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Absence of eigenvalues of Dirac type operators II – A gauge invariant condition –

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Abstract This is a continuation of the preceded result [4], which proposed a condition for the absence of eigenvalues of Dirac type operators in an exterior domain. Unfortunately, the condition given there is not gauge invariant. In this note we take an effect of magnetic vector potentials into consideration to give a gauge invariant condition for the absence of eigenvalues.

1 Introduction

If $U\subset \mathbf{R}^3$ is either an exterior domain or the whole space, the eigenvalue problem for the Dirac operator can be formulated as follows.

(D) $$\alpha\cdot pu+m\beta u+Vu+\lambda u=0,\ u\in L^2(U)^4,\ \lambda\in\mathbf{R},\ p=-i\nabla,$$

where $\{\alpha_j\}_{j=0}^3$ is a family of 4×4 matrices satisfying

$$\alpha_j^*=\alpha_j\ ,\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk},\ \forall j,\ k=0,\dots,3,\ \beta=\alpha_0,$$

$m(x)$ is a real-valued function and $V(x)$ is a matrix close to a scalar one at infinity.

In [2], the authors has shown, roughly speaking, that (D) admits no nontrivial solutions in $L^2(U)^4$ provided that there exists a positive spherically symmetric function q that may diverge at infinity but does not oscillate rapidly such that

$$\tilde{V}=V(x)+\lambda\sim q(|x|),\ m(x)=o(q),\quad \text{as } |x|\to\infty.$$

This result indicates that the nature of eigenvalue problems for systems is different from the one for Schrödinger operators when their potential grows at infinity.

In this paper we give a similar result to Dirac type operators with vector potential of external magnetic field

$$\{\frac{1}{2}(A\cdot(p-b)+(p-b)\cdot A)+mA_0+V+\lambda\}u=0,$$

where $\{A_j(x)\}_{j=0,1,2,3}$ is a family of symmetric matrices ($A_j^*=A_j$) such that

$$A_jA_k+A_kA_j\to 2\delta_{jk}\ \text{(Kronecker's Delta) as } |x|\to\infty.$$

$b\in C^1(U;\ \mathbf{R}^d)$ is a vector potential of external magnetic field $\nabla\times b$ and V is a matrix-valued potential. Our condition which guarantees the nonexistence of eigenvalues is invariant under any gauge transformation. (In [3] we treated the same problem when $b = 0$.)

Central method of our approach to this kind of problem consists of a series of weighted L^2 estimates based on a local version of the virial theorem. This kind of strategy was firstly employed in [5] and has been improved in [2] and [3]. We shall give a minor modification to the local version of the virial theorem in order to treat the Dirac type operators. Furthermore, at the final stage of our method, we shall use a new unique continuation theorem which is interesting in itself.

2 Main result

Let $\{A_k\}_{k=1}^3 \subset C^2(U)^{4\times 4}$ be a family of symmetric matrices such that

$$A_jA_k + A_kA_j = 2g^{jk}(x)I, \ \forall j, \ k = 1, 2, 3, \tag{2.1}$$

where $G = (g^{jk})$ satisfies

$$^{\exists}(G\xi, \xi) \geq \delta|\xi|^2, \ \forall x \in U, \ \xi \in \mathbf{C}^3 \tag{2.2}$$

and

$$g^{jk}(x) - \delta_{jk} = o(1), \ r = |x| \to \infty. \tag{2.3}$$

We are interested in the following Dirac type operator $\mathcal{D}$ in U,

$$\mathcal{D} = \sum_{k=1}^{3} \frac{1}{2}\{A_k(p_k - b_k) + (p_k - b_k)A_k\},$$

where $b_k(x) \in C^1(U;\ \mathbf{R})$, $k = 1, \ldots, 3$. We emphasize that the principal symbol of $\mathcal{D}^2$ is scalar by virtue of the assumption (2.1).

To state our further assumption on the derivatives of A_k and b, we shall introduce a class of scalar functions. If $I_a = (a, \infty)$ and $0 \leq \sigma \leq \frac{1}{2}$, we define

$$\begin{aligned}\mathcal{P}_\sigma(I_a) = \{q(r) \in C^2(I_a; \mathbf{R}); \inf_{I_a} q(r) = q_\infty > 0, \ [q']_- = o(r^{-1}q),\\ q'(r) = o(r^{-1/2}q^{2-\sigma}), \ q'' = o(r^{-1}q^2)\Big\}.\end{aligned}$$

Here,

$$[f(r)]_- = \max(0, -f(r)), \ f' = \frac{d}{dr}f(r), \ \text{etc..}$$

Remark 2.1 e^r, r^s, $(s \geq 0)$, $\log r \in \mathcal{P}(I_a)$.

Thus, we make the following assumptions on the derivatives of A_k $(k = 1, 2, 3)$ and b:

$$\nabla_x A_k(x) = o\left(\frac{1}{r}\right), \quad k = 1, 2, 3 \tag{2.4}$$

and for some element q of $\mathcal{P}_\sigma$,

$$\nabla_x^2 A_k(x) = o\left(\frac{q}{r}\right), \quad k = 1, 2, 3. \tag{2.5}$$

$$\nabla \times b = o(r^{-1}q). \tag{2.6}$$

In addition, $A_0 \in C^1(U)^{4\times 4}$ denotes a symmetric matrix satisfying that for a $c(x) \in C^1(U;\ \mathbf{R})^3$

$$A_j A_0 + A_0 A_j - 2c_j I = o(r^{-1/2}\sqrt{q}), \ j = 1, 2, 3 \tag{2.7}$$

and

$$|A_0(x)| + |c(x)| = o(q), \ |\nabla_x A_0(x)| + |\nabla_x b(x)| = o\left(\frac{q}{r}\right). \tag{2.8}$$

Let a be sufficiently large such that

$$U \supset D_a = \{x \in \mathbf{R}^3;\ |x| > a\}.$$

We shall make the following assumptions on the potential V.

$$V = V_1 + V_2, \ V_1^* = V_1, \ V_1 \in C^1(U)^{4\times 4}, \tag{A-1}$$

$$|V_2(x)| \le K_0/|x|, \tag{A-2}$$

$$V_1(x) - q(|x|)I = o\left(\frac{q^\sigma}{|x|^{1/2}}\right), \tag{A-3}$$

$$\partial_r\{V_1(x) - q(|x|)I\} = o\left(\frac{q}{|x|}\right). \tag{A-4}$$

$$\{\nabla_x - \frac{x}{|x|}\partial_r\}V_1(x) = \mathcal{O}(\frac{q}{|x|}) \text{ as } r \to \infty. \tag{A-5}$$

Theorem 2.1 *Suppose (2.1)–(2.8). If $V(x)$ satisfies (A-1)–(A-5) with $K_0 < 1/2$, then $\mathcal{D}u + A_0 u + Vu = 0$ admits no nontrivial solution in $L^2(U)^4$.*

Remark 2.2 *It is shown in [2] that the same conclusion as in Theorem 2.1 holds for the Dirac operator (D) if* $2K_0 < 1 - b_0$ *under the conditions (A-1)–(A-4) and (A-6)–(A-8):*

(A-6)
$$m(x) - m_1(|x|) = o\left(\frac{q^\sigma}{|x|^{1/2}}\right),$$

(A-7)
$$\partial_r\{m(x) - m_1'(|x|)\} = o\left(\frac{q}{|x|}\right),$$

(A-8)
$$|m_1 + rm_1'| \le b_0 q(r),\ b_0 < 1.$$

3 Proof of Theorem 2.1

3.1 Change of unknown functions

In what follows, $r = |x|$, $\omega = x/|x| \in \mathbf{S}^2$, $\langle u, v\rangle$ denotes the inner product of $\{L^2(\mathbf{S}^2)\}^4$, $\|u\| = \sqrt{\langle u, u\rangle}$ and $T(\mathbf{S}^2)$ stands for the tangent space of $\mathbf{S}^2$.

$$\partial_{x_j} = \omega_j \partial_r + r^{-1}\Omega_j,$$

where $\Omega_j \in T(\mathbf{S}^2)$. For $\Gamma_k = \Omega_k - irb_k$, we put

$$A_r = \sum_{j=1}^{3} A_j(x)\omega_j,\ A_\Gamma = \frac{1}{2}\sum_{j=1}^{3}\{A_j(x)\Gamma_j + \Gamma_j A_j(x)\},$$

$$S_\Gamma = A_\Gamma - A_r,\ S_\Gamma^* = -S_\Gamma,$$

$$J = \frac{1}{2}(S_\Gamma A_r^{-1} - A_r^{-1}S_\Gamma),\ K = \frac{1}{2}(S_\Gamma A_r^{-1} + A_r^{-1}S_\Gamma).$$

It turns out

$$\langle Jf, h\rangle = \langle f, Jh\rangle,\ \langle Kf, h\rangle = -\langle f, Kh\rangle,\ \forall f,\ h \in C^1(\mathbf{S}^2).$$

If $u \in L^2(U)^4$, the integral

$$\int_a^\infty \langle Vru/\sqrt{q}, ru/\sqrt{q}\rangle dr$$

is finite, so that $u/\sqrt{q}$ is more convenient than u itself.

Suppose

$$0 \le \chi \in C_0^\infty(\mathbf{R}_+),\ \mathrm{supp}\chi \subset [s-1, t+1],\ \chi(r) = 1,\ r \in [s, t],$$

$$\varphi \in C^3(\mathbf{R}_+),\ \varphi' \geq 0.$$

Let $u \in L^2(U)^4$ satisfy

$$\mathcal{D}u + (A_0 + V)u = 0, \text{ in } U.$$

Define

$$\zeta = \chi(r)e^{\varphi}v,\ v = \frac{ru}{\sqrt{q}}.$$

Then,

$$\begin{aligned}(3.1)\qquad \{-iA_r\partial_r - i(r^{-1}S_\Gamma - A_r\varphi')A_0 + V - iA_r q'/(2q)\}\zeta \\ = -iA_r\chi' e^{\varphi}v + ir[A_r, \partial_r]\zeta := f_\chi\end{aligned}$$

and

$$(3.2)\qquad [\partial_r - r^{-1}K - (r^{-1}J + \varphi') + i\{(A_0 + V)A_r^{-1} - iq'/(2q)\}]A_r\zeta = if_\chi.$$

To describe fundamental relations among K, L and A_r, we introduce a class of matrix of vector fields $L(r)$ on $\mathbf{S}^2$, depending smoothly in r as follows. Let $\tilde{T}(\mathbf{S}^2) = \{L_1(r) + L_0(r);\ L_1 \in T(\mathbf{S}^2),\ L_0 \in L^\infty(\mathbf{S}^2)\}$ and

$$\begin{aligned}\mathcal{V}^1_\sigma = \{L \in C(I_a;\ \tilde{T}(\mathbf{S}^2)^{4\times 4}); \exists C(r) > 0,\ C > 0,\ \forall u \in C^1(\mathbf{S}^2)^4, \\ \|Lu\| \leq M_1(r)\|Ju\| + M_2(r)\|u\|, \\ M_1(r) = o(r^{-\sigma}),\ M_2(r) = o(q) \text{ as } r \to \infty\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{V}^0_\sigma = \{B \in C^0(\mathbf{S}^2)^{4\times 4}; \exists M(r) > 0,\ M(r) = o(r^{-\sigma}) \\ \|Bu\| \leq M(r)\|u\|,\ \forall u \in C^0(\mathbf{S}^2)^4\}.\end{aligned}$$

Lemma 3.1

$$2A_rKA_r = S_\Gamma A_r + A_rS_\Gamma = (-ir\omega\cdot b + \sum_{j,k=1}^{3}(g_{jk} - \delta_{jk})\omega_k\Gamma_j)I + h_0(x),$$

where h_0 denotes some matrix-valued function asymptotically equal to zero. In particular, K has a scalar principal part, and if $g^{jk}(x) \equiv \delta_{jk}$, then $K = 0$.

Proof:

$$\begin{aligned}2A_rKA_r + 2A_r^2 &= S_\Gamma A_r + A_rS_\Gamma + 2A_r^2 \\ &= \frac{1}{2}\sum_{a,b=1}^{3}(A_aA_b\Gamma_a\omega_b + A_bA_a\omega_b\Gamma_a + \Gamma_a\omega_bA_aA_b + \omega_b\Gamma_aA_bA_a \\ &\quad + A_a[\Gamma_a, A_b]\omega_b + \omega_b[A_b, \Gamma_a]A_a)\end{aligned}$$

$$= \sum_{a,b=1}^{3} \{g^{ab}\omega_b\Gamma_a + \omega_b\Gamma_a g^{ab}\}$$

$$+ \frac{1}{2}\sum_{a,b=1}^{3} \{2A_aA_b[\Gamma_a, \omega_b] + A_a[\Gamma_a, A_b]\omega_b + \omega_b[A_b, \Gamma_a]A_a\}$$

$$= \sum_{a,b=1}^{3} \{g^{ab}\omega_b\Gamma_a + \omega_b\Gamma_a g^{ab}\}$$

$$+ \frac{1}{2}\sum_{a,b=1}^{3} \{2A_aA_b(\delta_{ab} - \omega_a\omega_b) + A_a[\Gamma_a, A_b]\omega_b + \omega_b[A_b, \Gamma_a]A_a\}$$

$$= \sum_{a,b=1}^{3} \{g^{ab}\omega_b\Gamma_a + \omega_b\Gamma_a g^{ab}\} + \sum_{a=1}^{3} g^{a,a}(1-\omega_a^2) - \sum_{a>b} g^{ab}\omega_a\omega_b$$

$$+ \frac{1}{2}\sum_{a,b=1}^{3} \{A_a[\Gamma_a, A_b]\omega_b - \omega_b[\Gamma_a, A_b]A_a\}.$$

In view of

$$\sum_{a=1}^{3} \omega_a\Gamma_a = -ir\omega \cdot b, \; A_r^2 - I = o(1), \; \sum_{a=1}^{3}(1-\omega_a^2) = 2$$

and

$$2A_rKA_r = A_r^{-1}[S_\Gamma A_r + A_rS_\Gamma]A_r^{-1}$$

the assumptions (D-3) and (D-4) give the first part of the conclusion. □

Lemma 3.2

$$2A_rJA_r = 2\sum_{j>k}(A_jA_k - g_{jk}I)(\omega_j\Gamma_k - \omega_k\Gamma_j) + h_0(x),$$

where h_0 is a similar function in the previous lemma.

Proof: Observe

$$2A_rJA_r = \sum_{j,k}(A_jA_k - g_{jk}I)\omega_j\Gamma_k + \sum_{j,k}(A_jA_k - g_{jk}I)\Gamma_k\omega_j$$

$$- \sum_{j,k} A_j[\Gamma_j, A_k]\omega_k + h_0.$$

In view of

$$[\Gamma_k, \omega_j] = \delta_{jk} - \omega_j\omega_k \quad \text{and} \quad \sum_{j=1}^{3}(\omega_j^2 - 1) = -2,$$

$$\sum_{\ell=1}^{3} \Gamma_\ell\omega_\ell = -2 - ir\omega \cdot b,$$

we arrive at the conclusion. □

Lemma 3.3

$$(3.3)\quad [\omega\cdot b, \omega_j\Gamma_k-\omega_k\Gamma_j] = -(\omega_j b_k-\omega_k b_j)-(\omega_j r\partial_r b_k-\omega_k r\partial_r b_j) - \sum_{\ell=1}^{3}\omega_\ell\omega_j(\Gamma_k b_\ell-\Gamma_\ell b_k)-\sum_{\ell=1}^{3}\omega_\ell\omega_k(\Gamma_\ell b_j-\Gamma_j b_\ell).$$

Proof:

$$[\omega\cdot b, \omega_j\Gamma_k-\omega_k\Gamma_j] = -(\omega_j b_k-\omega_k b_j)-\{\omega_j\sum_{i=1}^{3}\omega_i[\Gamma_k, b_i]-\omega_k\sum_{i=1}^{3}\omega_i[\Gamma_j, b_i]\}.$$

Using

$$\sum_{\ell=1}^{3}\omega_\ell[\Gamma_\ell, f]=0,\ \forall f\in C^1(\mathbf{S}^2),$$

we have

$$\begin{aligned}(3.4)\quad \omega_j\sum_{i=1}^{3}\omega_i\Gamma_k b_i-\omega_k\sum_{i=1}^{3}\omega_i\Gamma_j b_i &= (1-\sum_{\ell\neq j,k}\omega_\ell^2)(\Gamma_k b_j-\Gamma_j b_k)\\ &+\sum_{\ell\neq j,k}\omega_\ell\omega_j(\Gamma_k b_\ell-\Gamma_\ell b_k)+\sum_{\ell\neq j,k}\omega_\ell\omega_k(\Gamma_\ell b_j-\Gamma_j b_\ell)\\ &=\sum_{\ell=1}^{3}\omega_\ell\omega_j(\Gamma_k b_\ell-\Gamma_\ell b_k)+\sum_{\ell=1}^{3}\omega_\ell\omega_k(\Gamma_\ell b_j-\Gamma_j b_\ell)\\ &=\sum_{\ell=1}^{3}\omega_\ell\omega_j(r\partial_k b_\ell-r\partial_\ell b_k)+\sum_{\ell=1}^{3}\omega_\ell\omega_k(r\partial_\ell b_j-r\partial_j b_\ell)+(\omega_j r\partial_r b_k-\omega_k r\partial_r b_j).\end{aligned}$$

□

Lemma 3.4 *There exist positive constants δ and C such that*

$$\|Jf\|_{L^2}\geq\delta\sum_{j=1}^{3}\|\Gamma_j f\|_{L^2}-C\|f\|_{L^2},\ ^\forall f\in C^1(\mathbf{S}^2)^4.$$

Lemma 3.5

$$[r\partial_r-K, J]\in\mathcal{V}_0^1,\ K\in\mathcal{V}_0^1.$$
$$(JA_r+A_rJ)=[K, A_r]\in\mathcal{V}_0^0,$$
$$[\partial_r, A_r]\in\mathcal{V}_1^0.$$

Proof: From Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.4 we arrive at the conclusion.

Let

$$A_0A_r^{-1}=B_1+B_2,\ B_1=(A_0A_r^{-1}-A_r^{-1}A_0)/2,\ B_2=(A_0A_r^{-1}+A_r^{-1}A_0)/2.$$

Lemma 3.6

$$[K, A_0] \in q\mathcal{V}_0^0, \; B_1 \in q\mathcal{V}_0^0, \; \nabla B_1 \in \frac{q}{r}\mathcal{V}_0^0, \; B_2 - \omega \cdot cI \in \sqrt{\frac{q}{r}}\mathcal{V}_0^0, \; \nabla B_2 \in \frac{q}{r}\mathcal{V}_0^0.$$

Proof: The first two properties follow from the hypothesis (2.7) and Lemma 3.1. The remaining properties follow from the hypothesis (2.6). □

3.2 A local version of the virial theorem

Lemma 3.7 *If $L_r = \partial_r - r^{-1}K + i\omega \cdot c$ and $\tilde{A}_0 = A_0 - \omega \cdot cA_r^{-1}$, then*

$$\begin{aligned}\int_{s-1}^{t+1} \langle \partial_r\{r(\tilde{A}_0 + V_1)\}\zeta, \zeta\rangle &= 2\mathrm{Re}\int_{s-1}^{t+1} \langle r\{V_2 - iA_r\frac{q'}{2q}\}\zeta, L_r\zeta\rangle dr \\ &+ 2\mathrm{Re}\int_{s-1}^{t+1} \langle irA_r\varphi'\zeta, L_r\zeta\rangle dr - 2\mathrm{Re}\int_{s-1}^{t+1} \langle rf_\chi v, L_r\zeta\rangle dr \\ &- \mathrm{Re}\int_{s-1}^{t+1} \langle [L_r, A_r/i]\zeta, rL_r\zeta\rangle dr \\ &+ \mathrm{Re}\int_{s-1}^{t+1} \langle\{[K, \tilde{A}_0 + V_1] - i(L_rJA_r + A_rJL_r)\}\zeta, \zeta\rangle dr \\ &:= I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

Proof: This is a simple consequence of

$$\begin{aligned}2\mathrm{Re}\int_{s-1}^{t+1} \langle [L_r - (r^{-1}J + \varphi') + i\{(\tilde{A}_0 + V)A_r^{-1} - iq'/(2q)\}]A_r\zeta, riL_r\zeta\rangle dr \\ = 2\mathrm{Re}\int_{s-1}^{t+1} \langle irf_\chi, iL_r\zeta\rangle dr\end{aligned}$$

by use of an integration by parts. To see this, it suffices to check

$$\mathrm{Re}\int_{s-1}^{t+1} \langle L_rA_r\zeta, irL_r\zeta\rangle dr = \mathrm{Re}\int_{s-1}^{t+1} \langle [L_r, -iA_r]\zeta, irL_r\zeta\rangle dr,$$

$$-\mathrm{Re}\int_{s-1}^{t+1} \langle JA_r\zeta, iL_r\zeta\rangle dr = -\mathrm{Im}\int_{s-1}^{t+1} \langle (L_rJA_r + A_rJL_r)\zeta, \zeta\rangle dr$$

and

$$\begin{aligned}\mathrm{Re}\int_{s-1}^{t+1} \langle (\tilde{A}_0 + V_1)\zeta, rL_r\zeta\rangle dr \\ = \mathrm{Re}\int_{s-1}^{t+1} [-\langle \partial_r\{r(\tilde{A}_0 + V_1)\}\zeta, \zeta\rangle + \langle [K, \tilde{A}_0 + V_1]\zeta, \zeta\rangle] dr.\end{aligned}$$

□

Remark 3.1 *If $|x| > a \gg 1$, our assumptions imply*

$$(rV)' = q + (V - q) + rq' + r(V - q)' \geq (1 - \varepsilon)q, \; 1 \gg \varepsilon > 0.$$

3.3 L^2-weighted inequality

We shall estimate the integrals $\{I_j\}_{j=1}^5$ from above to obtain

Proposition 3.8 *If $t > s$ is large enough, then*

$$(3.5)\quad \int_{s-1}^{t+1} [\{(1-2K_0-o(1))q\}\|e^{\varphi}\chi v\|^2 + r\varphi'\|L_r(e^{\varphi}A_r\chi v/\sqrt{q})\|^2]dr$$
$$+\int_{s-1}^{t+1} k_{\varphi}\|e^{\varphi}A_r\chi v/\sqrt{q}\|^2 dr$$
$$\leq C\{\int_{s-1}^{s}+\int_{t}^{t+1}\}\left[rq+\{|\varphi'|+|\varphi''|\}rq^{-1}\right]\|e^{\varphi}A_r v\|^2 dr,$$

where C is a positive constant independent of choice of φ and

$$k_{\varphi} \simeq r\varphi'\{(\varphi''+(r^{-1}-o(r^{-1}))\varphi'\} - \frac{1}{2}(r\varphi'')'$$
$$-o(1)\varphi' - o(1)\{1+(\varphi')^2+(r|\varphi''|)^2\}.$$

The proof of Proposition 3.8 is given in the next section.

Once Proposition 3.8 is established, the proof of Theorem 2.1 follows the argument presented in [5] or [2]. We shall give a sketch of the proof.

Lemma 3.9 *Suppose that $v \in L^2(U)$. Let $0 < b < 1$. If s is large enough,*

$$\int_{s+1}^{\infty} e^{nr^b(\log r)^2}\|\sqrt{q}v\|^2 dr \leq \int_{s-1}^{s} e^{nr^b(\log r)^2}\|\sqrt{rq}v\|^2 dr.$$

Proof: Taking φ in Lemma 3.8 as $\varphi(r) = n\log\log r$, we see that

$$(3.6)\quad \int_s^t (\log r)^n\|q^{1/2}v\|^2 dr \leq C\{\int_{s-1}^{t+1} o(1)(1+n^2(\log r)^{-2})(\log r)^n\|q^{-1/2}v\|^2 dr$$
$$+\{\int_t^{t+1}+\int_{s-1}^{s}\}n(\log r)^{-1}(\log r)^n\|q^{-1/2}v\|^2 dr\}.$$

The induction hypothesis $(\log r)^{n-1}\sqrt{q}v \in L^2(D_s)^4$ gives

$$\liminf_{t\to\infty}\int_t^{t+1} r\|(\log r)^{n-1}\sqrt{q}v\|^2 dr = 0.$$

Therefore, we obtain

$$\int_s^{\infty}(\log r)^n\|\sqrt{q}v\|^2 dr < \infty.$$

In view of

$$r^m = \sum_{n=0}^{\infty}\frac{(m\log r)^n}{n!},$$

we can conclude that

$$\int_s^\infty r^m \|\sqrt{q}v\|^2 dr < \infty.$$

A similar procedure with $\varphi = n\log r$ gives

$$(3.7)\quad \int_s^\infty \sum_{n=2}^N \frac{1}{n!}(mr^b)^n \|q^{1/2}v\|^2 dr$$
$$\leq C\int_{s-1}^\infty o(1) r^{-2(1-b)} m^2 \sum_{n=2}^N \frac{1}{(n-2)!}(mr^b)^{n-2}\|q^{-1/2}v\|^2 dr$$
$$+ C_m \int_{s-1}^s \|v\|^2 dr$$

for all $N = 2, 3, \ldots$. Hence if $0 < b < 1$, it follows from

$$e^{r^b} = \sum_{n=0}^\infty \frac{(r^b)^n}{n!}$$

that

$$\int_{s+1}^\infty e^{nr^b}\|\sqrt{q}v\|^2 dr < +\infty,\ n = 1, 2, \ldots .$$

Finally if $\varphi = nr^b$, then $k_\varphi > 0$, so that the conclusion follows from Lemma 3.8. □

Letting $n \to \infty$ in the inequality in Lemma 3.9, we have $u = 0$ on $|x| \geq s+1$. Therefore, the proof of Theorem 2.1 is completed if we show the unique continuation property for $\mathcal{D}$, which will be derived in Section 5.

4 Proof of Proposition 3.8

We begin the proof by an elliptic estimate of the Dirac type operator in the polar coordinates.

Lemma 4.1 *If $k_0 \in C^1(U)^{4\times 4}$ is a symmetric matrix,*

$$(4.1)\quad \int_{s-1}^{t+1}\{\|L_r k(r) A_r\zeta\|^2 + \frac{1}{2}\|(r^{-1}J + \varphi' - \frac{q'}{2q} + k_0)kA_r\zeta\|^2\}dr$$
$$\leq \int_{s-1}^{t+1} k^2\|i\{f_\chi - (\tilde{A}_0 + V)\zeta\} + (k'k^{-1} - k_0)A_r\zeta\|^2 dr + \frac{1}{2}\int_{s-1}^{t+1}\frac{k^2}{r^2}\|A_r\zeta\|^2 dr$$
$$- \int_{s-1}^{t+1}\left\{r^{-1}k_0 + [L_r, k_0] + \frac{\varphi'}{r} + \varphi'' - (\frac{q'}{2q})' - \frac{q'}{2rq}\right\}\|kA_r\zeta\|^2 dr$$
$$+ \int_{s-1}^{t+1} o(\frac{1}{r})\{\varphi' + q + |k_0| + o(1)\}\|kA_r\zeta\|^2 dr.$$

Proof: The equation $k\zeta$ should satisfy is

$$\{L_r - (r^{-1}J + \psi(r) + k_0)\}A_r k\zeta = \xi. \tag{4.2}$$

Here

$$\psi(r) = \varphi' - q'/(2q),\ \xi = (k' - k_0)A_r\zeta - i(V + \tilde{A}_0)k\zeta + if_\chi.$$

Let

$$X = L_r,\ Y = (r^{-1}J + \psi(r) + k_0).$$

Then

$$\|XA_r k\zeta\|^2 + \|YA_r k\zeta\|^2 + 2\mathrm{Re}\langle X, Y\rangle = \|\xi\|^2$$

and

$$2\mathrm{Re}\int_{s-1}^{t+1}\langle XA_r k\zeta, YA_r k\zeta\rangle dr = \int_{s-1}^{t+1}\langle [Y,X]A_r k\zeta, A_r k\zeta\rangle dr.$$

The ellipticity of $J \in \tilde{T}(\mathbf{S}^2)$ and Lemma 3.5 imply

$$\langle -r^{-1}[\partial_r, J] + r^{-2}[J,K]v, v\rangle \le o\left(\frac{1}{r}\right)\{\|Yv\|\|v\| + \|\psi + k_0 v\|\|v\|\}$$

for any $v \in C^\infty(\mathbf{S}^2)^4$. In view of

$$[Y,X] = -\psi' - [L_r, k_0] + r^{-2}J - r^{-2}[r\partial_r - K, J] + [Y, i\omega\cdot c]$$

and

$$r^{-2}J = r^{-1}(r^{-1}J + \psi + k_0) - r^{-1}(\psi + k_0),$$

we obtain (4.1). □

Proposition 3.8 follows from the following Lemmas 4.2–4.4.

Lemma 4.2 *For any small $\varepsilon > 0$, it holds that*

$$\begin{aligned} I_1 =& 2\mathrm{Re}\int_{s-1}^{t+1}\langle r\{V_2 - iA_r\frac{q'}{2q}\}\zeta, L_r\zeta\rangle dr \\ \le& \int_{s-1}^{t+1}\{(2+\varepsilon)K_0 q + r[q']_+ + o(q) - \varphi''\frac{rq' + (1+\varepsilon)q}{q^2}\}\|\zeta\|^2 dr \\ &+ C\{\int_{s-1}^{s} + \int_{t}^{t+1}\}\left[rq + \{\varphi' + |\varphi''|\}rq^{-1}\right]\|e^{\varphi}\tilde{v}\|^2 dr. \end{aligned}$$

Lemma 4.3 *If $w = \zeta/\sqrt{q}$,*

$$\begin{aligned} I_2 \le \int_{s-1}^{t+1}\{&-k_\varphi\|A_r w\|^2 - r\varphi'\|L_r A_r w\|^2 + o(1)\varphi'\|w\|\|\sqrt{q}\zeta\| \\ &+ o(1)|\varphi''|\|\zeta\|\|A_r w\| + K_0 r^{-1}\|A_r w\|^2 + C\|\chi' e^{\varphi}v/\sqrt{q}\|^2\}dr. \end{aligned}$$

Lemma 4.4

$$I_4 + I_5 = \int_{s-1}^{t+1} o(1)[\{q + (\varphi')^2/q\}\|e^{\varphi}\tilde{v}\|^2 + \|\chi' e^{\varphi}\tilde{v}\|^2]dr. \tag{4.3}$$

$$I_3 \le C\{\int_{s-1}^{s} + \int_{t}^{t+1}\} \left[rq + \{\varphi' + |\varphi''|\}rq^{-1}|\right] \|e^{\varphi}\tilde{v}\|^2 dr. \tag{4.4}$$

Proof: Observe that if $M_r = r\partial_r - K$, then

$$A_r J M_r + M_r J A_r = A_r[J, M_r] + [A_r, M_r]J + M_r(JA_r + A_r J).$$

In addition, the conditions (A4) and (A5) give

$$[K, V_1] = [K, V_1 - q] + [K, q] \in q\mathcal{V}_0^0.$$

In view of these observations and Lemma 3.6, combining Lemma 3.5 with Lemma 4.1 with $k = 1$ and $k_0 = -\varphi' + q'q^{-1}$, we can conclude that

$$I_4 + I_5 = \int_{s-1}^{t+1} o(1)\{(q + \varphi')\|e^{\varphi}\tilde{v}\|^2 + \|\chi' e^{\varphi}\tilde{v}\|^2\}dr. \tag{4.5}$$

The Schwarz inequality gives

$$\varphi' \le \frac{1}{2}\{q + \frac{(\varphi')^2}{q}\},$$

so that (4.3) follows from (4.5). (4.4) can be easily verified by use of an integration by parts. □

5 A unique continuation theorem

In this section we shall show that $\mathcal{D}$ has the strong unique continuation property. We say that $u \in L^2_{\text{loc}}(U)$ vanishes of infinitely order at $x_0 \in U$ if

$$\int_{|x-x_0|<R} |u|^2 dx = \mathcal{O}(R^n), \ R \to 0, \ \forall n \in \mathbf{N}.$$

Theorem 5.1 *Suppose (2.1) and (2.2). If $u \in L^2_{\text{loc}}(U)$ satisfies*

$$\mathcal{D}u + Vu = 0, \ V \in L^\infty_{\text{loc}}(U)^{4\times 4} \tag{5.1}$$

and vanishes of infinitely order at $x_0 \in U$, then u is identically zero in U.

Proof: First of all, we shall reduce $\mathcal{D}$ into the classical Dirac operator at x_0. In fact, there exists an orthogonal transformation $T = (t_{jk})_{j,k=1}^3$ such that $TG(x_0)T^{-1}$ is a diagonal matrix H. Under the transformation $z = T(x - x_0)$ the operator $\mathcal{D}$ has the form

$$\mathcal{D} = -i\sum_{k=1}^{3}\frac{1}{2}\{\partial_{z_k}\tilde{A}_k + \tilde{A}_k\partial_{z_k}\},$$

where

$$\tilde{A}_j(x) = \sum_{k=1}^{3} t_{jk}A_k(x).$$

Then, it is easily verified that

$$\tilde{A}_j\tilde{A}_k + \tilde{A}_k\tilde{A}_j = \sum_{a,b=1}^{3} t_{ka}t_{jb}g_{ab}(x_0 + T^{-1}z)I.$$

The diagonal elements of H are denoted by $g_j > 0$, $j = 1, 2, 3$, and $E = (e_{jk})_{j,k=1}^3$ stands for the matrix

$$e_{jj} = 1/\sqrt{g_j},\ e_{jk} = 0,\ j \neq k.$$

Under the dilation $y = Ez$, $\mathcal{D}$ has the desired property. Namely,

$$\mathcal{D} = \frac{1}{2}\sum_{j=1}^{3}\{\hat{A}_jD_{y_j} + D_{y_j}\hat{A}_j\}$$

with

$$\hat{A}_j\hat{A}_k + \hat{A}_k\hat{A}_j = \hat{g}_{jk}(y)I,\ \hat{g}_{jk}(0) = \delta_{jk}.$$

In this new coordinates, it is written

$$\mathcal{D} = \mathcal{D}_0 + \sum_{j=1}^{3} B_j(y)D_{y_j} + C(y), \tag{5.2}$$

where $\mathcal{D}_0 = \sum_{j=1}^3 \hat{A}_j(0)D_{y_j}$ is the classical Dirac operator,

$$B_j(y) = \mathcal{O}(|y|),\ B_j(y) \in C^2(\tilde{U})^{4\times 4},\ C(y) \in C^1(\tilde{U})^{4\times 4},$$

and $\tilde{U}$ is a domain of $\mathbf{R}^3$ containing the origin. We introduce the polar coordinates

$$y = r\omega,\ r = |y|,\ \omega = y/|y|.$$

In what follows, we use the notation A_j instead of $\hat{A}_j$. Keeping the same notation as in Section 3.1, we have

Lemma 5.2

$$i\tilde{\mathcal{D}}ru = \{\partial_r - r^{-1}(K+J)\}A_r(ru).$$

Furthermore,

$$\|[K,J]v\| = \mathcal{O}(r\|Jv\| + \|v\|) \text{ as } r \to 0.$$

Proof: This can be verified in the same manner as in Lemma 3.5 because

$$A_jA_k + A_kA_j = 2\delta_{jk} + \mathcal{O}(r) \text{ as } r \to 0.$$

□

In [1], it has been proved that

$$\frac{1}{4}\int r^{-2n-2}|v|^2 dy \le \int r^{-2n}\|\mathcal{D}_0 v\|^2 dy, \quad v \in C_0^\infty(\tilde{U})^4 \tag{5.3}$$

for any $n \in \mathbf{N}$.

Lemma 5.3 *If $u \in H^1_{\text{loc}}(\tilde{U})^4$ is a solution to (5.2) vanishing of infinitely order at the origin, then*

$$\int_{|y|<R}\{|u|^2 + |\nabla_y u|^2\}dy \le C\exp\{-\delta R^{-1}\}$$

for any small positive R.

Proof: Suppose that $h(r) \in C^\infty([0,\infty)$ satisfies

$$0 \le h \le 1, \ \ h = 0, \text{ on } [2,\infty), \ \ h = 1 \text{ on } [0,1].$$

Let M be a large positive number determined later. Applying the inequality (5.3) to $v = h(nM|y|)u(y)$, we obtain

$$\frac{1}{4}\int r^{-2n-2}|h(nMr)u|^2 dy \le \int r^{-2n}|\mathcal{D}_0 h(nM|y|)u|^2 dy. \tag{5.4}$$

On the other hand, the ellipticity of $\tilde{\mathcal{D}}$ gives

$$\int |\nabla_y r^{-n}h(nMr)u|^2 dy \le C\int\{|\tilde{\mathcal{D}}r^{-n}h(nMr)u|^2 + |r^{-n}h(nMr)u|^2\}dy.$$

From the triangle inequality, it follows that

$$\begin{aligned}\frac{1}{2}\int r^{-2n}|\nabla_y h(nMr)u|^2 dy &\le n^2\int r^{-2n-2}|h(nMr)u|^2 dy \\ &\quad + C\int\{|\tilde{\mathcal{D}}r^{-n}h(nMr)u|^2 + \|r^{-n}h(nMr)u\|^2\}dy.\end{aligned} \tag{5.5}$$

From (5.2), (5.4) and (5.5)$\times n^{-2}/4$, it follows that

$$(5.6)\quad \int \{\frac{1}{8} r^{-2n-2}|h(nMr)u|^2 + \frac{1}{16} n^{-2} r^{-2n} |\nabla_y h(nMr)u|^2\} dy$$
$$\leq 4 \int r^{-2n} h(nM|y|)^2 |(\tilde{\mathcal{D}} + Q)u|^2 dy$$
$$+ C_1 \int r^{-2n} h(nM|y|)^2 \{|u|^2 + |y|^2 |\nabla_y u|^2\} dy$$
$$+ C_2 (nM)^2 \int_{1 \leq nMr \leq 2} |u|^2 dy.$$

Since

$$|y| \leq 2/(nM), \text{ on } \mathrm{supp}\{h(nM|y|)\},$$

we obtain

$$(5.7)\qquad \frac{1}{16} \int r^{-2n-2} |h(nMr)u|^2 dy \leq C_2 (nM)^2 \int_{1 \leq nMr \leq 2} |u|^2 dy$$

if M is large enough. Hence,

$$\int_{|y| < 1/2nM} |u|^2 dy \leq C e^{-n \log 2} \int_{1 \leq nMr \leq 2} |u|^2 dy.$$

For any small $R > 0$, one can find n such that $1/(n+1) < R < 1/n$, so that

$$\int_{|y| < R} |u|^2 dy = C' \exp\{-(\log 2)/R\}.$$

□

For the sake of Lemma 5.3, if $0 < b < 1$, then

$$\int \exp\left\{n r^{-b}\right\} \{|u|^2 + |\nabla_y u|^2\} dy < \infty.$$

Thus, we can use another Carleman inequality with a stronger weight function.

Lemma 5.4 *If $b > 0$, we have*

$$(5.8)\qquad \frac{b^2 n}{4} \int r^{-b} \exp\left\{n r^{-b}\right\} |A_r u|^2 dy \leq C \int \exp\left\{n r^{-b}\right\} |r(\tilde{\mathcal{D}} + Q)u|^2 dy$$

for any $u(x) \in C_0^\infty(U \backslash \{0\})^4$ and any large positive number n if U is small enough.

Proof: Let $\varphi = n r^{-b}/2$ with $1 > b > 0$. Note

$$M_r = r\partial_r + \frac{1}{2} - K$$

is skew symmetric. If $v = re^{\varphi}u$ and $u \in C_0^\infty(U)$ then

$$ir\tilde{\mathcal{D}}v = \{M_r - (J + \frac{1}{2} + r\varphi')\}A_r v.$$

Thus,

$$(5.9)\quad \int_0^\infty \|r(\tilde{\mathcal{D}} + Q)v\|^2 dr \geq \frac{1}{2}\int_0^\infty \{\|M_r A_r v\|^2 + \|(J + \frac{1}{2} + r\varphi')A_r v\|^2\} dr$$
$$- \operatorname{Re}\int_0^\infty \langle M_r A_r v, (J + \frac{1}{2} + r\varphi')A_r v\rangle dr - \sup_U |Q| \int_0^\infty \|rv\|^2 dr$$

and

$$(5.10)\quad -2\operatorname{Re}\int_0^\infty \langle M_r A_r v, (J + \frac{1}{2} + r\varphi')A_r v\rangle dr$$
$$= \int_0^\infty \langle \{r(r\varphi')' + [K, J] + [r\partial_r, J]\} A_r v, A_r v\rangle dr.$$

The ellipticity of J implies

$$(5.11)\quad \langle [K, J]A_r v, A_r v\rangle + \langle [r\partial_r, J])A_r v, A_r v\rangle$$
$$\leq C\{r\|(J + \frac{1}{2} + r\varphi')A_r v\| + r\|(\frac{1}{2} + r\varphi')A_r v\| + \|A_r v\|\}\|A_r v\|.$$

If U is shrunk sufficiently, it holds

$$r(r\varphi')' - Cr^2\varphi' \geq \frac{n}{2}b^2 r^{-b} - C\frac{n}{2}br^{-b+1} \geq \frac{n}{4}b^2 r^{-b}.$$

Therefore, (5.9)–(5.11) gives the conclusion (5.8) with aid of the the Schwarz inequality. □

The strong unique continuation property follows from Lemmata 5.4 and 5.3 by the standard procedure. This achieves the proof of Theorem 5.1. □

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