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ON THE MOMENT MATRIX $E(n)$

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1 Introduction and preliminaries

Consider a collection of complex numbers

$$\gamma \equiv \gamma^{(2n)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \cdots, \gamma_{0,2n}, \gamma_{1,2n-1}, \cdots, \gamma_{2n-1,1}, \gamma_{2n,0},$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$. The *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu(z) \qquad (0 \leq i + j \leq 2n); \tag{1.1}$$

μ is called a *representing measure* for γ. This truncated complex moment problem has been well-established ([CuF1], [CuF2], [JLLL]).

We recall first some notation from [CuF1] and [CuF2]. For $n \geq 1$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \cdots, Z^n, \bar{Z}Z^{n-1}, \cdots, \bar{Z}^{n-1}Z, \bar{Z}^n.$$

We define $M(n) := M(n)(\gamma) \in \mathcal{M}_m(\mathbb{C})$ as follows: for $0 \leq k + l \leq n, 0 \leq i + j \leq n$, the entry in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ is $M(n)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$. These matrices come from Bram-Halmos characterization for a cyclic operator T satisfying $\gamma_{ij} = (T^{*i}T^j x_0, x_0)$, where x_0 is a cyclic vector for T (cf. [Br] or [Con]). So it is nature to consider moment matrices corresponded by Embry characterization for subnormality of such operators (cf. [Em]. or [Con]). We will write such matrices by $E(n)$.

Consider a collection of complex numbers

$$\gamma \equiv \{\gamma_{ij}\}(0 \le i+j \le 2n, |i-j| \le n) \text{ with } \gamma_{00} > 0 \text{ and } \gamma_{ji} = \bar{\gamma}_{ij}.$$

For $n \in \mathbb{N}$, let

$$m = m[n] = \left(\left[\frac{n}{2}\right]+1\right)\left(\left[\frac{n+1}{2}\right]+1\right).$$

For $A \in M_m(\mathbb{C})$, we first introduce the following order on the rows and columns of A : $1, Z, Z^2, \bar{Z}Z, Z^3, \bar{Z}Z^2, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, Z^5, \cdots$. We denote the entry of A in row $\bar{Z}^kZ^l$ and column $\bar{Z}^iZ^j$ by $A_{(k,l)(i,j)}$. If $n = 2k$, $k = 1, 2, \cdots$, let

$$\mathcal{SP}_n = \{p(z,\bar{z}) = a_{00} + a_{01}z + a_{02}z^2 + a_{11}\bar{z}z + a_{03}z^3 + a_{12}\bar{z}z^2 + \cdots + a_{kk}\bar{z}^kz^k\};$$

if $n = 2k+1, k = 0, 1, 2, \cdots$, let

$$\mathcal{SP}_n = \{p(z,\bar{z}) = a_{00}+a_{01}z+a_{02}z^2+a_{11}\bar{z}z+a_{03}z^3+a_{12}\bar{z}z^2+\cdots+a_{k,k+1}\bar{z}^kz^{k+1}\},$$

where $a_{ij} \in \mathbb{C}$. It is clear that $\mathcal{SP}_n$ is a subspace of $\mathcal{P}_n$, the vector space of all complex polynomials in $z, \bar{z}$ of total degree $\le n$. For $p \in \mathcal{SP}_n$, let $\widehat{p} = [a_{00}, a_{01}, \cdots, a_{kk}]^T$ (which means the transposed) or $[a_{00}, a_{01}, \cdots, a_{k,k+1}]^T$ in $\mathbb{C}^m$. We define a sesquilinear form $\langle \cdot, \cdot \rangle_A$ on $\mathcal{SP}_n$ by $\langle p, q\rangle_A := \langle A\widehat{p}, \widehat{q}\rangle$ $(p, q \in \mathcal{SP}_n)$. In particular, $\langle \bar{z}^iz^j, \bar{z}^kz^l\rangle_A = A_{(k,l)(i,j)}$, for $0 \le i+j \le n, i \le j$ and $0 \le k+l \le n, k \le l$. For γ, we define the moment matrix $E(n) \equiv E(n)(\gamma) \in M_m(\mathbb{C})$ as follows: $E(n)_{(k,l)(i,j)} := \gamma_{l+i,j+k}$.

The following provides a motivation to study the truncated moment theory of $E(n)$, whose proof can be found in [JKLP].

Theorem 1.1. *Let S be a contractive subnormal operator with a cyclic vector x_0 in $\mathcal{H}$ and let $\gamma_{ij} = (S^{*i}S^jx_0, x_0)$. Then the following assertions are equivalent:*

(i) $M(n) \ge 0$ *for any* $n \in \mathbb{N}$;

(ii) $E(n) \ge 0$, *for any* $n \in \mathbb{N}$;

(iii) *there exists a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that*

$$\gamma_{ij} = \int_{\mathbb{D}} \bar{z}^iz^jd\mu(z) \qquad \text{for any } i,j \in \mathbb{N}\cup\{0\},$$

where $\mathbb{D}$ is the closed unit disc in $\mathbb{C}$.

We may give the following conjecture, as in [CuF1].

Conjecture 1.2. *Let* $\gamma \equiv \{\gamma_{ij}\}(0 \le i+j \le 2n, |i-j| \le n)$ *be a truncated moment sequence. The following statements are equivalent:*

(i) γ *has a rank* $E(n)$*-atomic representing measure;*

(ii) $E(n) \ge 0$ *and* $E(n)$ *admits a flat extension* $E(n+1)$.

In this article, we will consider the conjecture concretely and give the double flat extension theorem.

2 Moment matrices $E(n)$ and representing measures

If μ is the representing measure for γ, then $\langle E(n)\widehat{p}, \widehat{p}\rangle = \int |p(z,\bar{z})|^2 \, d\mu$, for $p(z,\bar{z}) \in \mathcal{SP}_n$. Hence $E(n) \ge 0$. But the converse implication is not always true (see Example 2.2 below). We first introduce an analogous statement with that of $M(n)$. For $p \in \mathcal{SP}_n$, let $\mathcal{Z}(p) = \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$.

Lemma 2.1. ([JKLP]) *Let* $\gamma \equiv \{\gamma_{ij}\}(0 \le i+j \le 2n, |i-j| \le n)$. *Assume that* γ *has a representing measure* μ. *For* $p \in \mathcal{SP}_n$, supp $\mu \subseteq \mathcal{Z}(p) \Longleftrightarrow p(Z,\bar{Z}) = 0$.

Example 2.2. Consider

$$M := E(3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1+i & 0 \\ 0 & 1 & 0 & 0 & 1+i & 2 \\ 0 & 0 & 2 & 1-i & 0 & 0 \\ 1 & 0 & 1+i & 2 & 1+i & 0 \\ 1-i & 1-i & 0 & 1-i & 4 & 2(1-i) \\ 0 & 2 & 0 & 0 & 2(1+i) & 4 \end{bmatrix}.$$

It is easy to show that $E(3)$ is positive and rank $E(3) = 3$. In fact, $\det([M]_4) = \det([M]_5) = \det M = 0$, where $[M]_k$ is the left upper $k \times k$ submatrix. Furthermore, we have

$$\begin{aligned} \bar{Z}Z &= \mathit{1} + \frac{1-i}{2} Z^2, \\ \bar{Z}Z^2 &= 2Z, \\ Z^3 &= (1+i)\mathit{1} + (1+i)Z. \end{aligned}$$

$$\begin{aligned} p_1(z,\bar{z}) &= 1+\frac{1-i}{2}z^2-\bar{z}z, \\ p_2(z,\bar{z}) &= 2z-\bar{z}z^2, \\ p_3(z,\bar{z}) &= (1+i)+(1+i)z-z^3. \end{aligned}$$

Then $\mathcal{Z}(p_1,p_2,p_3) = \{z \in \mathbb{C} : p_i(z,\bar{z}) = 0,\ i = 1,2,3\} = \varnothing$. Thus for the given moment sequence γ in $E(3)$, there is no representing measure for γ.

Theorem 2.3. ([CuF1]) *If* $\gamma \equiv \{\gamma_{ij}\}(0 \le i+j \le 2n)$ *is flat and* $M(n) \geq 0$, *then* $M(n)$ *admits a unique flat extension of the form* $M(n+1)$.

The above theorem produces Conjecture 1.2.

We showed this conjecture is true in the case of even numbers [JKLP]. We can provide a counter example for Conjecture 1.2 in the case $n = 3$.

Example 2.4. (Example 2.2 revisited) Since rank $E(2) =$ rank $E(3) = 3$, $E(3)$ is flat. If $E(3)$ admits a flat extension $E(4)$, then

$$Z^4 = (1+i)Z+(1+i)Z^2, \qquad \bar{Z}Z^3 = 2Z^2. \tag{2.1}$$

From the first equality of (2.1), we obtain $\gamma_{34} = 2$, and from the second equality of (2.1), we obtain $\gamma_{34} = 0$. Hence $E(3)$ has no flat extension of $E(4)$.

So we have the following theorems in sharpness whose proof can be found in [JKLP].

Theorem 2.5. *Let* $n \geq 2$. *If* γ *is double flat* (*i.e.*, rank $E(n) =$ rank $E(n-2)$) *and* $E(n) \geq 0$, *then* $E(n)$ *admits a unique flat extension of the form* $E(n+1)$.

Theorem 2.6. *The truncated complex moment sequence* $\gamma \equiv \{\gamma_{ij}\}(0 \le i+j \le 2n, |i-j| \le n)$ *has a* rank $E(n)$*-atomic representing measure if and only if* $E(n) \geq 0$ *and* $E(n)$ *admits a double flat extension* $E(n+2)$, *i.e.*, rank $E(n) =$ rank $E(n+2)$.

Finally, we give the following example that affirm Theorem 2.6.

Example 2.7. Let

$$E(2) = \begin{bmatrix} 1 & 0 & i & 1 \\ 0 & 1 & 1+i & 1-i \\ -i & 1-i & 3 & -3i \\ 1 & 1+i & 3i & 3 \end{bmatrix}.$$

Then $E(2)$ admits a double flat extension $E(4)$ as the following

$$E(4)=\begin{bmatrix} E(3) & B^* \\ B & C \end{bmatrix},$$

where

$$E(3) = \begin{bmatrix} 1 & 0 & i & 1 & i-1 & 1+i \\ 0 & 1 & 1+i & 1-i & 3i & 3 \\ -i & 1-i & 3 & -3i & 4+4i & 4-4i \\ 1 & 1+i & 3i & 3 & -4+4i & 4+4i \\ -1-i & -3i & 4-4i & -4-4i & 11 & -11i \\ 1-i & 3 & 4+4i & 4-4i & 11i & 11 \end{bmatrix},$$

$$B = \begin{bmatrix} -3 & -4-4i & -11i & -11 & 15(1-i) & -15(1+i) \\ -3i & 4-4i & 11 & -11i & 15(1+i) & 15(1-i) \\ 3 & 4+4i & 11i & 11 & -15(1-i) & 15(1+i) \end{bmatrix},$$

$$C = \begin{bmatrix} 41 & -41i & -41 \\ 41i & 41 & -41i \\ -41 & 41i & 41 \end{bmatrix}.$$

In fact, rank $E(2) =$ rank $E(4) = 2$. Since

$$\begin{cases} z^2 = i + (1+i)z, \\ \bar{z}z = 1 + (1-i)z, \end{cases}$$

we obtain two atoms $z_0 = (1-\sqrt{3})(1+i)/2$ and $z_1 = (1+\sqrt{3})(1+i)/2$. According to

$$\begin{bmatrix} 1 & 1 \\ z_0 & z_1 \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} = \begin{bmatrix} \gamma_{00} \\ \gamma_{01} \end{bmatrix},$$

we have $\rho_0 = \frac{1+\sqrt{3}}{2\sqrt{3}}$, $\rho_1 = \frac{-1+\sqrt{3}}{2\sqrt{3}}$. Thus we obtain the representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$.

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