j \neq k$, where $\| \ \|$ denotes the norm of bounded operators acting on L^2.

The work [7] has studied the same problem in the case of potential scattering for the operator $-\Delta + \sum_{j=1}^{N} V_j(x - d_j)$ with potentials falling off rapidly at infinity. The obtained result is that the scattering amplitude

$$f_d(\omega \to \tilde{\omega}; E) = \sum_{j=1}^{N} f_j(\omega \to \tilde{\omega}; E) + o(1)$$

is completely split into the sum of amplitudes $f_j(\omega \to \tilde{\omega}; E)$ corresponding to potentials $V_j(\cdot - d_j)$, and we do not have to modify the phase factors. A new difficulty arises in the case of magnetic scattering. Roughly speaking, this is due to the long–range property of magnetic potentials and several new devices are required to overcome such a difficulty. We work in the phase space and the microlocal analysis plays an important role in proving the theorems. We conclude the section by stating that in the scattering by point–like magnetic fields, the fields interact with one another through long–range magnetic potentials by the Aharonov–Bohm effect, although the trapping effect between fields is weak, as is seen from resolvent estimate (13).

References

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Appendix : Figures of differential cross sections

The differential cross section is a quantity observable through actual experiments and it is one of the most important quantities in the scattering theory. We here figure the approximate values for $|f_d(\omega_1 \to \tilde{\omega}; E)|^2$, $\omega_1 = (1,0)$, to see how the intensity of scattering changes with three flux parameters α_1, α_2 and α_3 and with positions of centers d_1, d_2, d_3 as stated in Remark 1. We first consider the case that the three centers $d_1 = (0,0), d_2 = (100,0)$ and $d_3 = (200,0)$ are at even intervals along direction $\omega_1 = (1,0)$, i.e., $|d_{12}| = |d_{23}| = d = 100$. The figures are drawn for $|f_d(\omega_1 \to \tilde{\omega}; E)|^2$ with $E = 1$ and $4\pi/9 < \gamma(\tilde{\omega}; \omega_1) < 5\pi/9$. For example, the scattering angle $\gamma(\tilde{\omega}; \omega_1) = \pi/2$ corresponds to the value $\pi/2 = 1.57....$ on the horizontal axis in the figures below.

Figure 1 $(\alpha_1 = 1/4,\ \alpha_2 = 1/4,\ \alpha_3 = 1/2)$: three coefficients do not vanish.
Figure 2 $(\alpha_1 = 1/2,\ \alpha_2 = 1/2,\ \alpha_3 = 1/2)$: coefficient of $f_2(\omega_1 \to \tilde{\omega}; E)$ vanishes.
Figure 3 $(\alpha_1 = 1/4,\ \alpha_2 = (\arctan 2)/\pi,\ \alpha_3 = 1/2)$: coefficient of $f_3(\omega_1 \to \tilde{\omega}; E)$ vanishes.

Next we consider the case that d_3 moves only a little from $(200,0)$ to $(200,2)$ so that three centers are not on the same line. Fluxes are the same as in the case of Figure 3. Figure 4 represents this case. Though the movement of d_3 is small, there is remarkable differences between them.

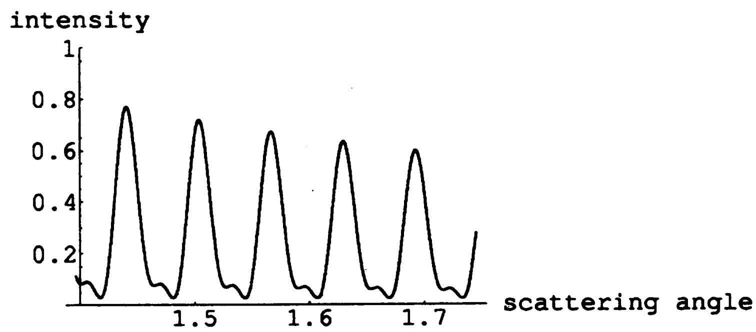

Figure 1 $\alpha_1 = 1/4,\ \alpha_2 = 1/4,\ \alpha_3 = 1/2;\ d_1 = (0,0),\ d_2 = (100,0),\ d_3 = (200,0)$

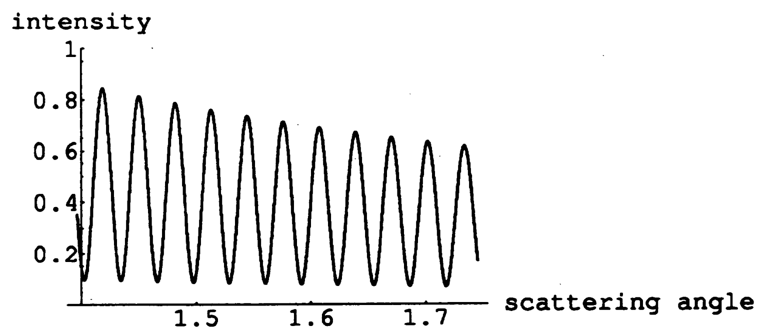

Figure 2 $\alpha_1 = 1/2,\ \alpha_2 = 1/2,\ \alpha_3 = 1/2;\ d_1 = (0,0),\ d_2 = (100,0),\ d_3 = (200,0)$

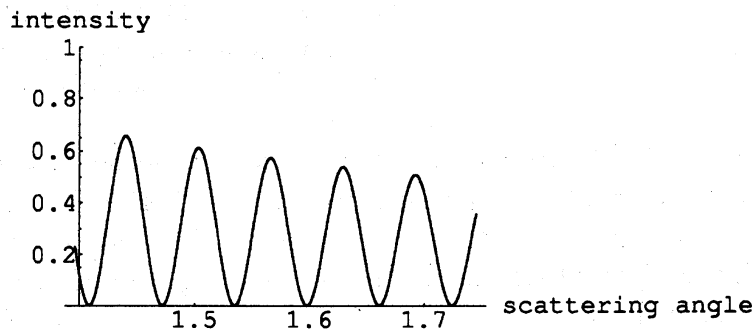

Figure 3
$\alpha_1 = 1/4,\ \alpha_2 = \pi^{-1} \arctan 2,\ \alpha_3 = 1/2; d_1 = (0,0),\ d_2 = (100,0),\ d_3 = (200,0)$

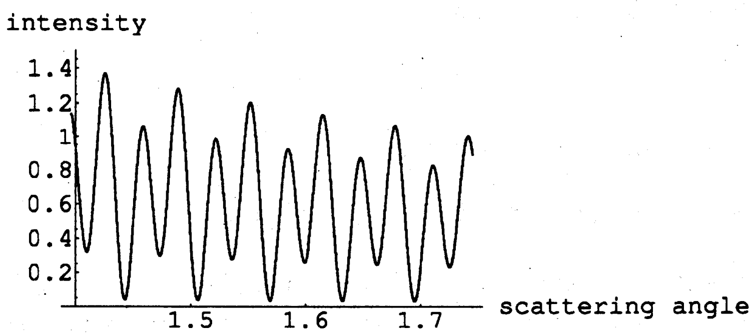

Figure 4
$\alpha_1 = 1/4,\ \alpha_2 = \pi^{-1} \arctan 2,\ \alpha_3 = 1/2; d_1 = (0,0),\ d_2 = (100,0),\ d_3 = (200,2)$