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Eigenvalue problems on domains with cracks II

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0. Introduction. Let Ω be a bounded domain in $\mathbf{R}^2$ with a smooth boundary and let $\gamma : [0, t_0] \to \mathbf{R}^2$ be a smooth curve without self-intersection. We assume that

$$(A.1) \qquad \gamma((0,t_0)) \subset \Omega, \qquad \gamma(0) = 0 \in \partial\Omega, \qquad \gamma(t_0) \in \partial\Omega.$$

For $\epsilon \in [0, t_0)$, we put

$$\Omega_\epsilon = \Omega \backslash \gamma([\epsilon, t_0)).$$

Let $\alpha \in (0, \pi)$. For $b > 0$, we define

$$\Pi_\alpha^b = \{(x_1, x_2) \in \mathbf{R}^2; \quad x_2 > 0\} \backslash \{(r\cos\alpha, r\sin\alpha) \in \mathbf{R}^2; \quad r \geq b\}.$$

For $a \in \mathbf{R}^2$ and $r > 0$, we denote by $D(a,r)$ the open planar disk of radius r centered at a. We impose the following assumptions on Ω and γ.

$(A.2)$ There exist $r_0 > 0$ and $\epsilon_0 \in (0, r_0)$ such that

$$\Omega_\epsilon \cap D(0, r_0) = \Pi_\alpha^\epsilon \cap D(0, r_0) \quad \text{for all} \quad \epsilon \in (0, \epsilon_0].$$

The set Ω_0 consists of two connected components. Let Ω_+ and Ω_- be the connected components of Ω_0 which satisfy $(\epsilon_0, 0) \in \partial\Omega_+$ and $(-\epsilon_0, 0) \in \partial\Omega_-$, respectively. We define

$$Q_\epsilon = \{u \in H^1(\Omega_\epsilon); \quad u = 0 \quad \text{on} \quad \partial\Omega\},$$

$$Q^\pm = \{u \in H^1(\Omega_\pm); \quad u = 0 \quad \partial\Omega \cap \partial\Omega_\pm\},$$

$$q_\epsilon(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)} \quad \text{for} \quad u, v \in Q_\epsilon,$$

$$q^\pm(u,v) = (\nabla u, \nabla v)_{L^2(\Omega_\pm)} \quad \text{for} \quad u, v \in Q^\pm.$$

Let L_ϵ be the self-adjoint operator associated with the quadratic form q_ϵ. The operator L_ϵ is the negative laplacian on Ω_ϵ subject to the Dirichlet boundary condition on $\partial\Omega$ and the Neumann boundary condition of the crack $\gamma((0,t_0))$. By $\lambda_j(\epsilon)$ we denote by the jth eigenvalue of L_ϵ counted with multiplicity. The aim of this paper is to find the asymptotic form of the first eigenvalue $\lambda_1(\epsilon)$ as ϵ tends to zero. Let L^+ and L^- be the self-adjoint operators associated with the quadratic forms q^+ and q^-, respectively. Let $\lambda_1^\pm < \lambda_2^\pm \leq \lambda_3^\pm \leq \cdots$ be the eigenvalues of $L^\pm$ repeated according to multiplicity. We assume that

$$(A.3) \qquad \lambda_1^+ < \lambda_1^-.$$

We put

$$\beta = \frac{\alpha}{\pi}.$$

We further impose the following assumption on α.

$$(A.4) \qquad \frac{l}{\beta} + \frac{m}{1-\beta} \notin \mathbf{Z} \quad \text{for all} \quad (l,m) \in \mathbf{Z}^2 \backslash \{(0,0)\}.$$

Let $\Psi_0(x)$ be the eigenvector of L^+ associated with the eigenvalue λ_1^+ which is normalized by the conditions

$$\Psi_0(x) > 0 \quad \text{in} \quad \Omega_+, \qquad \|\Psi_0\|_{L^2(\Omega_+)} = 1. \tag{0.1}$$

The function $\Psi_0(x)$ admits the following asymptotic expansion which can be differentiated term by term arbitrary times.

$$\Psi_0(x) \sim \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} C_{j,k} r^{\frac{2j-1}{2\beta}+2k} \sin\frac{(2j-1)\theta}{2\beta} \quad \text{as} \quad r \to 0, \tag{0.2}$$

$$C_{1,0} > 0, \tag{0.3}$$

where (r,θ) stand for the polar coordinates of $x \in \Omega_+$. Our main result is the following claim.

THEOREM 0.1. *The function $\lambda_1(\epsilon)$ admits the asymptotic expansion of the form*

$$\lambda_1(\epsilon) \sim \lambda_1^+ + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \lambda_{m,n,p}\epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p} \quad \text{as} \quad \epsilon \to 0, \tag{0.4}$$

where

$$\lambda_{1,0,0} = \frac{\pi}{4}\beta^{-1+1/\beta}\left(\int_{-1}^{0} \frac{-x}{(x+1)^{1-\beta}(\frac{1-\beta}{\beta}-x)^{\beta}}\,dx\right)^{1/\beta} C_{1,0}^2. \tag{0.5}$$

Our work is inspired and motivated by that of M. Dauge and B. Helffer. By using the method of variation, they proved in [2] that

$$\lim_{\epsilon\to 0}\lambda_j(\epsilon) = \nu_j \quad \text{for} \quad j \in \mathbb{N},$$

where $\nu_1 \leq \nu_2 \leq \cdots$ are the rearrangement of $\{\lambda_j^+\}_{j=1}^{\infty} \cup \{\lambda_j^-\}_{j=1}^{\infty}$ counted with multiplicity. This result interests us in the asymptotic behavior of $\lambda_j(\epsilon)$ as ϵ tends to zero. In our previous work [8], the full asymptotic expansions of $\lambda_1(\epsilon)$ and $\lambda_2(\epsilon)$ are obtained in the case when $\alpha = \pi/2$ and $\lambda_1^+ = \lambda_1^-$. In the derivation of these asymptotic expansions, we made use of the reflection symmetry of Ω_ϵ in the vicinity of the origin. The scope of this paper is to obtain the full asymptotic expansion of the eigenvalue of L_ϵ as ϵ tends to zero in the case when $\alpha \neq \pi/2$. In the proof of Theorem 0.1 we need a tool which differs from the reflection argument used in [8] because the region Ω_ϵ has no symmetry in any neighborhood of the origin.

Throughout this paper we use the following expedient about summations and sets. For k, $l \in \mathbb{Z}$ with $k > l$, we define $\sum_{j=k}^{l} a_j = 0$ and $\{b_j\}_{k\leq j\leq l} = \emptyset$. A formula that contains either $\pm$ or $\mp$ means two formulae which correspond to the upper sign and the lower sign, respectively. For example, the formula $a^{\pm} = b^{\mp}$ means that $a^+ = b^-$ and $a^- = b^+$.

We prove the main theorem by using the method of matched asymptotic expansion (see [5] and [3]). We define

$$\xi = \epsilon^{-1}x.$$

We look for the approximate first eigenvalue of L_ϵ and the associated eigenvector in the following form.

$$\lambda(\epsilon) = \lambda_1^+ + \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \lambda_{m,n,p}\epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p}, \tag{0.6}$$

$$\Psi_{\text{out}}^+(x) = \Psi_0(x) + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\Psi_{j,k,l}^+(x) \quad \text{in} \quad \Omega_+\backslash D(0,\sqrt{\epsilon}), \tag{0.7}$$

$$\Psi_{\text{out}}^-(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l}\Psi_{j,k,l}^-(x) \quad \text{in} \quad \Omega_-\backslash D(0,\sqrt{\epsilon}), \tag{0.8}$$

$$\Psi_{\text{in}}(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2l} v_{j,k,l}(\xi) \quad \text{in} \quad \Omega_\epsilon \cap D(0,2\sqrt{\epsilon}). \tag{0.9}$$

Inserting (0.6) and (0.7) into the equation $(\Delta_x + \lambda(\epsilon))\Psi^+_{\text{out}}(x) = 0$ and identifying the power of ϵ in view of $(A.4)$, we obtain

$$(\Delta_x + \lambda_1^+)\Psi^+_{j,k,l}(x) = -\lambda_{j,k,l}\Psi_0(x) - \sum_{m=1}^{j-1}\sum_{n=0}^{k}\sum_{p=0}^{l}\lambda_{m,n,p}\Psi^+_{j-m,k-n,l-p}(x) \quad \text{in} \quad \Omega_+, \qquad (0.10)_{j,k,l}$$

$$\Psi^+_{j,k,l}(x) = 0 \quad \text{on} \quad \partial\Omega_+ \cap \partial\Omega, \qquad \frac{\partial}{\partial n}\Psi^+_{j,k,l}(x) = 0 \quad \text{on} \quad \gamma((0,t_0)).$$

In a similar way, we obtain the following equations from (0.6) and (0.8).

$$(\Delta_x + \lambda_1^+)\Psi^-_{j,k,l}(x) = -\sum_{m=1}^{j-1}\sum_{n=0}^{k-1}\sum_{p=0}^{l}\lambda_{m,n,p}\Psi^-_{j-m,k-n,l-p}(x) \quad \text{in} \quad \Omega_-, \qquad (0.11)_{j,k,l}$$

$$\Psi^-_{j,k,l}(x) = 0 \quad \text{on} \quad \partial\Omega_- \cap \partial\Omega, \qquad \frac{\partial}{\partial n}\Psi^-_{j,k,l}(x) = 0 \quad \text{on} \quad \gamma((0,t_0)).$$

Plugging (0.6) and (0.9) into the equation $(\Delta_x + \lambda(\epsilon))\Psi_{\text{in}}(x) = 0$ and equating the powers of ϵ in view of $(A.4)$, we get

$$\Delta_\xi v_{j,k,l}(\xi) = -\lambda_1^+ v_{j,k,l-1}(\xi) - \sum_{m=1}^{j-1}\sum_{n=0}^{k}\sum_{p=0}^{l-1}\lambda_{m,n,p}v_{j-m,k-n,l-p-1}(\xi) \quad \text{in} \quad \Pi^1_\alpha, \qquad (0.12)_{j,k,l}$$

$$v_{j,k,l}(\cdot,0) = 0 \quad \text{on} \quad \mathbf{R}, \qquad \frac{\partial}{\partial n_\pm}v_{j,k,l}(\xi) = 0 \quad \text{for} \quad \xi \in \partial\Pi^1_\alpha\backslash(\mathbf{R}\times\{0\}),$$

where

$$\frac{\partial}{\partial n_\pm}v_{j,k,l}(\xi) := \lim_{h\to\pm 0}\frac{v_{j,k,l}(\xi + hn_0) - v_{j,k,l}(\xi)}{h} \quad \text{for} \quad \xi \in \partial\Pi^1_\alpha\backslash(\mathbf{R}\times\{0\}),$$

and $n_0 = (\sin\alpha, -\cos\alpha)$ is the unit normal vector to $\partial\Pi^1_\alpha\backslash(\mathbf{R}\times\{0\})$. We shall construct Ψ^+_{out}, Ψ^-_{out}, and Ψ_{in} in such a way that Ψ^+_{out}, Ψ^-_{out} asymptotically coincide with Ψ_{in} on the intermidiate regions $\Omega_+ \cap (D(0,2\sqrt{\epsilon})\backslash D(0,\sqrt{\epsilon}))$ and $\Omega_- \cap (D(0,2\sqrt{\epsilon})\backslash D(0,\sqrt{\epsilon}))$, respectively. We organize this paper as follows. In section 1, we solve the outer equations $(0.10)_{j,k,l}$ and $(0.11)_{j,k,l}$. We also analyze the asymptotic behavior of the solutions to the equations $(0.10)_{j,k,l}$ and $(0.11)_{j,k,l}$ in a neighborhood of the origin. For this purpose we use the standard L^2–theory of elliptic differential equations on coner domains which was originated by V. A. Kondrat'ev. In section 2, we solve the inner equation $(0.12)_{j,k,l}$. We give an explicit formula for the solution to this equation. Using this formula, we derive the asymptotic expansion of the solution to $(0.12)_{j,k,l}$ as $|\xi| \to \infty$. To construct this formula, we need a special conformal map. Thanks to this map, we can derive the explicit formula (0.5). This map is the most significant tool in the proof of Theorem 0.1. In section 3 we construct the coefficients of (0.6)–(0.9) by using an induction procedure and the results in the previous sections. In the construction we need matching conditions which ensures the mentioned coincidence of the expansions (0.7)–(0.9) on the intermidiate regions. On inequalities we denote inessential constants by C.

1. Outer equations. In order to solve the outer equations (0.10) and (0.11), we use the L^2–theory of elliptic differential equations on domains with conic singularities which was inspired by V. A. Kondrat'ev (see [6] and [7]). For $\mu \in (0, 2\pi)$, we put

$$\mathbb{K}_\mu = \{(r\cos\theta, r\sin\theta) \in \mathbf{R}^2; \quad r > 0, \quad 0 < \theta < \mu\}.$$

Let $\mathbf{R}_+$ be the set of all positive real numbers. By $(r,\theta) \in \mathbf{R}_+ \times (0,\mu)$ we denote the polar coordinates of $x \in \mathbb{K}_\mu$. Let us consider the equation

$$\begin{cases} -\Delta_x u(r,\theta) = f(r,\theta) \quad \text{in} \quad \mathbb{K}_\mu, \\ u(\cdot,0) = 0 \quad \text{on} \quad \mathbf{R}_+, \quad \dfrac{\partial}{\partial\theta}u(\cdot,\mu) = 0 \quad \text{on} \quad \mathbf{R}_+. \end{cases} \qquad (1.1)$$

By $\mathbf{Z}_+$ we denote the set of all non-negative integers. For $l \in \mathbf{Z}_+$ and $\gamma \in \mathbf{R}$, we define

$$V^l_\gamma(\mathbb{K}_\mu) = \{u \in \mathcal{D}'(\mathbb{K}_\mu); \quad r^{\gamma - l + |\delta|}\partial^\delta_x u(x) \in L^2(\mathbb{K}_\mu) \quad \text{for} \quad \delta \in \mathbf{Z}^2_+, \quad |\delta| \le l\}.$$

From [6] we recall the following two theorems (see also [7, Chapter 2]).

THEOREM 1.1 (V. A. Kondrat'ev). *Let $l \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$. Assume that $\gamma - l - 1 \notin \{\frac{\pi}{2\mu}(2j-1);\quad j \in \mathbb{Z}\}$ and $f \in V^l_\gamma(\mathbb{K}_\mu)$. Then the equation* (1.1) *has a unique solution in $V^{l+2}_\gamma(\mathbb{K}_\mu)$.*

THEOREM 1.2 (V. A. Kondrat'ev). *Let $l \in \mathbb{Z}_+$, $\gamma_1 < \gamma_2$, $\gamma_k - l - 1 \notin \{\frac{\pi}{2\mu}(2j-1);\quad j \in \mathbb{Z}\}$ for $k = 1, 2$, and $f \in V^l_{\gamma_1}(\mathbb{K}_\mu) \cap V^l_{\gamma_2}(\mathbb{K}_\mu)$. For $k = 1, 2$, let $u_k \in V^{l+2}_{\gamma_k}(\mathbb{K}_\mu)$ be the solution of* (1.1). *Then we have*

$$u_1(x) - u_2(x) = \sum_{n \in \mathcal{A}(\gamma_1,\gamma_2,l)} c_n r^{\frac{\pi}{2\mu}(2n-1)} \sin \tfrac{(2n-1)\pi}{2\mu}\theta \quad in \quad \mathbb{K}_\mu,$$

where

$$\mathcal{A}(\gamma_1, \gamma_2, l) = \{n \in \mathbb{Z};\quad l + 1 - \gamma_2 < \frac{\pi}{2\mu}(2n-1) < l + 1 - \gamma_1\}.$$

Now we introduce function spaces which we need in the sequel. For $j \in \mathbb{N}$, we define

$$\begin{aligned} S^j(\mathbb{K}_\mu) &:= \bigcap_{l=0}^{\infty} V^{l+2j}_{l+1}(\mathbb{K}_\mu) \\ &= \{u \in \mathcal{D}'(\mathbb{K}_\mu);\quad r^{-2j+1+|\delta|}\partial_x^\delta u \in L^2(\mathbb{K}_\mu) \quad \text{for all} \quad \delta \in \mathbb{Z}^2_+\}. \end{aligned}$$

For an open set Σ in $\mathbb{R}^2$ and a finite subset S of $\partial\Sigma$, we define

$$C^k(\overline{\Sigma}\backslash S) = \{u : \Sigma \to \mathbb{R};\quad u \in C^k(\overline{\Sigma\backslash A}) \quad \text{for any open covering} \quad A \quad \text{of} \quad S\}, \quad k \in \mathbb{N},$$

$$C^\infty(\overline{\Sigma}\backslash S) = \cap_{k=1}^\infty C^k(\overline{\Sigma}\backslash S).$$

Choose $\chi \in C^\infty([0,\infty))$ such that

$$x(r) = 1 \quad \text{on} \quad [0, r_0/4], \quad x(r) = 0 \quad \text{on} \quad [r_0/2, \infty).$$

For $m \in \mathbb{Z}_+$, we define

$$\begin{aligned} J^+_m = \{&u \in C^\infty(\overline{\Omega_+}\backslash\{0, \gamma(t_0)\}); \\ &(1 - \chi(r))u \in L^2(\Omega_+), \quad u = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_+, \quad \frac{\partial}{\partial n}u = 0 \quad \text{on} \quad \gamma((0,t_0)), \\ &\text{the function} \quad u(x) \quad \text{admits the asymptotic expansion of the form} \\ &u(x) \sim \sum_{j=0}^\infty \sum_{k=0}^\infty D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k} \sin \tfrac{(2j-2m+1)\theta}{2\beta} \quad \text{as} \quad r \to 0, \quad x \in \Omega_+, \\ &\text{which can be differentiated term by term infinitely many times.}\} \end{aligned}$$

For f, $g \in \cap_{r\in(0,r_0)} L^2(\Omega_\pm \backslash D(0,r))$, we define

$$(f, g)_{\Omega_\pm} = \lim_{r\to+0} (f, g)_{L^2(\Omega_\pm\backslash D(0,r))}$$

if and only if the limit exists.

In this section we are mainly aimed to prove the following lemma.

LEMMA 1.3. *Let $m \in \mathbb{Z}_+$, $f \in J^+_m$, and $\{a_j\}_{j=0}^m \subset \mathbb{R}$. Then there exists $\mu \in \mathbb{R}$ such that the equation*

$$\begin{cases} (\Delta + \lambda_1^+)\varphi = -\mu\Psi_0 + f \quad in \quad \Omega_+, \\ \varphi = 0 \quad on \quad \partial\Omega \cap \partial\Omega_+, \\ \dfrac{\partial}{\partial n}\varphi = 0 \quad on \quad \gamma((0,t_0)), \\ (\varphi, \Psi_0)_{\Omega_+} = 0 \end{cases} \tag{1.2}$$

has a solution $\varphi \in J^+_{m+1}$ *which admits the asymptotic expansion*

$$\varphi(x) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} E_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta} \quad \textit{as} \quad r \to 0, \quad x \in \Omega_+ \tag{1.3}$$

with

$$E_{j,0} = a_j \quad \textit{for} \quad 0 \le j \le m. \tag{1.4}$$

In order to prove this Lemma, we need the asymptotic representation (0.2) of the function Ψ_0. Supposing this formula for a moment, we shall complete the proof of this Lemma.

Proof of Lemma 1.3. Since $f \in J^+_m$, the function f admits the asymptotic expansion

$$f(x) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k} \sin\frac{(2j-2m+1)\theta}{2\beta} \quad \text{as} \quad r \to 0. \tag{1.5}$$

Let F be the partial sum of the formal power series on the right side of (1.5):

$$F(x) = \sum_{j=0}^{m-1}\sum_{k=0}^{\infty} D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k} \sin\frac{(2j-2m+1)\theta}{2\beta}.$$

We introduce a formal power series Ψ satisfying $(\Delta + \lambda_1^+)\Psi = F$ term by term as follows. We put

$$\Psi(x) = \sum_{j=0}^{m}\sum_{k=0}^{\infty} E_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta},$$

where the coefficients $\{E_{j,k}\}$ are defined by the recurrent formulae

$$E_{j,0} := a_j \quad \text{for} \quad 0 \le j \le m,$$

$$E_{0,k+1} := -\frac{\beta\lambda_1^+}{2(k+1)(2j-2m-1)} E_{0,k} \quad \text{for} \quad k \in \mathbf{Z}_+,$$

$$E_{j,k+1} := \frac{\beta}{2(k+1)(2j-2m-1)}(D_{j-1,k} - \lambda_1^+ E_{j,k}) \quad \text{for} \quad k \in \mathbf{Z}_+, \quad 1 \le j \le m.$$

For $N, j \in \mathbf{Z}_+$, we define

$$M(j,N) = \left[\frac{N+m-j+1}{2\beta}\right] + 1.$$

Then we have

$$\frac{2j-2m-1}{2\beta} + 2M(j,N) > \frac{2N+1}{2\beta}. \tag{1.6}$$

We introduce the following partial sum of Ψ:

$$\Psi^N(x) := \sum_{j=0}^{m}\sum_{k=0}^{M(j,N)} E_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}.$$

We seek a solution of (1.2) which admits the form

$$\varphi(x) = \chi(r)\Psi^N(x) + \phi_N(x), \quad \phi_N \in \mathcal{D}(L_+). \tag{1.7}$$

Inserting this into the equation (1.2), we obtain the equation for ϕ_N:

$$\begin{cases} (\Delta + \lambda_1^+)\phi_N = -\mu\Psi_0 + g_N \quad \text{in} \quad \Omega_+, \\ \phi_N = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_+, \\ \dfrac{\partial}{\partial n}\phi_N = 0 \quad \text{on} \quad \gamma((0,t_0)), \\ (\phi_N, \Psi_0)_{L^2(\Omega_+)} = -(\chi(r)\Psi^N(x), \Psi_0)_{\Omega_+}, \end{cases} \tag{1.8}$$

$$g_N = (1-\chi)f - \Psi^N\Delta\chi - 2\nabla\chi\cdot\nabla\Psi^N + \chi\left(f - \sum_{j=0}^{m-1}\sum_{k=0}^{M(j+1,N)-1} D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k}\sin\frac{(2j-2m+1)\theta}{2\beta}\right.$$
$$\left. -\lambda_1^+\sum_{j=0}^{m} E_{j,M(j,N)} r^{\frac{2j-2m-1}{2\beta}+2M(j,N)}\sin\frac{(2j-2m-1)\theta}{2\beta}\right).$$

From (1.5) and (1.6), we have $g_N \in L^2(\Omega_+)$. Since $-\mu\Psi_0 + g_N \in L^2(\Omega_+)$ and λ_1^+ is a simple eigenvalue of L_+, the equation (1.8) has a solution in $\mathcal{D}(L_+)$ if and only if $(-\mu\Psi_0 + g_N, \Psi_0)_{L^2(\Omega_+)} = 0$; i.e. $\mu = (g_N, \Psi_0)_{L^2(\Omega_+)}$. We define $\mu_N = (g_N, \Psi_0)_{L^2(\Omega_+)}$. For $\mu = \mu_N$, let $\phi_N \in \mathcal{D}(L_+)$ be the unique solution of the equation (1.8). We put $\varphi_N(x) = \chi(r)\Psi^N(x) + \phi_N(x)$.

Let us show that μ_N and φ_N are independent of the choice of $N \in \mathbb{Z}_+$. For $N, M \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&(\Delta + \lambda_1^+)(\varphi_N - \varphi_M) = (\mu_N - \mu_M)\Psi_0 \quad \text{in} \quad \Omega_+,\\
&(\varphi_N - \varphi_M, \Psi_0)_{\Omega_+} = 0,\\
&\varphi_N(x) - \varphi_M(x) = \chi(r)(\Psi^N(x) - \Psi^M(x)) + \phi_N(x) - \phi_M(x) \in \mathcal{D}(L_+).
\end{aligned}$$

Since λ_1^+ is a simple eigenvalue of L_+, we get $\mu_N - \mu_M = 0$ and $\varphi_N - \varphi_M = 0$. Thus μ_N and φ_N are independent of the choice of $N \in \mathbb{Z}_+$, which we denote by μ and φ, respectively.

Our next task is to prove that $\varphi \in J_{m+1}^+$. As a preliminary, we first prove that $\chi\phi_0 \in S^1(\mathbb{K}_\alpha)$ by induction. Since $\phi_0 \in \mathcal{D}(L_+)$, we have the Hardy inequality

$$\begin{aligned}
\int_{\Omega_+\cap D(0,r_0)} |\nabla_x\phi_0|^2\,dx &\geq \int_{\Omega_+\cap D(0,r_0)} r^{-2}|\partial_\theta\phi_0|^2\,dx\\
&= \int_0^\alpha\int_0^{r_0} r^{-1}|\partial_\theta\phi_0|^2\,drd\theta\\
&\geq \frac{\pi^2}{4\alpha^2}\int_0^\alpha\int_0^{r_0} r^{-1}|\phi_0|^2\,drd\theta\\
&= \frac{\pi^2}{4\alpha^2}\int_{\Omega_+\cap D(0,r_0)} r^{-2}|\phi_0|^2\,dx.
\end{aligned}$$

So we get $\chi\phi_0 \in V_0^1(\mathbb{K}_\alpha)$. Now we assume that $\chi\phi_0 \in V_k^{k+1}(\mathbb{K}_\alpha)$ for some $k \in \mathbb{Z}_+$. For $N \in \mathbb{Z}_+$, we obtain

$$\begin{cases}
\Delta(\chi\phi_N) = \chi(-\lambda_1^+\phi_N - \mu\Psi_0 + g_N) + 2\nabla\chi\cdot\nabla\phi_N + \phi_N\Delta\chi =: h_N \quad \text{in} \quad \mathbb{K}_\alpha,\\
(\chi\phi_N)(\cdot,0) = 0 \quad \text{on} \quad \mathbb{R}_+, \quad \dfrac{\partial}{\partial\theta}(\chi\phi_N)(\cdot,\alpha) = 0 \quad \text{on} \quad \mathbb{R}_+.
\end{cases} \tag{1.9$_N$}$$

Since $h_0 \in V_{k+1}^k(\mathbb{K}_\alpha)$, we infer from Theorem 1.1 that there exists $v \in V_{k+1}^{k+2}(\mathbb{K}_\alpha)$ such that

$$\begin{cases}
\Delta v = h_0 \quad \text{in} \quad \mathbb{K}_\alpha,\\
v(\cdot,0) = 0 \quad \text{on} \quad \mathbb{R}_+, \quad \dfrac{\partial}{\partial\theta}v(\cdot,\alpha) = 0 \quad \text{on} \quad \mathbb{R}_+.
\end{cases}$$

Since $\Delta(\chi\phi_0 - v) = 0$ and $v \in V_{k+1}^{k+2}(\mathbb{K}_\alpha)$, we have $\int_{\mathbb{K}_\alpha} |\nabla(\chi\phi_0 - v)|^2\,dx = 0$. So we get $v = \chi\phi_0 \in V_{k+1}^{k+2}(\mathbb{K}_\alpha)$. Thus we obtain $\chi\phi_0 \in V_k^{k+1}(\mathbb{K}_\alpha)$ for all $k \in \mathbb{Z}_+$ and hence $\chi\phi_0 \in S^1(\mathbb{K}_\alpha)$.

For $n \in \mathbb{Z}_+$, we define $l(n) = [\frac{(2n-1)\pi}{4\alpha}] + 1$. Then we get

$$2(l(n)-1) < \frac{(2n-1)\pi}{2\alpha} < 2l(n). \tag{1.10}$$

Let us demonstrate the following claim.

CLAIM. *For any $n \in \mathbf{Z}_+$, the function $\chi\phi_n$ admits the representation*

$$\chi\phi_n = \chi\Big(\sum_{j=m+1}^{m+n}\sum_{k=0}^{M(j,n)} A_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}\Big) + w_n, \qquad (1.11)_n$$

where $w_n \in S^{l(n+1)}(\mathbb{K}_\alpha)$ and

$$\Delta\Big(A_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}\Big) = (-\lambda_1^+ A_{j,k-1} - \mu C_{j-m,k-1} + D_{j-1,k-1}) r^{\frac{2j-2m-1}{2\beta}+2k-2} \sin\frac{(2j-2m-1)\theta}{2\beta}$$

for $m+1 \le j \le m+n$, $1 \le k \le M(j,n)$.

We prove this Claim by induction on n. Let us show that $(1.11)_n$ holds for $n = 0$. Note that $\chi\phi_0 \in S^1(\mathbb{K}_\alpha) \cap S^0(\mathbb{K}_\alpha)$. By induction, let us prove that $\chi\phi_0 \in S^j(\mathbb{K}_\alpha)$ for $j \le l(1)$. Let $1 \le k < l(1)$ and assume that $\chi\phi_0 \in S^k(\mathbb{K}_\alpha)$. Since $\chi\phi_0 \in S^0(\mathbb{K}_\alpha) \cap S^k(\mathbb{K}_\alpha)$, we have $h_0 \in S^0(\mathbb{K}_\alpha) \cap S^k(\mathbb{K}_\alpha)$. Combining this with Theorem 1.2, $(1.9)_0$, and the fact that $-\frac{\pi}{2\alpha} < 0 < 2k < \frac{\pi}{2\alpha}$, we obtain $\chi\phi_0 \in S^{k+1}(\mathbb{K}_\alpha)$. Hence we have $\chi\phi_0 \in S^{l(1)}(\mathbb{K}_\alpha)$.

Assume that $(1.11)_n$ is valid for some $n \in \mathbf{Z}_+$. Inserting $(1.11)_n$ into $(1.9)_n$, we obtain the equation for w_n:

$$\begin{cases} \Delta w_n = -\lambda_1^+ w_n + \tilde{h}_n \quad \text{in} \quad \mathbb{K}_\alpha, \\ w_n(\cdot,0) = 0 \quad \text{on} \quad \mathbf{R}_+, \quad \dfrac{\partial}{\partial\theta} w_n(\cdot,\alpha) = 0 \quad \text{on} \quad \mathbf{R}_+, \end{cases} \qquad (1.12)$$

where

$$\begin{aligned} \tilde{h}_n = &- \lambda_1^+ \chi \sum_{j=m+1}^{m+n} A_{j,M(j,n)} r^{\frac{2j-2m-1}{2\beta}+2M(j,n)} \sin\frac{(2j-2m-1)\theta}{2\beta} \\ &- \mu\chi\Big(\Psi_0 - \sum_{j=1}^{n}\sum_{k=0}^{M(j+m,n)-1} C_{j,k} r^{\frac{2j-1}{2\beta}+2k} \sin\frac{(2j-1)\theta}{2\beta}\Big) \\ &+ \chi\Big(g_n - \sum_{j=m}^{n+m-1}\sum_{k=0}^{M(j+1,n)-1} D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k} \sin\frac{(2j-2m+1)\theta}{2\beta}\Big) + 2\nabla\chi\cdot\nabla\phi_n + \phi_n\Delta\chi \\ &- (\Delta\chi) \sum_{j=m+1}^{m+n}\sum_{k=0}^{M(j,n)} A_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta} \\ &- 2\nabla\chi\cdot\nabla\Big(\sum_{j=m+1}^{m+n}\sum_{k=0}^{M(j,n)} A_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}\Big). \end{aligned}$$

Using (1.6) and (1.10), we have $\tilde{h}_n \in S^{l(n+1)}(\mathbb{K}_\alpha)$. So we get

$$-\lambda_1^+ w_n + \tilde{h}_n \in S^{l(n+1)}(\mathbb{K}_\alpha) \cap S^{l(n)}(\mathbb{K}_\alpha).$$

This together with Theorem 1.2 and (1.12) implies that w_n admits the representation

$$w_n = c_n r^{\frac{2n+1}{2\beta}} \sin\frac{(2n+1)\theta}{2\beta} + q_n, \quad q_n \in S^{l(n+1)+1}(\mathbb{K}_\alpha). \qquad (1.13)$$

Notice that the asymptotic expansion of $\tilde{h}_n(x)$ as $r \to 0$ is given by the formal power series

$$\begin{aligned} H_n = &- \lambda_1^+ \sum_{j=m+1}^{m+n} A_{j,M(j,n)} r^{\frac{2j-2m-1}{2\beta}+2M(j,n)} \sin\frac{(2j-2m-1)\theta}{2\beta} \\ &- \mu\Big(\sum_{j=1}^{n}\sum_{k=M(j+m,n)}^{\infty} + \sum_{j=n+1}^{n+1}\sum_{k=0}^{\infty}\Big) C_{j,k} r^{\frac{2j-1}{2\beta}+2k} \sin\frac{(2j-1)\theta}{2\beta} \\ &- \lambda_1^+ \sum_{j=0}^{m} E_{j,M(j,n)} r^{\frac{2j-2m-1}{2\beta}+2M(j,n)} \sin\frac{(2j-2m-1)\theta}{2\beta} \\ &+ \Big(\sum_{j=0}^{m-1}\sum_{k=M(j+1,n)}^{\infty} + \sum_{j=m}^{n+m-1}\sum_{k=M(j+1,n)}^{\infty} + \sum_{j=n+m}^{n+m}\sum_{k=0}^{\infty}\Big) D_{j,k} r^{\frac{2j-2m+1}{2\beta}+2k} \sin\frac{(2j-2m+1)\theta}{2\beta}. \end{aligned}$$

$$G=\sum_{k=0}^{\infty} g_{n,k} r^{\frac{2n+1}{2\beta}+2k} \sin\frac{(2n+1)\theta}{2\beta}+\sum_{j=0}^{m+n} \sum_{k=M(j,n)+1}^{\infty} B_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}$$

be the formal power series satisfying

$$g_{n,0}=c_n \tag{1.14}$$

and

$$(\Delta+\lambda_1^+)G=H_n. \tag{1.15}$$

By the construction of the formal power series G and Ψ, we have

$$B_{j,k}=E_{j,k} \quad \text{for} \quad 0\le j\le m, \quad k\ge M(j,n)+1. \tag{1.16}$$

We introduce the following partial sum of G.

$$\tilde{G}=\sum_{k=0}^{M(m+n+1,n+1)} g_{n,k} r^{\frac{2n+1}{2\beta}+2k} \sin\frac{(2n+1)\theta}{2\beta}+\sum_{j=0}^{m+n} \sum_{k=M(j,n)+1}^{M(j,n+1)} B_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k} \sin\frac{(2j-2m-1)\theta}{2\beta}.$$

We put

$$\tilde{q}_n=w_n-\chi\tilde{G}. \tag{1.17}$$

From (1.13) and (1.14), we have $\tilde{q}_n\in S^{l(n+1)+1}(\mathbb{K}_\alpha)$. Inserting $w_n=\tilde{q}_n+\chi\tilde{G}$ into the equation (1.12), we obtain

$$\begin{aligned}\Delta\tilde{q}_n&=-\lambda_1^+\tilde{q}_n+\tilde{h}_n-\chi(\Delta+\lambda_1^+)\tilde{G}-2\nabla\chi\cdot\nabla\tilde{G}-\tilde{G}\Delta\chi\\ &=:-\lambda_1^+\tilde{q}_n+k_n.\end{aligned} \tag{1.18}$$

From (1.6), (1.10), and (1.15), we have $k_n\in S^{l(n+2)}(\mathbb{K}_\alpha)$. By induction, let us show that $\tilde{q}_n\in S^j(\mathbb{K}_\alpha)$ for $l(n+1)+1\le j\le l(n+2)$. Assume that $\tilde{q}_n\in S^k(\mathbb{K}_\alpha)$ for some $l(n+1)+1\le k\le l(n+2)-1$. Since $-\lambda_1^+\tilde{q}_n+k_n\in S^{k-1}(\mathbb{K}_\alpha)\cap S^k(\mathbb{K}_\alpha)$ and since $\frac{(2n+1)\pi}{2\alpha}<2(k-1)<2k<\frac{(2n+3)\pi}{2\alpha}$, Theorem 1.2 and (1.18) imply that $\tilde{q}_n\in S^{k+1}(\mathbb{K}_\alpha)$. So we get $\tilde{q}_n\in S^{l(n+2)}(\mathbb{K}_\alpha)$. Notice that

$$\begin{aligned}\chi\phi_{n+1}&=\chi\phi_n+\chi^2(\Psi^n-\Psi^{n+1})\\ &=\chi\phi_n+\chi(\Psi^n-\Psi^{n+1})-\chi(1-\chi)(\Psi^n-\Psi^{n+1}).\end{aligned}$$

This together with $(1.11)_n$, (1.16), and (1.17) implies that

$$\begin{aligned}\chi\phi_{n+1}=&\chi\Big(\sum_{j=m+1}^{m+n}\sum_{k=0}^{M(j,n)} A_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k}\sin\frac{(2j-2m-1)\theta}{2\beta}+\sum_{k=0}^{M(m+n+1,n+1)} g_{n,k} r^{\frac{2n+1}{2\beta}+2k}\sin\frac{(2n+1)\theta}{2\beta}\\ &+\sum_{j=m+1}^{m+n}\sum_{k=M(j,n)+1}^{M(j,n+1)} B_{j,k} r^{\frac{2j-2m-1}{2\beta}+2k}\sin\frac{(2j-2m-1)\theta}{2\beta}\Big)+\tilde{q}_n-(1-\chi)\chi(\Psi^n-\Psi^{n+1}).\end{aligned}$$

Combining this with (1.15), we infer that $(1.11)_{n+1}$ is valid. Hence we obtain the assertion of the Claim.

Since $w_n\in S^{l(n+1)}(\mathbb{K}_\alpha)$, Sobolev's imbedding theorem implies that

$$w_n\in C^{2l(n+1)-3}(\overline{\mathbb{K}_\alpha})$$

and

$$|\partial_x^\delta w_n|\le C_\delta r^{2l(n+1)-3-|\delta|} \quad \text{on} \quad \mathbb{K}_\alpha \quad \text{for} \quad \delta\in\mathbb{Z}_+^2, \quad |\delta|\le 2l(n+1)-3.$$

This together with $(1.11)_n$ and (1.7) implies that $\varphi\in J_{m+1}^+$. □

Proof of (0.2) *and* (0.3). We omit the proof of (0.2) because it is easier than that of (1.3). It remains to prove (0.3). We extend $\Psi_0(r,\theta)$ to the function $\tilde{\Psi}_0(r,\theta)$ on $W = \{(r,\theta);\quad 0 < r < r_0,\quad 0 < \theta < 2\alpha\}$ by the formula

$$\tilde{\Psi}_0(r,\theta) = \begin{cases} \Psi_0(r,\theta) & \text{for} \quad 0 < \theta \le \alpha, \\ \Psi_0(r, 2\alpha - \theta) & \text{for} \quad \alpha \le \theta < 2\alpha. \end{cases}$$

Then we have $\tilde{\Psi}_0 \in H^1(W) \cap C^\infty(W \backslash \{0\})$, $\tilde{\Psi}_0 > 0$ on W, $\tilde{\Psi}_0(\cdot,0) = \tilde{\Psi}_0(\cdot, 2\alpha) = 0$ on $(0, r_0)$, and the asymptotic representation

$$\tilde{\Psi}_0(r,\theta) \sim \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} C_{j,k} r^{\frac{2j-1}{2\beta}+2k} \sin\frac{(2j-1)\theta}{2\beta} \quad \text{as} \quad r \to 0, \quad (r,\theta) \in W.$$

By mimicking the proof of [1, Proposition 19.2], we have (0.3). □

Next we look at the other outer equation. For $m \in \mathbf{Z}_+$, we define

$$\begin{aligned} J_m^- = \{&u \in C^\infty(\overline{\Omega_-} \backslash \{0, \gamma(t_0)\}); \\ &(1-\chi(r))u \in L^2(\Omega_-), \quad u = 0 \quad \text{on} \quad \partial\Omega \cap \partial\Omega_-, \quad \frac{\partial}{\partial n} u = 0 \quad \text{on} \quad \gamma((0,t_0)), \\ &\text{the function} \quad u(x) \quad \text{admits the asymptotic expansion of the form} \\ &u(x) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} D_{j,k} r^{\frac{2j-2m+1}{2(1-\beta)}+2k} \sin\frac{(2j-2m+1)(\pi-\theta)}{2(1-\beta)} \quad \text{as} \quad r \to 0, \quad x \in \Omega_-, \\ &\text{which can be differentiated term by term infinitely many times.}\} \end{aligned}$$

As in the proof of Lemma 1.3, we have the following claim.

LEMMA 1.4. *Let* $m \in \mathbf{Z}_+$, $f \in J_m^-$, *and* $\{a_j\}_{j=0}^{m-1} \subset \mathbf{R}$. *Then the equation*

$$\begin{cases} (\Delta + \lambda_1^+)\varphi = f & \text{in} \quad \Omega_-, \\ \qquad \varphi = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_-, \\ \quad \dfrac{\partial}{\partial n}\varphi = 0 & \text{on} \quad \gamma((0,t_0)), \\ (\varphi, \Psi_0)_{\Omega_-} = 0 \end{cases}$$

has a solution $\varphi \in J_m^-$ *which admits the asymptotic expansion*

$$\varphi(x) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} D_{j,k} r^{\frac{2j-2m+1}{2(1-\beta)}+2k} \sin\frac{(2j-2m+1)(\pi-\theta)}{2(1-\beta)} \quad \text{as} \quad r \to 0, \quad x \in \Omega_-$$

with

$$D_{j,0} = a_j \quad \text{for} \quad 0 \le j \le m-1.$$

2. Inner equations. Our first task in this section is to derive an explicit formula for a solution to the equation

$$\begin{cases} \Delta_\xi u(\xi) = f(\xi) \quad \text{in} \quad \Pi_\alpha^1, \\ u(\cdot, 0) = 0 \quad \text{on} \quad \mathbf{R}, \quad \dfrac{\partial}{\partial n_\pm} u(\xi) = 0 \quad \text{for} \quad \xi \in \partial\Pi_\alpha^1 \backslash (\mathbf{R} \times \{0\}). \end{cases} \tag{2.1}$$

For this purpose we use the technics of conformal maps. We identify $\mathbf{R}^2$ with $\mathbf{C}$ by the map $\mathbf{R}^2 \ni (x,y) \mapsto x + iy \in \mathbf{C}$. We put

$$P = \{z \in \mathbf{C};\quad \operatorname{Im} z > 0\}.$$

We have

$$\Pi_\alpha^1 = P \backslash \{re^{i\alpha};\quad r > 1\}.$$

It is readily seen that the Green function for the equation

$$\begin{cases} \Delta_z u(z) = g(z) \quad \text{in} \quad \mathbb{R}^2_+, \\ u(\cdot, 0) = 0 \quad \text{on} \quad \mathbb{R}_+, \quad \dfrac{\partial}{\partial z_1} u(0, \cdot) = 0 \quad \text{on} \quad \mathbb{R}_+ \end{cases}$$

is given by the formula

$$G(z, w) = \frac{1}{4\pi} \ln \frac{|z - w||z + \overline{w}|}{|z + w||z - \overline{w}|}.$$

Thus, it suffices to construct a conformal map Ψ from Π^1_α onto $\mathbb{R}^2_+$ which maps $\mathbb{R}\times\{0\}$ and $\partial\Pi^1_\alpha\backslash(\mathbb{R}\times\{0\})$ onto $\mathbb{R}_+ \times \{0\}$ and $\{0\} \times \mathbb{R}_+$, respectively.

We shall construct the conformal map Ψ by composing some elementary conformal maps. We first define the Schwarts-Christoffel map F by the formula

$$F(w) = \int_0^w \frac{z}{(z+1)^{1-\beta}(z - \frac{1-\beta}{\beta})^\beta} \, dz, \quad w \in P,$$

where $z^t = \exp(t \ln z)$ for $z \in \mathbb{C}\backslash\{0\}$ and the branch cut of $\ln z$ is $\mathbb{R}_+$. We put

$$T = -\int_{-1}^0 \frac{x}{(x+1)^{1-\beta}(\frac{1-\beta}{\beta} - x)^\beta} \, dx,$$

$$Q = \{z \in \mathbb{C};\quad \operatorname{Im} z > -T \sin\alpha\}\backslash\{re^{-i\alpha} \in \mathbb{C};\quad 0 < r < T\}.$$

Let us demonstrate the following claim.

PROPOSITION 2.1. *The function F is a conformal map from P onto Q which maps $\mathbb{R}\backslash(-1, \frac{1-\beta}{\beta})$ and $(-1, \frac{1-\beta}{\beta})$ onto $\{z \in \mathbb{C};\quad \operatorname{Im} z = -T \sin\alpha\}$ and $\{re^{-i\alpha} \in \mathbb{C};\quad 0 < r < T\}$, respectively.*

Proof. By the Schwartz-Christoffel theorem, we see that F maps P onto a polygon with vertices $F(-1)$, $F(0)$, and $F(\frac{1-\beta}{\beta})$. The angles of the polygon at the vertices $F(-1)$, $F(0)$, and $F(\frac{1-\beta}{\beta})$ are α, 2π, and $\pi - \alpha$, respectively. We have

$$\begin{aligned} F(\frac{1-\beta}{\beta}) - F(-1) &= \int_{-1}^{\frac{1-\beta}{\beta}} \frac{z}{(z+1)^{1-\beta}(z - \frac{1-\beta}{\beta})^\beta} \, dz \\ &= e^{-i\pi\beta} \int_{-1}^{\frac{1-\beta}{\beta}} \frac{x}{(x+1)^{1-\beta}(\frac{1-\beta}{\beta} - x)^\beta} \, dx \\ &= e^{-i\pi\beta} \int_0^{\frac{1}{\beta}} \frac{y-1}{y^{1-\beta}(\frac{1}{\beta} - y)^\beta} \, dy \\ &= e^{-i\pi\beta} \left[\int_0^{\frac{1}{\beta}} \frac{1}{y^{-\beta}(\frac{1}{\beta} - y)^\beta} \, dy - \int_0^{\frac{1}{\beta}} \frac{1}{y^{1-\beta}(\frac{1}{\beta} - y)^\beta} \, dy \right] \\ &= e^{-i\pi\beta} \left[\frac{1}{\beta} B(1+\beta, 1-\beta) - B(\beta, 1-\beta) \right] \\ &= 0, \end{aligned}$$

where $B(\cdot, \cdot)$ stands for the beta function. So we get

$$F(\frac{1-\beta}{\beta}) = F(-1) = e^{-i\alpha} T.$$

Since $F'(w) > 0$ on $\mathbb{R}\backslash[-1, \frac{1-\beta}{\beta}]$, $F(\mathbb{R}\backslash[-1, \frac{1-\beta}{\beta}])$ is a line parallel to the real axis. This completes the

Next we define the Möbius map S by the formula

$$S(z) = \frac{-T}{z - e^{-i\alpha}T}.$$

Then S is a conformal map from Q onto Π_α^1 which maps $\{z \in \mathbb{C};\quad \operatorname{Im} z = -T\sin\alpha\}$ and $\{re^{-i\alpha} \in \mathbb{C};\quad 0 < r < T\}$ onto $\mathbf{R} \times \{0\}$ and $\partial\Pi_\alpha^1 \backslash (\mathbf{R} \times \{0\})$, respectively. Finally, we define

$$K(z) = \sqrt{\frac{z+1}{z - \frac{1-\beta}{\beta}}} \quad \text{for} \quad z \in P.$$

Then K is a conformal map from P onto $\mathbf{R}_+^2$ which maps $\mathbf{R}\backslash(-1, \frac{1-\beta}{\beta})$ and $(-1, \frac{1-\beta}{\beta})$ onto $i\mathbf{R}_+$ and $\mathbf{R}_+$, respectively. Now, we define

$$\Psi = K \circ F^{-1} \circ S^{-1}.$$

Then the function Ψ is a conformal map from Π_α^1 onto $\mathbf{R}_+^2$ which maps $\mathbf{R} \times \{0\}$ and $\partial\Pi_\alpha^1 \backslash (\mathbf{R} \times \{0\})$ onto $\mathbf{R}_+ \times \{0\}$ and $\{0\} \times \mathbf{R}_+$, respectively. Thus a solution to the equation (2.1) is given by the formula

$$u(w) = \int_{\Pi_\alpha^1} G(\Psi(w), \Psi(\xi)) f(\xi)\, d\xi, \quad w \in \Pi_\alpha^1. \tag{2.2}$$

Thanks to this formula, we can get the asymptotic expansion of a solution to the equation (2.1). Let

$$\tau = \min\{1-\beta, \beta\}.$$

We define

$$H^1_{\text{comp}}(\Pi_\alpha^1) = \{u : \Pi_\alpha^1 \to \mathbf{R};\quad u \in H^1(A) \quad \text{for any non-void bounded open subset} \quad A \quad \text{of} \quad \Pi_\alpha^1\}.$$

PROPOSITION 2.2. *Let $N \in \mathbb{N}$ and $N \geq 2 + \frac{5}{\tau}$. Assume that $f \in L^\infty(\Pi_\alpha^1)$ and f is locally Lipschitz continuous in Π_α^1. We also suppose that f obeys the condition*

$$f(\xi) = \mathcal{O}(|\xi|^{-N}) \quad as \quad |\xi| \to \infty.$$

Then the function $u(w)$ from (2.2) admits the following asymptotic expansions as $|w| \to \infty$ which can be differentiated term by term one time:

$$u(w) = \sum_{j=1}^{M} c_j \rho^{-\frac{2j-1}{2\beta}} \sin\frac{(2j-1)\theta}{2\beta} + \mathcal{O}(\rho^{-\frac{2M+1}{2\beta}} + \rho^{-N+2}\ln\rho) \quad for \quad 0 < \theta < \alpha, \tag{2.3}$$

$$u(w) = \sum_{j=1}^{M} d_j \rho^{-\frac{2j-1}{2(1-\beta)}} \sin\frac{(2j-1)\theta}{2(1-\beta)} + \mathcal{O}(\rho^{-\frac{2M+1}{2(1-\beta)}} + \rho^{-N+2}\ln\rho) \quad for \quad \alpha < \theta < \pi, \tag{2.4}$$

where $(\rho, \theta) \in \mathbf{R}_+ \times (0, \pi)$ are the polar coordinates of w and $M = [\frac{\tau(N-2)-1}{2}] - 1\,(\geq 1)$. Moreover, we have $u \in H^1_{\text{comp}}(\Pi_\alpha^1) \cap C^2(\Pi_\alpha^1) \cap L^\infty(\Pi_\alpha^1)$ and $u|_{\Lambda_\pm} \in C^1(\overline{\Lambda_\pm} \backslash \{e^{i\alpha}\})$.

Proof. We have

$$4\pi G(z, \zeta) = \operatorname{Re}\,[\ln(1 - \frac{z}{\zeta}) + \ln(1 + \frac{z}{\bar{\zeta}}) - \ln(1 + \frac{z}{\zeta}) - \ln(1 - \frac{z}{\bar{\zeta}})].$$

Since

$$|\ln(1-t) + \sum_{j=1}^{n} \frac{t^j}{j}| \leq C_n |t|^{n+1} \quad \text{for} \quad t \in \mathbf{C}, \quad |t-1| \geq \frac{1}{2},$$

$$|t|^j \le C_j |\ln|1-t|| \quad \text{for} \quad t \in \mathbb{C}, \quad |t-1| \le \frac{1}{2},$$

we infer that the kernel $G(z,\zeta)$ admits the following expression:

$$G(z,\zeta) = \sum_{j=1}^{M} \frac{1}{(2j-1)\pi} \mathrm{Im}(\zeta^{-2j+1})\mathrm{Im}(z^{2j-1}) + H_M(z,\zeta), \tag{2.5}$$

$$|H_M(z,\zeta)| \le C_M |\zeta|^{-2M-1}|z|^{2M+1} \quad \text{for} \quad \zeta \in M_z := \Omega_z \cap \Omega_{-z} \cap \Omega_{\overline{z}} \cap \Omega_{-\overline{z}},$$

$$|H_M(z,\zeta)| \le C_M[|\ln|1-\frac{z}{\zeta}|| + |\ln|1+\frac{z}{\overline{\zeta}}|| + |\ln|1+\frac{z}{\zeta}|| + |\ln|1-\frac{z}{\overline{\zeta}}||] \quad \text{for} \quad \zeta \in M_z^c,$$

where

$$\Omega_z := \{\zeta \in \mathbb{C}; \quad |z-\zeta| \ge \frac{1}{2}|\zeta|\}$$
$$= \{\zeta \in \mathbb{C}; \quad |\zeta - \frac{4}{3}z| \ge \frac{2}{3}|z|\}.$$

Notice that $F^{-1}(z)$ admits the Puiseux series expansions

$$F^{-1}(z) = -1 + \sum_{j=1}^{\infty} p_j (z - e^{-i\alpha}T)^{\frac{j}{\beta}}, \quad \pi - \alpha < \arg(z - e^{-i\alpha}T) < \pi, \quad |z - e^{-i\alpha}T| < T,$$

$$F^{-1}(z) = \frac{1-\beta}{\beta} + \sum_{j=1}^{\infty} q_j (z - e^{-i\alpha}T)^{\frac{j}{1-\beta}}, \quad 0 < \arg(z - e^{-i\alpha}T) < \pi - \alpha, \quad |z - e^{-i\alpha}T| < T.$$

Combining these with $S^{-1}(w) = -\frac{T}{w} + Te^{-i\alpha}$ and $\Psi = K \circ F^{-1} \circ S^{-1}$, we claim that Ψ admits the Puiseux series expansions

$$\Psi(w) = \sum_{j=1}^{\infty} p_j (-\frac{T}{w})^{\frac{2j-1}{2\beta}} \quad |w| > 1, \quad 0 < \arg w < \alpha, \tag{2.6}$$

$$\Psi(w) = \sum_{j=0}^{\infty} q_j (-\frac{T}{w})^{\frac{2j-1}{2(1-\beta)}} \quad |w| > 1, \quad \alpha < \arg w < \pi. \tag{2.7}$$

Let $0 < \arg w < \alpha$ and $|w| > 1$. Using (2.5), (2.6), and (2.7), we express the kernel $G(\Psi(w),\Psi(\xi))$ as follows.

$$G(\Psi(w),\Psi(\xi)) = \sum_{j=1}^{M} K_j(\xi)\rho^{\frac{-2j+1}{2\beta}} \sin\frac{(2j-1)\theta}{2\beta} + L_M(w,\xi),$$

$$|K_j(\xi)| \le C_j(1+|\xi|)^{\frac{2j-1}{2\beta}},$$

$$|L_M(w,\xi)| \le C_M(1+|\xi|)^{\frac{2M+1}{2\beta}}|w|^{-\frac{2M+1}{2\beta}} \quad \text{for} \quad \xi \in \Psi^{-1}(M_{\Psi(w)}), \tag{2.8}$$

$$|L_M(w,\xi)| \le C_M[|\ln|1-\frac{\Psi(w)}{\Psi(\xi)}|| + |\ln|1+\frac{\Psi(w)}{\overline{\Psi(\xi)}}|| + |\ln|1+\frac{\Psi(w)}{\Psi(\xi)}|| + |\ln|1-\frac{\Psi(w)}{\overline{\Psi(\xi)}}||] \tag{2.9}$$

for $\xi \in \Psi^{-1}(M^c_{\Psi(w)})$. From (2.2), we obtain

$$u(w) = \sum_{j=1}^{M} c_j \rho^{\frac{-2j+1}{2\beta}} \sin\frac{(2j-1)\theta}{2\beta} + \int_{\Psi^{-1}(M_{\Psi(w)})} L_M(w,\xi)f(\xi)\,d\xi + \int_{\Psi^{-1}(M^c_{\Psi(w)})} L_M(w,\xi)f(\xi)\,d\xi, \tag{2.10}$$

where $c_j = \int_{\mathbf{R}^2_+} K_j(\xi) f(\xi)\, d\xi$. It follows from (2.8) that

$$|\int_{\Psi^{-1}(M_{\Psi(w)})} L_M(w,\xi) f(\xi)\, d\xi| \le C\rho^{-\frac{2M+1}{2\beta}} \int_{\mathbf{R}^2_+} (1+|\xi|)^{\frac{2M+1}{\beta}-N}\, d\xi. \tag{2.11}$$

Since $M = [\frac{\beta(N-2)-1}{2}] - 1$, we have $\int_{\mathbf{R}^2_+} (1+|\xi|)^{\frac{M}{\beta}-N}\, d\xi < \infty$. From (2.9), we get

$$\begin{aligned} &|\int_{\Psi^{-1}(M^c_{\Psi(w)})} L_M(w,\xi) f(\xi)\, d\xi| \\ \le& C\int_{M^c_{\Psi(w)}} [|\ln|1-\frac{\Psi(w)}{y}|| + |\ln|1+\frac{\Psi(w)}{\overline{y}}|| + |\ln|1+\frac{\Psi(w)}{y}|| + |\ln|1-\frac{\Psi(w)}{\overline{y}}||] \\ &\times |f(\Psi^{-1}(y))||(\Psi^{-1})'(y)|^2\, dy. \end{aligned}$$

Since $M^c_{\Psi(w)} \subset \{\zeta \in \mathbf{C};\quad |\zeta| \le 2|\Psi(w)|\}$ and $|\Psi(w)| \le K|w|^{-\frac{1}{2\beta}}$, we have

$$\begin{aligned} &|\int_{\Psi^{-1}(M^c_{\Psi(w)})} L_M(w,\xi) f(\xi)\, d\xi| \\ \le& C\int_{D(0,K\rho^{-\frac{1}{2\beta}})} [|\ln|1-\frac{\Psi(w)}{y}|| + |\ln|1+\frac{\Psi(w)}{\overline{y}}|| + |\ln|1+\frac{\Psi(w)}{y}|| + |\ln|1-\frac{\Psi(w)}{\overline{y}}||] \\ &\times |y|^{2\beta N}|y|^{2(-2\beta-1)}\, dy \\ \le& C\rho^{-N-\frac{-2\beta-1}{\beta}} \int_{D(0,K\rho^{-\frac{1}{2\beta}})} [|\ln|y-\Psi(w)|| + |\ln|\overline{y}+\Psi(w)|| + |\ln|y+\Psi(w)|| \\ &+ |\ln|\overline{y}-\Psi(w)|| + 4|\ln|y||]\, dy. \end{aligned}$$

Since $\int_{D(0,K\rho^{-\frac{1}{2\beta}})} |\ln|y||\, dy = O(\rho^{-\frac{1}{\beta}}\ln\rho)$, we get

$$|\int_{\Psi^{-1}(\Omega^c_{\Psi(w)})} L_M(w,\xi) f(\xi)\, d\xi| = O(\rho^{-N+2}\ln\rho).$$

Thus we obtain

$$u(w) = \sum_{j=1}^{M} c_j \rho^{-\frac{2j-1}{2\beta}} \sin\tfrac{(2j-1)\theta}{2\beta} + O(\rho^{-\frac{2M+1}{2\beta}} + \rho^{-N+2}\ln\rho) \quad \text{for} \quad \arg w \in (0,\alpha).$$

Applying a similar method to the derivatives of u, we arrive at (2.3). The proof of (2.4) is similar to that of (2.3).

From (2.2) we have

$$u(\Psi^{-1}(p)) = \int_{\mathbf{R}^2_+} G(p,q) f(\Psi^{-1}(q))|(\Psi^{-1})'(q)|^2\, dq.$$

Since $f(\Psi^{-1}(q))|(\Psi^{-1})'(q)|^2 = O(|q|^{-2N(1-\beta)+2(1-2\beta)})$ as $|q| \to \infty$ and since $f(\Psi^{-1}(\cdot))|(\Psi^{-1})'(\cdot)|^2$ is bounded and locally Lipschitz continuous in $\mathbf{R}^2_+$, we claim from the regularity theorem for the Newtonian potential (see [4, Lemmas 4.1 and 4.2]) that $u(\Psi^{-1}(\cdot)) \in C^1(\overline{\mathbf{R}^2_+}) \cap C^2(\mathbf{R}^2_+)$. This implies that $u \in H^1_{\mathrm{comp}}(\Pi^1_\alpha) \cap C^2(\Pi^1_\alpha) \cap L^\infty(\Pi^1_\alpha)$ and $u|_{\Lambda_\pm} \in C^1(\overline{\Lambda_\pm}\setminus\{e^{i\alpha}\})$. □

Finally we introduce harmonic functions in Π^1_α which we need in the sequel. For $j \in \mathbf{Z}_+$, we put

$$V_j^+(\eta) = \mathrm{Im}(\eta^{-1-2j}),$$

$$V_j^-(\eta) = \mathrm{Im}(\eta^{1+2j}).$$

We immediately see that

$$\Delta V_j^{\pm} = 0 \quad \text{in} \quad \mathbb{R}_+^2,$$

$$V_j^{\pm}(\cdot, 0) = 0 \quad \text{on} \quad \mathbb{R}_+, \quad \frac{\partial}{\partial \eta_1} V_j^{\pm}(0, \cdot) = 0 \quad \text{on} \quad \mathbb{R}_+.$$

We define

$$Y_j^{\pm}(\xi) = V_j^{\pm}(\Psi(\xi)).$$

Since Ψ is a conformal map from Π_α^1 onto $\mathbb{R}_+^2$ which maps $\mathbb{R} \times \{0\}$ and $\partial \Pi_\alpha^1 \backslash (\mathbb{R} \times \{0\})$ onto $\mathbb{R}_+ \times \{0\}$ and $\{0\} \times \mathbb{R}_+$, respectively, we have

$$\Delta Y_j^{\pm} = 0 \quad \text{in} \quad \Pi_\alpha^1,$$

$$Y_j^{\pm}(\cdot, 0) = 0 \quad \text{on} \quad \mathbb{R}, \quad \frac{\partial}{\partial n} Y_j^{\pm}(\xi) = 0 \quad \text{for} \quad \xi \in \partial \Pi_\alpha^1 \backslash (\mathbb{R} \times \{0\}).$$

We put

$$\Lambda_+ = \mathbb{K}_\alpha,$$

$$\Lambda_- = \{(\rho \cos \theta, \rho \cos \theta) \in \mathbb{R}^2; \quad \rho > 0, \quad \alpha < \theta < \pi\}.$$

By a straightforward computation, we infer that the function $Y_j^{\pm}(\xi)$ admits the following power series expansions for $\rho > 1$ which can be differentiated term by tern infinitely many times.

$$Y_j^{+}(\xi) = \begin{cases} \sum_{k=0}^{\infty} A_{j,k}^{+} \rho^{\frac{2j-2k+1}{2\beta}} \sin \frac{(2j-2k+1)\theta}{2\beta} & \text{for} \quad \xi \in \Lambda_+, \\ \sum_{k=0}^{\infty} B_{j,k}^{+} \rho^{\frac{-2k-1}{2(1-\beta)}} \sin \frac{(-2k-1)(\pi-\theta)}{2(1-\beta)} & \text{for} \quad \xi \in \Lambda_-, \end{cases}$$

$$Y_j^{-}(\xi) = \begin{cases} \sum_{k=0}^{\infty} A_{j,k}^{-} \rho^{\frac{2j-2k+1}{2(1-\beta)}} \sin \frac{(2j-2k+1)(\pi-\theta)}{2(1-\beta)} & \text{for} \quad \xi \in \Lambda_-, \\ \sum_{k=0}^{\infty} B_{j,k}^{-} \rho^{\frac{-2k-1}{2\beta}} \sin \frac{(-2k-1)\theta}{2\beta} & \text{for} \quad \xi \in \Lambda_+, \end{cases}$$

where

$$A_{j,0}^{+} = -\beta^{-\frac{1+2j}{2\beta}} T^{-\frac{1+2j}{2\beta}},$$

$$A_{j,0}^{-} = -\beta^{-\frac{1+2j}{2(1-\beta)}} T^{-\frac{1+2j}{2(1-\beta)}},$$

$$A_{0,1}^{+} = -\frac{1}{2} \beta^{\frac{1-2\beta}{2\beta}} T^{\frac{1}{2\beta}}.$$

It is convenient to normalize the functions $Y_j^{\pm}(\xi)$ $(j \geq 0)$. We inductively define harmonic functions $X_j^{\pm}(\xi)$ $(j \in \mathbb{Z}_+)$ by the formulae

$$X_0^{\pm}(\xi) = (A_{0,0}^{\pm})^{-1} Y_0^{\pm}(\xi),$$

$$X_j^{\pm}(\xi) = (A_{j,0}^{\pm})^{-1} (Y_j^{\pm}(\xi) - \sum_{k=1}^{j} A_{j,k}^{\pm} X_{j-k}^{\pm}(\xi)) \quad \text{for} \quad j \geq 1.$$

Then $X_j^{\pm}(\xi)$ admits the following power series expansions for $\rho > 1$.

$$X_j^{+}(\xi) = \begin{cases} \rho^{\frac{2j+1}{2\beta}} \sin \frac{(2j+1)\theta}{2\beta} + \sum_{k=0}^{\infty} \tilde{A}_{j,k}^{+} \rho^{\frac{-1-2k}{2\beta}} \sin \frac{(-1-2k)\theta}{2\beta} & \text{for} \quad \xi \in \Lambda_+, \\ \sum_{k=0}^{\infty} \tilde{B}_{j,k}^{+} \rho^{\frac{-2k-1}{2(1-\beta)}} \sin \frac{(-2k-1)(\pi-\theta)}{2(1-\beta)} & \text{for} \quad \xi \in \Lambda_-, \end{cases}$$

$$X_j^{-}(\xi) = \begin{cases} \rho^{\frac{2j+1}{2(1-\beta)}} \sin \frac{(2j+1)\theta}{2(1-\beta)} + \sum_{k=0}^{\infty} \tilde{A}_{j,k}^{-} \rho^{\frac{-1-2k}{2(1-\beta)}} \sin \frac{(-1-2k)(\pi-\theta)}{2(1-\beta)} & \text{for} \quad \xi \in \Lambda_-, \\ \sum_{k=0}^{\infty} \tilde{B}_{j,k}^{-} \rho^{\frac{-2k-1}{2\beta}} \sin \frac{(-2k-1)\theta}{2\beta} & \text{for} \quad \xi \in \Lambda_+, \end{cases}$$

where

$$\tilde{A}_{0,0} = \frac{1}{2} \beta^{\frac{1-\beta}{\beta}} T^{\frac{1}{\beta}}.$$

3. Matching procedure. In this section, we are mainly aimed to prove the following theorem.

THEOREM 3.1. *There exist* $\{\Psi^+_{j,k,l}\}_{j\geq 1,k\geq 0,l\geq 0}$, $\{\Psi^-_{j,k,l}\}_{j\geq 1,k\geq 1,l\geq 0}$, $\{v_{j,k,l}\}_{j\geq 1,k\geq 0,l\geq 0}$, *and* $\{\lambda_{m,n,p}\}_{m\geq 1,n\geq 0,p\geq 0}$ *satisfying* (0.10), (0.11), (0.12), *and the conditions below.*

(i) *The functions* $\Psi^+_{j,k,l}\in J^+_j$, $\Psi^-_{j,k,l}\in J^-_k$, *and* $v_{j,k,l}$ *admit the following asymptotic expansions.*

$$\Psi^+_{l,m,n}(x)\sim\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}C^+_{l,m,n,j,k}r^{\frac{2j-2l+1}{2\beta}+2k}\sin\frac{(2j-2l+1)\theta}{2\beta}\quad as\quad r\to 0,\quad x\in\Omega_+. \qquad (3.1)_{l,m,n}$$

$$\Psi^-_{l,m,n}(x)\sim\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}C^-_{l,m,n,j,k}r^{\frac{2j-2m+1}{2(1-\beta)}+2k}\sin\frac{(2j-2m+1)(\pi-\theta)}{2(1-\beta)}\quad as\quad r\to 0,\quad x\in\Omega_-. \qquad (3.2)_{l,m,n}$$

$$\begin{aligned}v_{l,0,n}(\xi)\sim&C_{l.n}\rho^{\frac{2l-1}{2\beta}+2n}\sin\frac{(2l-1)\theta}{2\beta}\\&+\sum_{k=0}^{\infty}\sum_{s=0}^{n}K^+_{l,0,n,k,s}\rho^{\frac{2l-2k-3}{2\beta}+2s}\sin\frac{(2l-2k-3)\theta}{2\beta}\quad as\quad\rho\to\infty,\quad\xi\in\Lambda_+,\end{aligned} \qquad (3.3)_{l,0,n}$$

$$v_{l,0,n}(\xi)\sim\sum_{k=0}^{\infty}\sum_{s=0}^{n}K^-_{l,0,n,k,s}\rho^{\frac{-2k-1}{2(1-\beta)}+2s}\sin\frac{(-2k-1)(\pi-\theta)}{2(1-\beta)}\quad as\quad\rho\to\infty,\quad\xi\in\Lambda_-. \qquad (3.4)_{l,0,n}$$

For $m\neq 0$, *we have*

$$v_{l,m,n}(\xi)\sim\sum_{k=0}^{\infty}\sum_{s=0}^{n}K^+_{l,m,n,k,s}\rho^{\frac{2l-2k-3}{2\beta}+2s}\sin\frac{(2l-2k-3)\theta}{2\beta}\quad as\quad\rho\to\infty,\quad\xi\in\Lambda_+, \qquad (3.5)_{l,m,n}$$

$$v_{l,m,n}(\xi)\sim\sum_{k=0}^{\infty}\sum_{s=0}^{n}K^-_{l,m,n,k,s}\rho^{\frac{2m-2k-1}{2(1-\beta)}+2s}\sin\frac{(2m-2k-1)(\pi-\theta)}{2(1-\beta)}\quad as\quad\rho\to\infty,\quad\xi\in\Lambda_-. \qquad (3.6)_{l,m,n}$$

The above asymptotic expansion for $\Psi^{\pm}_{j,k,l}$ *can be differentiated term by term arbitrary times and that for* $v_{j,k,l}$ *can be differentiated term by term one time. Moreover we have* $v_{j,k,l}\in H^1_{\mathrm{comp}}(\Pi^1_\alpha)\cap C^2(\Pi^1_\alpha)$ *and* $v_{j,k,l}|_{\Lambda_\pm}\in C^1(\overline{\Lambda_\pm}\backslash\{e^{i\alpha}\})$.

(ii) *For* $l\geq 1$, $m\geq 0$, $n\geq 0$, $k\geq 0$, *and* $0\leq s\leq n$, *the matching conditions hold:*

$$K^+_{l,m,n,k,s}=C^+_{k+1,m,n-s,l-1,s} \qquad (3.7)_{l,m,n,k,s}$$

$$K^-_{l,m,n,k,s}=C^-_{l,k+1,n-s,m,s}. \qquad (3.8)_{l,m,n,k,s}$$

This theorem immediately follows form the following lemma and induction.

LEMMA 3.2. *Let* $L+1$, $K+1$, $J\in\mathbf{Z}_+$. *Assume that there exist sequences*

$$\begin{array}{lll}\{\Psi^+_{j,k,l}\}_{j\geq 1,\,k\geq 0,\,0\leq l\leq L}, & \{\Psi^+_{j,k,L+1}\}_{0\leq k\leq K,\,j\geq 1}, & \{\Psi^+_{j,K+1,L+1}\}_{1\leq j\leq J},\\ \{\Psi^-_{j,k,l}\}_{j\geq 1,\,k\geq 1,\,0\leq l\leq L}, & \{\Psi^-_{j,k,L+1}\}_{1\leq k\leq K+1,\,j\geq 1}, & \{\Psi^-_{j,K+2,L+1}\}_{1\leq j\leq J},\\ \{v_{j,k,l}\}_{j\geq 1,\,k\geq 0,\,0\leq l\leq L}, & \{v_{j,k,L+1}\}_{0\leq k\leq K,\,j\geq 1}, & \{v_{j,K+1,L+1}\}_{1\leq j\leq J},\\ \{\lambda_{j,k,l}\}_{j\geq 1,\,k\geq 0,\,0\leq l\leq L}, & \{\lambda_{j,k,L+1}\}_{0\leq k\leq K,\,j\geq 1}, & \{\lambda_{j,K+1,L+1}\}_{1\leq j\leq J}\end{array}$$

which satisfy (0.10), (0.11), (0.12), (i) *in Theorem 3.1, and the conditions*

$(3.7)_{l,m,n,k,s}$ *for* $l\geq 1$, $m\geq 0$, $0\leq n\leq L$, $k\geq 0$, $0\leq s\leq n$,
$(3.7)_{l,m,L+1,k,s}$ *for* $l\geq 1$, $0\leq m\leq K$, $k\geq 0$, $0\leq s\leq L+1$,
$(3.7)_{l,K+1,L+1,k,0}$ *for* $1\leq l\leq J$, $0\leq k\leq J-1$,
$(3.7)_{l,K+1,L+1,k,s}$ *for* $1\leq l\leq J$, $0\leq k$, $1\leq s\leq L+1$,
$(3.8)_{l,m,n,k,s}$ *for* $l\geq 1$, $m\geq 0$, $0\leq n\leq L$, $k\geq 0$, $0\leq s\leq n$,
$(3.8)_{l,m,L+1,k,s}$ *for* $l\geq 1$, $0\leq m\leq K$, $k\geq 0$, $1\leq s\leq L+1$,
$(3.8)_{l,m,L+1,k,0}$ *for* $l\geq 1$, $0\leq m\leq K$, $0\leq k\leq K$,
$(3.8)_{l,K+1,L+1,k,s}$ *for* $1\leq l\leq J$, $k\geq 0$, $1\leq s\leq L+1$,
$(3.8)_{l,m,L+1,K+1,0}$ *for* $1\leq l\leq J$, $0\leq m\leq K$,
$(3.8)_{l,K+1,L+1,k,0}$ *for* $1\leq l\leq J$, $0\leq k\leq K+1$.

Then there exist $\Psi^+_{J+1,K+1,L+1}$, $\Psi^-_{J+1,K+2,L+1}$, $v_{J+1,K+1,L+1}$, *and* $\lambda_{J+1,K+1,L+1}$ *satisfying* (0.10), (0.11), (0.12), (i) *in Theorem 3.1, and the conditions*

$$\begin{array}{ll}(3.7)_{l,K+1,L+1,k,s} & \textit{for}\ \ 1\le l\le J+1,\ 0\le k,\ 1\le s\le L+1,\\ (3.7)_{l,K+1,L+1,k,0} & \textit{for}\ \ 1\le l\le J+1,\ 0\le k\le J,\\ (3.8)_{l,K+1,L+1,k,s} & \textit{for}\ \ 1\le l\le J+1,\ k\ge 0,\ 1\le s\le L+1,\\ (3.8)_{l,K+1,L+1,k,0} & \textit{for}\ \ 1\le l\le J+1,\ 0\le k\le K+1,\\ (3.8)_{l,m,L+1,K+1,0} & \textit{for}\ \ 1\le l\le J+1,\ 0\le m\le K.\end{array}$$

Proof. We first construct $v_{J+1,K+1,L+1}$ which is a solution to the equation

$$\Delta_\xi v_{J+1,K+1,L+1}=-\lambda_1^+ v_{J+1,K+1,L}-\sum_{m=1}^{J}\sum_{n=0}^{K+1}\sum_{p=0}^{L}\lambda_{m,n,p}v_{J+1-m,K+1-n,L-p}\quad\text{in}\quad \Pi^1_\alpha,\tag{3.9}$$

$$v_{J+1,K+1,L+1}(\cdot,0)=0\quad\text{on}\quad\mathbb{R},\qquad \frac{\partial}{\partial n_\pm}v_{J+1,K+1,L+1}(\xi)=0\quad\text{for}\quad \xi\in\partial\Pi^1_\alpha\backslash(\mathbb{R}\times\{0\}).$$

By $H(\xi)$ we denote the right side of (3.9). The function $H(\xi)$ admits the asymptotic expansions

$$H(\xi)\sim\sum_{k=0}^{\infty}\sum_{s=0}^{L}H^+_{k,s}\rho^{\frac{2J-2k-1}{2\beta}+2s}\sin\tfrac{2J-2k-1}{2\beta}\theta\quad\text{as}\quad\rho\to\infty,\quad \xi\in\Lambda_+,\tag{3.10}$$

$$H(\xi)\sim\sum_{k=0}^{\infty}\sum_{s=0}^{L}H^-_{k,s}\rho^{\frac{2K-2k-1}{2(1-\beta)}+2s}\sin\tfrac{2K-2k-1}{2(1-\beta)}(\pi-\theta)\quad\text{as}\quad\rho\to\infty,\quad \xi\in\Lambda_-,\tag{3.11}$$

where

$$H^+_{k,s}=-\lambda_1^+K^+_{J+1,K+1,L,k,s}-\lambda_{k+1,0,L-s}C_{J-k,s}-\sum_{m=1}^{\min\{J,k\}}\sum_{n=0}^{K+1}\sum_{p=0}^{L-s}\lambda_{m,n,p}K^+_{J+1-m,K+1-n,L-p,k-m,s},$$

$$H^-_{k,s}=-\lambda_1^+K^-_{J+1,K+1,L,k,s}-\sum_{m=1}^{J}\sum_{n=0}^{\min\{K+1,k\}}\sum_{p=0}^{L-s}\lambda_{m,n,p}K^-_{J+1-m,K+1-n,L-p,k-n,s}.$$

By $H^+(\xi)$ and $H^-(\xi)$ we denote the formal power series on the right sides of (3.10) and (3.11), respectively. Let

$$L^+(\xi)=\sum_{k=0}^{\infty}\sum_{s=1}^{L+1}L^+_{k,s}\rho^{\frac{2J-2k-1}{2\beta}+2s}\sin\tfrac{2J-2k-1}{2\beta}\theta$$

and

$$L^-(\xi)=\sum_{k=0}^{\infty}\sum_{s=1}^{L+1}L^-_{k,s}\rho^{\frac{2K-2k+1}{2(1-\beta)}+2s}\sin\tfrac{2K-2k+1}{2(1-\beta)}(\pi-\theta)$$

be formal power series which satisfy $\Delta L^\pm(\xi)=H^\pm(\xi)$. We put

$$L^+_N(\xi)=\begin{cases}\sum_{k=0}^{N}\sum_{s=1}^{L+1}L^+_{k,s}\rho^{\frac{2J-2k-1}{2\beta}+2s}\sin\frac{2J-2k-1}{2\beta}\theta & \text{on}\quad\Lambda^+,\\ 0 & \text{on}\quad\Lambda^-,\end{cases}$$

$$L^-_N(\xi)=\begin{cases}\sum_{k=0}^{N}\sum_{s=1}^{L+1}L^-_{k,s}\rho^{\frac{2K-2k+1}{2(1-\beta)}+2s}\sin\frac{2K-2k+1}{2(1-\beta)}(\pi-\theta) & \text{on}\quad\Lambda^-,\\ 0 & \text{on}\quad\Lambda^+.\end{cases}$$

We choose $\chi_0\in C^\infty[0,\infty)$ such that

$$\chi_0=0\quad\text{on}\quad[0,2],\qquad\chi_0=1\quad\text{on}\quad[3,\infty).$$

We seek a solution $\tilde{v}_N$ to the equation (3.9) which takes the form

$$\tilde{v}_N = \chi_0(\rho)L_N^+(\xi) + \chi_0(\rho)L_N^-(\xi) + w_N. \tag{3.12}$$

Inserting this into the equation (3.9), we derive the equation for w_N:

$$\Delta w_N = H_N \quad \text{in} \quad \Pi_\alpha^1, \tag{3.13}$$

$$w_N(\cdot,0) = 0 \quad \text{on} \quad \mathbf{R}, \quad \frac{\partial}{\partial n_\pm} w_N(\xi) = 0 \quad \text{for} \quad \xi \in \partial\Pi_\alpha^1 \backslash (\mathbf{R}\times\{0\}),$$

where

$$H_N(\xi) = H(\xi) - \chi_0(\rho)\Delta_\xi(L_N^+(\xi)+L_N^-(\xi)) - 2\nabla_\xi\chi_0(\rho)\cdot\nabla_\xi(L_N^+(\xi)+L_N^-(\xi)) - 2(L_N^+(\xi)+L_N^-(\xi))\Delta_\xi\chi_0(\rho).$$

We have

$$H_N(\xi) \sim \begin{cases} \displaystyle\sum_{k=N+1}^{\infty}\sum_{s=0}^{L} H_{k,s}^+ \rho^{\frac{2J-2k-1}{2\beta}+2s} \sin\frac{2J-2k-1}{2\beta}\theta & \text{as} \quad \rho\to\infty, \quad \xi\in\Lambda_+, \\ \displaystyle\sum_{k=N+1}^{\infty}\sum_{s=0}^{L} H_{k,s}^- \rho^{\frac{2K-2k+1}{2(1-\beta)}+2s} \sin\frac{(2K-2k+1)(\pi-\theta)}{2(1-\beta)} & \text{as} \quad \rho\to\infty, \quad \xi\in\Lambda_-. \end{cases}$$

From Proposition 2.2, we infer that the equation (3.13) has a solution w_N which admits the asymptotic expansions

$$w_N(\xi) = \begin{cases} \displaystyle\sum_{j=1}^{M(N)} A_{j,k}^+ \rho^{\frac{1-2j}{2\beta}} \sin\frac{1-2j}{2\beta}\theta + O(\rho^{\frac{-1-2M(N)}{2\beta}}) & \text{as} \quad \rho\to\infty, \quad \xi\in\Lambda_+, \\ \displaystyle\sum_{j=1}^{M(N)} A_{j,k}^- \rho^{\frac{1-2j}{2(1-\beta)}} \sin\frac{(1-2j)(\pi-\theta)}{2(1-\beta)} + O(\rho^{\frac{-1-2M(N)}{2(1-\beta)}}) & \text{as} \quad \rho\to\infty, \quad \xi\in\Lambda_-. \end{cases}$$

Next we shall show that the function $\tilde{v}_N$ from (3.12) is independent of the choice of N. We get

$$\Delta(\tilde{v}_N - \tilde{v}_M) = 0 \quad \text{in} \quad \Pi_\alpha^1,$$

$$(\tilde{v}_N - \tilde{v}_M)(\cdot,0) = 0 \quad \text{on} \quad \mathbf{R}_+, \quad \frac{\partial}{\partial n_\pm}(\tilde{v}_N - \tilde{v}_M)(\xi) = 0 \quad \text{for} \quad \xi\in\partial\Pi_\alpha^1\backslash(\mathbf{R}\times\{0\}).$$

Since $\tilde{v}_N - \tilde{v}_M$ is bounded in Π_α^1, we have $\tilde{v}_N - \tilde{v}_M = 0$. Thus the function $\tilde{v}_N$ is independent of the choice of N, which we denote by $\tilde{v}_{J+1,K+1,L+1}$. We define

$$v_{J+1,K+1,L+1}(\xi) = \tilde{v}_{J+1,K+1,L+1}(\xi) + \sum_{k=0}^{J-1} C_{k+1,K+1,L+1,J,0}^+ X_k^+(\xi) + \sum_{k=0}^{K} C_{J+1,k+1,L+1,K+1,0}^- X_k^-(\xi).$$

Then $v_{J+1,K+1,L+1}(\xi)$ admits the asymptotic expansion $(3.5)_{J+1,K+1,L+1}$ and $(3.6)_{J+1,K+1,L+1}$. Besides, $(3.7)_{J+1,K+1,L+1,k,0}$ holds for $0\le k\le J-1$ and $(3.8)_{J+1,K+1,L+1,k,0}$ holds for $0\le k\le K$.

We shall prove that $(3.7)_{J+1,K+1,L+1,k,s}$ holds for $k\ge 0$, $1\le s\le L+1$. Identifying the coefficients of $\rho^{\frac{2J-2k-1}{2\beta}+2(s-1)}\sin\frac{2J-2k-1}{2\beta}\theta$ in the asymptotic expansions of the both sides of (3.9) as $\rho\to\infty$, $\xi\in\Lambda_+$, we get

$$\begin{aligned} &\Delta(K_{J+1,K+1,L+1,k,s}^+ \rho^{\frac{2J-2k-1}{2\beta}+2s}\sin\frac{2J-2k-1}{2\beta}\theta) \\ &= -\lambda_1^+ K_{J+1,K+1,L,k,s-1}^+ \rho^{\frac{2J-2k-1}{2\beta}+2(s-1)}\sin\frac{2J-2k-1}{2\beta}\theta \\ &\quad - \sum_{m=1}^{J}\sum_{n=0}^{K+1}\sum_{p=0}^{L} \lambda_{m,n,p} K_{J+1-m,K+1-n,L-p,k-m,s-1}^+ \rho^{\frac{2J-2k-1}{2\beta}+2(s-1)}\sin\frac{2J-2k-1}{2\beta}\theta. \end{aligned} \tag{3.14}$$

Note that the function $\Psi^+_{k+1,K+1,L+1-s}$ solves the equation

$$\begin{aligned}(\Delta+\lambda_1^+)&\Psi^+_{k+1,K+1,L+1-s}\\ =&-\lambda_{k+1,K+1,L+1-s}\Psi_0-\sum_{m=1}^{k}\sum_{n=0}^{K+1}\sum_{p=0}^{L+1-s}\lambda_{m,n,p}\Psi^+_{k+1-m,K+1-n,L+1-s-p}\quad\text{in}\quad\Omega_+.\end{aligned}\tag{3.15}$$

Equating the coefficients of $r^{\frac{2J-2k-1}{2\beta}+2(s-1)}\sin\frac{2J-2k-1}{2\beta}\theta$ in the asymptotic expansions of the both sides of (3.15) as $r\to 0$, we get

$$\begin{aligned}\Delta(&C^+_{k+1,K+1,L+1-s,J,s}r^{\frac{2J-2k-1}{2\beta}+2s}\sin\tfrac{2J-2k-1}{2\beta}\theta)\\ =&(-\lambda_1^+C^+_{k+1,K+1,L+1-s,J,s-1}-\sum_{m=1}^{k}\sum_{n=0}^{K+1}\sum_{p=0}^{L+1-s}\lambda_{m,n,p}C^+_{k+1-m,K+1-n,L+1-s-p,J-m,s-1})\\ &\times r^{\frac{2J-2k-1}{2\beta}+2(s-1)}\sin\tfrac{2J-2k-1}{2\beta}\theta.\end{aligned}\tag{3.16}$$

Since $(3.7)_{l,m,n,k',s'}$ holds for $l\geq 1$, $m\geq 0$, $0\leq n\leq L$, $k'\geq 0$, $0\leq s'\leq n$, it follows from (3.14) and (3.16) that

$$\Delta_x((K^+_{J+1,K+1,L+1,k,s}-C^+_{k+1,K+1,L+1-s,J,s})r^{\frac{2J-2k-1}{2\beta}+2s}\sin\tfrac{2J-2k-1}{2\beta}\theta)=0.$$

This implies that $(3.7)_{J+1,K+1,L+1,k,s}$ holds for $k\geq 0$, $1\leq s\leq L+1$. In a similar manner, we infer that $(3.8)_{J+1,K+1,L+1,k,s}$ holds for $k\geq 0$, $1\leq s\leq L+1$.

Next we shall construct $\Psi^+_{J+1,K+1,L+1}$, $\lambda_{J+1,K+1,L+1}$, and $\Psi^-_{J+1,K+2,L+1}$. It follows from Lemma 1.3 that there exist $\Psi^+_{J+1,K+1,L+1}\in J^+_{J+1}$ and $\lambda_{J+1,K+1,L+1}\in\mathbb{R}$ which satisfy $(0.10)_{J+1,K+1,L+1}$, $(3.1)_{J+1,K+1,L+1}$, and $(3.7)_{j+1,K+1,L+1,J,0}$ for $0\leq j\leq J$. From Lemma 1.4, we claim that there exists $\Psi^-_{J+1,K+2,L+1}\in J^-_{K+2}$ satisfying $(0.11)_{J+1,K+2,L+1}$, $(3.2)_{J+1,K+2,L+1}$, and $(3.8)_{J+1,j,L+1,K+1,0}$ for $0\leq j\leq K+1$. This completes the proof of Theorem 3.1. □

LEMMA 3.3. *The number* $\lambda_{1,0,0}$ *is given by the formula* (0.5).

Proof. The procedure in the proof of Lemma 3.2 with $(L,K,J)=(-1,-1,0)$ shows that

$$v_{1,0,0}(\xi)=C_{1,0}X_0^+(\xi).$$

It follows from $(3.3)_{1,0,0}$ and $(3.7)_{1,0,0,0,0}$ that

$$C^+_{1,0,0,0,0}=K^+_{1,0,0,0,0}=C_{1,0}\tilde{A}_{0,0}.$$

Since $(\Delta+\lambda_1^+)\Psi^+_{1,0,0}(x)=-\lambda_{1,0,0}\Psi_0(x)$ in Ω_+, we have

$$\begin{aligned}\lambda_{1,0,0}=&-\lim_{\delta\to 0}\int_{\Omega_+\setminus D(0,\delta)}\Psi_0(x)(\Delta+\lambda_1^+)\Psi^+_{1,0,0}(x)\,dx\\ =&-\lim_{\delta\to 0}\int_0^\alpha(\Psi_0(\delta,\theta)\frac{\partial}{\partial r}\Psi^+_{1,0,0}(\delta,\theta)-\Psi^+_{1,0,0}(\delta,\theta)\frac{\partial}{\partial r}\Psi_0(\delta,\theta))\delta\,d\theta\\ =&\frac{\pi}{2}C^+_{1,0,0,0,0}C_{1,0}\\ =&\frac{\pi}{2}C^2_{1,0}\tilde{A}_{0,0},\end{aligned}$$

where we used an integration by parts in the second line and we used (0.2) and $(3.1)_{1,0,0}$ in the third line. □

Proof of Theorem 0.1. Let $N\in\mathbb{N}$. We define the approximate eigenvector of L_ϵ by the formula

$$\begin{aligned}\Phi^N_\epsilon(x)=&(1-\chi(\epsilon^{-1/2}r))(\Psi_0(x)+\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\Psi^+_{j,k,l}(x)+\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l}\Psi^-_{j,k,l}(x))\\ &+\chi(\epsilon^{-1/2}r)\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2l}v_{j,k,l}(\xi).\end{aligned}$$

We immediately obtain $\Phi^N_\epsilon(x)\in\mathcal{D}(L_\epsilon)$ from the following claim.

CLAIM. *Let $n_\pm$ be the interior unit normal to $\partial\Omega_\pm$. Assume that $f \in Q_\epsilon \cap C^2(\Omega_\epsilon)$, $\Delta f \in L^2(\Omega)$, $f|_{\Omega_\pm} \in C^1(\overline{\Omega_\pm}\backslash\{\gamma(\epsilon),\gamma(t_0)\})$, $f=0$ on $\partial\Omega$, and $\frac{\partial f}{\partial n_\pm}(x)=0$ for $x \in \gamma((\epsilon,t_0))$. Then we have $f \in \mathcal{D}(L_\epsilon)$.*

We first prove this claim. Let $u \in Q_\epsilon$. The standard mollifier technique shows that there exist two sequences $\{v_j^+\}_{j=1}^\infty \subset C^\infty(\overline{\Omega_+})$ and $\{v_j^-\}_{j=1}^\infty \subset C^\infty(\overline{\Omega_-})$ such that $v_j^\pm \to u|_{\Omega_\pm}$ in $H^1(\Omega_\pm)$ as $j\to\infty$, $v_j^+ = v_j^-$ on $\gamma((0,\epsilon))$, and $v_j^\pm = 0$ on $\partial\Omega_\pm \cap \partial\Omega$. Combining these with the density argument in the proof of [8, Proposition D.1], we claim that there exist two sequences $\{u_j^+\}_{j=1}^\infty \subset C^\infty(\overline{\Omega_+})$ and $\{u_j^-\}_{j=1}^\infty \subset C^\infty(\overline{\Omega_-})$ such that $u_j^\pm \to u|_{\Omega_\pm}$ in $H^1(\Omega_\pm)$ as $j\to\infty$, $u_j^+ = u_j^-$ on $\gamma((0,\epsilon))$, $u_j^\pm = 0$ on $\partial\Omega_\pm \cap \partial\Omega$, and $u_j^\pm$ vanish on an open covering of $\{\gamma(\epsilon),\gamma(t_0)\}$. Pick $\delta_j > 0$ such that $u^\pm = 0$ on $\overline{D(\gamma(\epsilon),\delta_j)} \cup \overline{D(\gamma(t_0),\delta_j)}$. We put $\Omega_\pm^j = \Omega_\pm \backslash \overline{D(\gamma(\epsilon),\delta_j)} \cup \overline{D(\gamma(t_0),\delta_j)}$. We obtain

$$
\begin{aligned}
&(\nabla f, \nabla u)_{L^2(\Omega)}\\
&= \lim_{j\to\infty} \int_{\Omega_+^j} \nabla f \cdot \nabla u_j^+ \, dx + \lim_{j\to\infty} \int_{\Omega_-^j} \nabla f \cdot \nabla u_j^- \, dx
\end{aligned}
$$

by using $f \in C^1(\overline{\Omega_\pm^j}) \cap C^2(\Omega_\pm^j)$, $\Delta f \in L^2(\Omega)$, $u_j^\pm \in C^\infty(\overline{\Omega_\pm})$, and Green's theorem

$$
= \lim_{j\to\infty} \Big(-\int_{\partial\Omega_+^j} u_j^+ \frac{\partial}{\partial n_+} f \, dS - \int_{\Omega_+} u_j^+ \Delta f \, dx - \int_{\partial\Omega_-^j} u_j^- \frac{\partial}{\partial n_-} f \, dS - \int_{\Omega_-} u_j^- \Delta f \, dx\Big).
$$

We conclude that

$$
\int_{\partial\Omega_+^j} u_j^+ \frac{\partial}{\partial n_+} f \, dS + \int_{\partial\Omega_-^j} u_j^- \frac{\partial}{\partial n_-} f \, dS = 0
$$

because $\frac{\partial}{\partial n_\pm} f = 0$ on $\gamma((\epsilon,t_0))$, $u_j^\pm = 0$ on $\partial\Omega \cap \partial\Omega_\pm$, $u_j^\pm = 0$ on $\overline{D(\gamma(\epsilon),\delta_j)} \cup \overline{D(\gamma(t_0),\delta_j)}$, and $u_j^+ \frac{\partial}{\partial n_+} f + u_j^- \frac{\partial}{\partial n_-} f = 0$ on $\gamma((0,\epsilon-\delta_j))$. Hence we obtain

$$
(\nabla f, \nabla u)_{L^2(\Omega)} = -(u, \Delta f)_{L^2(\Omega)} \quad \text{for all} \quad u \in Q_\epsilon.
$$

Thus we get the assertion of the Claim.

Next we shall show that there exist $P > 0$ and $Q \in \mathbf{R}$ such that the estimate

$$
\|(\Delta_x + \lambda_1^+ + \sum_{m=1}^N \sum_{n=0}^N \sum_{p=0}^N \lambda_{m,n,p} \epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p}) \Phi_\epsilon^N(x)\|_{L^2(\Omega)} = \mathcal{O}(\epsilon^{PN+Q}) \quad \text{as} \quad \epsilon \to 0 \tag{3.17}
$$

holds for all $N \in \mathbf{N}$. Using (0.10), (0.11), and (0.12), we obtain

$$
(\Delta_x + \lambda_1^+ + \sum_{m=1}^N \sum_{n=0}^N \sum_{p=0}^N \epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p}) \Phi_\epsilon^N(x) = I_{\epsilon,1} + I_{\epsilon,2} + I_{\epsilon,3} + I_{\epsilon,4},
$$

where

$$
\begin{aligned}
I_{\epsilon,1} =& \chi(\epsilon^{-1/2} r)[\lambda_1^+ \sum_{j=1}^N \sum_{k=0}^N \epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2N} v_{j,k,N}(\xi)\\
&+ \sum_{(j,k,l)\in T_1} \epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2(l-1)} \sum_{\substack{\max\{j-N,1\}\le m\le \min\{N,j-1\}\\ \max\{k-N,0\}\le n\le\min\{N,k\}\\ \max\{l-N-1,0\}\le p\le \min\{N,l-1\}}} \lambda_{m,n,p} v_{j-m,k-n,l-p-1}(\xi)],
\end{aligned}
$$

$$
\begin{aligned}
I_{\epsilon,2} =& (1-\chi(\epsilon^{-1/2} r))\\
&\times [\sum_{(j,k,l)\in T_2} \epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l} \sum_{\substack{\max\{j-N,1\}\le m\le \min\{N,j-1\}\\ \max\{k-N,0\}\le n\le\min\{N,k\}\\ \max\{l-N,0\}\le p\le \min\{N,l\}}} \lambda_{m,n,p} \Psi^+_{j-m,k-n,l-p}(x)\\
&+ \sum_{(j,k,l)\in T_2} \epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l} \sum_{\substack{\max\{j-N,1\}\le m\le \min\{N,j-1\}\\ \max\{k-N,0\}\le n\le\min\{N,k-1\}\\ \max\{l-N,0\}\le p\le \min\{N,l\}}} \lambda_{m,n,p} \Psi^-_{j-m,k-n,l-p}(x)],
\end{aligned}
$$

$$\begin{aligned}I_{\epsilon,3} =&2\epsilon^{-1/2}(\nabla\chi)(\epsilon^{-1/2}r)\cdot\nabla_x[\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2l}v_{j,k,l}(\xi)-\Psi_0(x)\\&-\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\Psi^+_{j,k,l}(x)-\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l}\Psi^-_{j,k,l}(x)],\end{aligned}$$

$$\begin{aligned}I_{\epsilon,4} =&\epsilon^{-1}(\Delta\chi)(\epsilon^{-1/2}r)[\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2l}v_{j,k,l}(\xi)-\Psi_0(x)\\&-\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\Psi^+_{j,k,l}(x)-\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l}\Psi^-_{j,k,l}(x)],\end{aligned}$$

$$\begin{aligned}T_1 =&\{(j,k,l)\in\mathbb{Z}^3;\quad 2\le j\le 2N,\quad 0\le k\le 2N,\quad 1\le l\le 2N+1\}\\&\setminus\{(j,k,l)\in\mathbb{Z}^3;\quad 2\le j\le N,\quad 0\le k\le N,\quad 1\le l\le N\},\end{aligned}$$

$$\begin{aligned}T_2 =&\{(j,k,l)\in\mathbb{Z}^3;\quad 2\le j\le 2N,\quad 1\le k\le 2N,\quad 0\le l\le 2N\}\\&\setminus\{(j,k,l)\in\mathbb{Z}^3;\quad 2\le j\le N,\quad 1\le k\le N,\quad 0\le l\le N\}.\end{aligned}$$

So it suffices to show that, for $j = 1, 2, 3, 4$, there exist $P_j > 0$ and $Q_j > 0$ such that the estimate

$$\|I_{\epsilon,j}\|_{L^2(\Omega)} = \mathcal{O}(\epsilon^{P_jN+Q_j}) \quad\text{as}\quad \epsilon\to 0 \tag{3.18$_j$}$$

holds for all $N\in\mathbb{N}$.

We first estimate $I_{\epsilon,1}$. By (i) of Theorem 3.1 we have

$$|v_{l,m,n}(\xi)|\le C(1+\rho^{\frac{2l-1}{2\beta}}+\rho^{\frac{2m-1}{2(1-\beta)}})\quad\text{in}\quad\Pi^1_\alpha.$$

Using this estimate, we have, for $1\le j\le N$ and $0\le k\le N$,

$$\begin{aligned}\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2N}\|\chi(\epsilon^{-1/2}r)v_{j,k,N}(\xi)\|_{L^2(\Omega)} \le& C\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2N}[\int_{|x|\le 2\epsilon^{1/2}}(1+|\epsilon^{-1}x|^{\frac{2j-1}{\beta}}+|\epsilon^{-1}x|^{\frac{2k-1}{1-\beta}})\,dx]^{1/2}\\ \le& C\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2N}(\epsilon^{1/8}+\epsilon^{-\frac{2j-1}{4\beta}+\frac{1}{2}}+\epsilon^{-\frac{2k-1}{4(1-\beta)}+\frac{1}{2}})\\ \le& C\epsilon^{2N+\frac{1}{8}+\frac{1}{4\beta}}.\end{aligned}$$

Similarly, we obtain, for $(j,k,l)\in T_1$, $m\ge 1$, and $n\ge 0$,

$$\begin{aligned}&\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2(l-1)}\|\chi(\epsilon^{-1/2}r)v_{j-m,k-n,l-p-1}(\xi)\|_{L^2(\Omega)}\\ \le& C\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2(l-1)}(\epsilon^{1/8}+\epsilon^{-\frac{2(j-m)-1}{4\beta}+\frac{1}{2}}+\epsilon^{-\frac{2(k-n)-1}{4(1-\beta)}+\frac{1}{2}})\end{aligned}$$

since $m\ge 1$ and $n\ge 0$

$$\begin{aligned}\le& C\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2(l-1)}(\epsilon^{1/8}+\epsilon^{-\frac{2j-3}{4\beta}+\frac{1}{2}}+\epsilon^{-\frac{2k-1}{4(1-\beta)}+\frac{1}{2}})\\ \le& C\epsilon^{\frac{2j-1}{4\beta}+\frac{k}{2(1-\beta)}+2(l-1)+\frac{1}{8}}\\ \le& C\epsilon^{\min\{\frac{1}{2\beta},\frac{1}{2(1-\beta)},2\}(j+k+l)-\frac{1}{4\beta}+\frac{1}{8}}\end{aligned}$$

since $j+k+l\ge N+1$

$$\le C\epsilon^{\min\{\frac{1}{2\beta},\frac{1}{2(1-\beta)},2\}(N+1)-\frac{1}{4\beta}+\frac{1}{8}}.$$

Hence we obtain $(3.18)_1$.

Next we estimate $I_{\epsilon,2}$. By (i) of Theorem 3.1 we have

$$|\Psi^+_{l,m,n}(x)| \le C r^{\frac{-2l+1}{2\beta}} \quad \text{on} \quad \Omega_+ \cap D(0,r_0),$$

and hence

$$\|(1-\chi(\epsilon^{-1/2}r))\Psi^+_{l,m,n}(x)\|_{L^2(\Omega_+)} \le C(1+\epsilon^{\frac{-2l+1}{4\beta}+\frac{1}{2}}).$$

Using this estimate, we have, for $(j,k,l) \in T_2$ and $m \ge 1$,

$$\epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\|(1-\chi(\epsilon^{-1/2}r))\Psi^+_{j-m,k-n,l-p}\|_{L^2(\Omega_+)} \le C\epsilon^{(N+1)\min\{\frac{1}{2\beta},\frac{1}{1-\beta},2\}}.$$

In a similar fashion, we have, for $(j,k,l) \in T_2$ and $n \ge 0$,

$$\epsilon^{\frac{2j-1}{2\beta}+\frac{2k-1}{2(1-\beta)}+2l}\|(1-\chi(\epsilon^{-1/2}r))\Psi^-_{j-m,k-n,l-p}\|_{L^2(\Omega_-)} \le C\epsilon^{(N+1)\min\{\frac{1}{\beta},\frac{1}{2(1-\beta)},2\}-\frac{1}{2\beta}-\frac{1}{4(1-\beta)}}.$$

Thus $(3.18)_2$ holds.

Next we estimate $I_{\epsilon,4}$. It follows from (ii) of Theorem 3.1 that

$$I_{\epsilon,4} = \epsilon^{-1}(\Delta\chi)(\epsilon^{-1/2}r)[\sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{2j-1}{2\beta}+\frac{k}{1-\beta}+2l}\tilde{v}^+_{j,k,l}(\xi) - \tilde{\Psi}_0(x) - \sum_{j=1}^{N}\sum_{k=0}^{N}\sum_{l=0}^{N}\epsilon^{\frac{j}{\beta}+\frac{k}{1-\beta}+2l}\tilde{\Psi}^+_{j,k,l}(x)]$$

on Ω_+, where

$$\tilde{v}^+_{j,k,l}(\xi) := v_{j,k,l}(\xi) - \sum_{p=0}^{N-1}\sum_{s=0}^{l}K^+_{j,k,l,p,s}\rho^{\frac{2j-2p-3}{2\beta}+2s}\sin\frac{(2j-2p-3)\theta}{2\beta} \quad \text{for} \quad k \ne 0,$$

$$\tilde{v}^+_{j,0,l}(\xi) := v_{j,0,l}(\xi) - C_{j,l}\rho^{\frac{2j-1}{2\beta}+2l}\sin\frac{(2j-1)\theta}{2\beta} - \sum_{p=0}^{N-1}\sum_{s=0}^{l}K^+_{j,0,l,p,s}\rho^{\frac{2j-2p-3}{2\beta}+2s}\sin\frac{(2j-2p-3)\theta}{2\beta},$$

$$\tilde{\Psi}_0(x) := \Psi_0(x) - \sum_{j=1}^{N}\sum_{l=0}^{N}C_{j,l}r^{\frac{2j-1}{2\beta}+2l}\sin\frac{(2j-1)\theta}{2\beta},$$

$$\tilde{\Psi}^+_{j,k,l}(x) := \Psi^+_{j,k,l}(x) - \sum_{p=0}^{N-1}\sum_{s=0}^{N-l}C^+_{j,k,l,p,s}r^{\frac{2p-2j+1}{2\beta}+2s}\sin\frac{(2p-2j+1)\theta}{2\beta}.$$

By (i) of Theorem 3.1 we have

$$|\tilde{v}_{j,k,l}(\xi)| \le C(1+\rho^{\frac{2j-2N-3}{2\beta}+2l}) \quad \text{on} \quad \Lambda_+,$$

$$|\tilde{\Psi}_0(x)| \le C r^{(N+1)\min\{2,\frac{2}{\beta}\}-\frac{1}{\beta}} \quad \text{on} \quad \Omega_+ \cap D(0,r_0),$$

$$|\tilde{\Psi}^+_{j,k,l}(x)| \le C r^{\frac{-2j+1}{2\beta}+(N-l)\min\{\frac{1}{\beta},2\}} \quad \text{on} \quad \Omega_+ \cap D(0,r_0).$$

Using these estimates, we have

$$\|I_{\epsilon,4}\|_{L^2(\Omega_+)} \le C\epsilon^{\frac{1}{2}N\min\{\frac{1}{\beta},2\}+\frac{1}{2}-\frac{1}{2\beta}}.$$

Similarly, we get

$$\|I_{\epsilon,4}\|_{L^2(\Omega_-)} \le C\epsilon^{\frac{N}{2}\min\{\frac{1}{1-\beta},2\}+\frac{1}{2}+\frac{1}{2\beta}}.$$

Therefore $(3.18)_4$ holds.

The proof of $(3.18)_3$ is similar to that of $(3.18)_3$. Thus we conclude that the estimate (3.17) holds It is readily seen that

$$\|\Phi^N_\epsilon\|_{L^2(\Omega)} = 1 + o(1) \quad \text{as} \quad \epsilon \to 0.$$

Combining this with (3.17) and the fact that $\Phi_\epsilon^N \in \mathcal{D}(L_\epsilon)$, we get

$$\begin{aligned}
&\mathrm{dist}(\sigma(L_\epsilon),\ \lambda_1^+ + \sum_{m=1}^{N}\sum_{n=0}^{N}\sum_{p=0}^{N}\lambda_{m,n,p}\epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p}) \\
\le&\|(\Delta_x + \lambda_1^+ + \sum_{m=1}^{N}\sum_{n=0}^{N}\sum_{p=0}^{N}\lambda_{m,n,p}\epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p})\Phi_\epsilon^N(x)\|_{L^2(\Omega)}/\|\Phi_\epsilon^N(x)\|_{L^2(\Omega)} \\
=&\mathcal{O}(\epsilon^{PN+Q}).
\end{aligned} \tag{3.19}$$

On the other hand, we have $\lambda_2(\epsilon) \geq \min\{\lambda_2^+, \lambda_1^-\} > \lambda_1^+$ for $\epsilon \in (0, t_0]$, because $Q_\epsilon \subset Q^+ \oplus Q^-$. This together with the estimate (3.19) implies that

$$\lambda_1^+(\epsilon) = \lambda_1^+ + \sum_{m=1}^{N}\sum_{n=0}^{N}\sum_{p=0}^{N}\lambda_{m,n,p}\epsilon^{\frac{m}{\beta}+\frac{n}{1-\beta}+2p} + \mathcal{O}(\epsilon^{PN+Q}) \quad \text{as} \quad \epsilon \to 0.$$

The proof is complete. □

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