数理解析研究所講究録 1259 巻 2002 年 71-78

Monotonicity of Sequences of Operator Means

福岡教育大学　内山　充　(Mitsuru Uchiyama)
Department of Mathematics, Fukuoka University of Education
Munakata,Fukuoka, 811-4192

1 Introduction

In this paper we denote bounded positive semidefinite operators on a Hilbert space by A, B, C and so on. A real valued continuous function $\varphi(x)$ on $[0,\infty)$ is called an *operator monotone function* if $0\leq A\leq B$ implies $\varphi(A)\leq\varphi(B)$. The fact that x^{a} $(0<a\leq 1)$ is operator monotone is called the *Löwner-Heinz* inequality.

In [8] (see p.76 of [9] for the relavant topics) a quadratic operator equation $B=XAX$ was studied and it was shown that if A is nonsingular, then there is a solution T with $0\leq T\leq 1$ if and only if $(A^{1/2}BA^{1/2})^{1/2}\leq A$ and that T is then given by the formula $T=A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$ if A is invertible. The solution of $B=XA^{-1}X$ is therefore given by $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. On the other hand, in [7] it was shown that if A is invertible, the maximum of all X such that

$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0$$

equals $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, which is called the *geometric mean* of A and B and denoted by $A\#B$. Therefore, by using this symbol, the solution T of $B=XAX$ is given by $T=A^{-1}\#B$ if A is invertible. For $0<\lambda<1$ and for invertible A the weighted geometric mean is defined as:

$$A \underset{\lambda}{\#} B := A^{1/2}(A^{-1/2}BA^{-1/2})^{\lambda}A^{1/2}.$$

Furuta [3, 4] showed that $A\leq B$ implies for $1\leq s,p$ and $0<r$

$$A^{1+r} \leq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1+r}{p+r}}, \tag{1}$$

$$A^{1-t+r} \leq \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{ps-ts+r}} \quad (0\leq t\leq 1, \quad t\leq r). \tag{2}$$

Further, in [1, 2, 10] it was shown that $A\leq B$ implies for $0<p,r$

$$e^{rA} \leq (e^{\frac{rA}{2}}e^{pB}e^{\frac{rA}{2}})^{\frac{r}{r+p}}. \tag{3}$$

These inequalities can be rewritten with the symbol #; for instance, (1) is equivalent to $A \le A^{-r} \#_{\frac{1+r}{p+r}} B^p$.

Now let us state a simple fact on numerical weighted geometric means: For positive numbers a, b, c, x and y, if $(x^a)^{\frac{bc}{a+bc}}(y^b)^{\frac{a}{a+bc}} \le 1$, then for any d with $-a \le d \le bc$, $(x^r)^{\frac{sc-d}{r+sc}}(y^s)^{\frac{r+d}{r+sc}}$ is decreasing for $r \ge a$ and for $s \ge b$. We will show that this result is true even if x and y are replaced by A and B and that (1), (2) and (3) follow simply from it.

We study in a more general situation. Namely, we treat operator connections (or means) which include every weighted geometric mean. Kubo and Ando [6] defined a *connection,* which is denoted by σ, and showed that there is a one to one correspondence between σ and an operator monotone function $\varphi \ge 0$ on $[0, \infty)$ by the formula

$$A\sigma B = A^{1/2}\varphi(A^{-1/2}BA^{-1/2})A^{1/2} \tag{4}$$

if A is invertible; σ is called an *operator mean* if $A\sigma A = A$, which is equivalent to $\varphi(1) = 1$. The operator mean corresponding to $\varphi(x) = x^{1/2}$ is clearly geometric mean.

In this paper we write σ_φ for σ corresponding to φ. In [11], to extend (1) and (3) we constructed a family $\{\phi_r(x)\}_{r>0}$ of non-negative operator monotone functions which satisfies

$$\phi_r(g(x)f(x)^r) = f(x)^{c+r} \quad (0 \le c \le 1),$$

where g and f are appropriate increasing functions; here by replacing $f(x)$ by x and $g(f^{-1}(x))$ by another function $g(x)$, ϕ_r satisfies $\phi_r(g(x)x^r) = x^r x^c$. In [12] we also studied the operator monotone function $\phi_{r,t}(x)$ defined by

$$\phi_{r,t}(x) = x^{\frac{r}{r+t}} f(x^{\frac{t}{r+t}}), \quad i.e., \quad \phi_{r,t}(x^r x^t) = x^r f(x^t),$$

where $f \ge 0$ is a given operator monotone function and $r > 0$ and $t > 0$. These investigations have led us to set up a pair of operator monotone functions $\{\psi_r\}$ and $\{\phi_r\}$ with the following situation:

$$\psi_r(x^r g(x)) = x^r, \qquad i.e., \quad x^{-r}\,\sigma_{\psi_r}\,g(x) = 1, \tag{5}$$
$$\phi_r(x^r g(x)) = x^r h(x), \quad i.e., \quad x^{-r}\,\sigma_{\phi_r}\,g(x) = h(x). \tag{6}$$

In this situation, ψ_r may be considered to be the subsidiary function of ϕ_r.

From now on, we assume that $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ are families of non-negative functions on $[0, \infty)$ satisfying (5) and (6) respectively, where g and h are continuous and g is increasing and that ψ_r and ϕ_r are both operator monotone for every r which is not less than a non-negative real number. Note that ψ_r is strictly increasing on $[0, \infty)$ with $\psi_r(0) = 0$ and $\psi_r(\infty) = \infty$, so the inverse function ψ_r^{-1} on $[0,\infty)$ exists. We remark that $h(x)$ is not necessarily increasing and that the region of r for which ψ_r is operator monotone is not necessarily coincident with that of r for which ϕ_r is: for instance, in (5) and (6) set $g(x) = x^t$ for a fixed $t > 0$ and $h(x) = x^{-1}$, then $\psi_r(x) = x^{r/(t+r)}$ is operator monotone for $r > 0$; on the other hand $\phi_r(x) = x^{(-1+r)/(t+r)}$ is operator monotone for $r \geq 1$.

2 Criteria for Monotonicity

Theorem 2.1. *Let $\{\psi_r\}_{r\geq a}$ and $\{\phi_r\}_{r\geq a}$ $(a > 0)$ be families of non-negative operator monotone functions satisfying* (5) *and* (6)*. Then the following hold:*

(a) *if $A^a \,\sigma_{\psi_a}\, B \geq 1$, then $A^r \,\sigma_{\psi_r} B$ and $A^r \,\sigma_{\phi_r}\, B$ are increasing for $r \geq a$;*

(b) *if A and B are invertible and if $A^a \,\sigma_{\psi_a}\, B \leq 1$, then $A^r \,\sigma_{\psi_r}\, B$ and $A^r \,\sigma_{\phi_r}\, B$ are decreasing for $r \geq a$.*

Proof. We only prove the first statement of (a). To do it, it suffices to show

$$A^s \,\sigma_{\psi_s}\, B \geq 1 \text{ for some } s \geq a \Rightarrow A^r \,\sigma_{\psi_r} B \geq A^s \,\sigma_{\psi_s}\, B \text{ for every } r \in [s, 2s].$$

Indeed, from $A^a \,\sigma_{\psi_a}\, B \geq 1$ it follows that $A^r \,\sigma_{\psi_r}\, B$ is increasing in $[a, 2a]$ and hence not less than 1; by the mathematical induction, we can see the statement. Since $A^r = (A^s)^{r/s}$, we may show that

$$A \,\sigma_{\psi_s}\, B \geq 1 \text{ for some } s \geq a \Rightarrow A^{r/s} \,\sigma_{\psi_r}\, B \geq A \,\sigma_{\psi_s}\, B \text{ for every } r \in [s, 2s]. \quad (7)$$

Notice $(A + \epsilon) \,\sigma_{\psi_s}\, (B + \epsilon) \geq A \,\sigma_{\psi_s}\, B \geq 1$ for $\epsilon > 0$. If we could show $(A + \epsilon)^{r/s} \,\sigma_{\psi_r}\, (B + \epsilon) \geq (A + \epsilon) \,\sigma_{\psi_s}\, (B + \epsilon)$, then we would get (7) as $\epsilon \to +0$. We therefore assume that A and B are invertible. Put $y = x^s$ in $\psi_s(x^s g(x)) = x^s$ and $\psi_r(x^r g(x)) = x^r$. Then, by setting $b = \frac{r-s}{s}$, we obtain

$$\psi_r(y^b \psi_s^{-1}(y)) = y^b y, \quad i.e., \quad y^{-b} \,\sigma_{\psi_r} \psi_s^{-1}(y) = y. \quad (8)$$

The assumption $A \,\sigma_{\psi_s}\, B \geq 1$ implies $\psi_s(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \geq A^{-1}$. Here, denote the left-hand side by H and the right-hand side by K. Since $H \geq K$ and $0 \leq b \leq 1$, by the Löwner-Heinz inequality, $K^{-b} \geq H^{-b}$. Hence we have

$$K^{-b} \,\sigma_{\psi_r} \psi_s^{-1}(H) \geq H^{-b} \,\sigma_{\psi_r} \psi_s^{-1}(H) = H.$$

Multiplying the above from the left and the right with $A^{1/2}$ yields

$$A^{b+1} \,\sigma_{\psi_r} B \geq A \,\sigma_{\psi_s}\, B.$$

Consequently, we have (7). □

In the second statement (b) of the above theorem, we assumed A and B are invertible, because the norm of $(A+\epsilon)^a \sigma_{\psi_b} (B+\epsilon)$ may not necessarily converge to that of $A^a \sigma_{\psi_b} B$ as $\epsilon \to +0$. We do not know if the invertibility of A and B can be removed.

Theorem 2.2. *Let $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ be families of non-negative operator monotone functions satisfying* (5) *and* (6). *If $A \le B$ or if $\log A \le \log B$ for invertible A and B, then for $r > 0$*

$$A^r \le \psi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}), \quad \psi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) \le B^r, \tag{9}$$

$$A^{\frac{r}{2}} h(B) A^{\frac{r}{2}} \le \phi_r(A^{\frac{r}{2}} g(B) A^{\frac{r}{2}}), \quad \phi_r(B^{\frac{r}{2}} g(A) B^{\frac{r}{2}}) \le B^{\frac{r}{2}} h(A) B^{\frac{r}{2}}. \tag{10}$$

Remark 2.1. In the above theorems, we assumed that the families $\{\psi_r\}_{r>0}$ and $\{\phi_r\}_{r>0}$ satisfy (5) and (6) respectively. However their proofs are still valid if

$$\phi_r(y^b \psi_s^{-1}(y)) = y^b \phi_s(\psi_s^{-1}(y)) \quad (y > 0), \tag{11}$$

and (8) hold. Therefore, theorems are true even if we assume that ψ_r and ϕ_r are non-negative operator monotone functions on $[0,\infty)$ with $\psi_r(0) = 0$ and $\psi_r(\infty) = \infty$ and that for all r and s with $r > s > 0$

$$\psi_r(\psi_s(x)^{\frac{r-s}{s}} x) = \psi_s(x)^{\frac{r}{s}} \quad \text{and} \quad \phi_r(\psi_s(x)^{\frac{r-s}{s}} x) = \psi_s(x)^{\frac{r-s}{s}} \phi_s(x)$$

instead of (5) and (6); because they satisfy

$$\psi_r(y^{\frac{r-s}{s}} \psi_s^{-1}(y)) = y^{\frac{r}{s}} \quad \text{and} \quad \phi_r(y^{\frac{r-s}{s}} \psi_s^{-1}(y)) = y^{\frac{r-s}{s}} \phi_s(\psi_s^{-1}(y)),$$

from which (8) and (11) follow.

Remark 2.2. Let $\{A_r\}_{r>0}$ be a weakly continuous semi-group of positive semidefinite operators, that is, $A_{r+s} = A_r A_s$. Then we get $(A_r)^a = A_{ra}$ for $a > 0$. Thus from Theorem 2.2 we obtain

(a) *if $A_a \sigma_{\psi_b} B \ge 1$, then $A_{ar} \sigma_{\psi_{br}} B$ is increasing for $r \ge 1$;*

(b) *if $A_a \sigma_{\psi_b} B \le 1$ for invertible A_a and B, then $A_{ar} \sigma_{\psi_{br}} B$ is decreasing for $r \ge 1$.*

3 Weighted Geometric Means

Our objective in this section is to apply the results we got in the preceding section to the weighted geometric means. As we mentioned in the first section the symbols $\overset{\lambda}{\#}$ and σ_{x^λ} express the same weighted geometric mean for $0 < \lambda \leq 1$. We have $A \underset{\lambda}{\#} B = B \underset{1-\lambda}{\#} A$.

Lemma 3.1. *Let $a > 0$, $c > 0$ and $c > d$. Then the following hold:*

(a) *if A and B are invertible and if $A^a \underset{\frac{a}{a+c}}{\#} B \leq 1$, then $A^r \underset{\frac{r+d}{r+c}}{\#} B$ is decreasing for $r \geq \max(a, -d)$;*

(b) *if $A^a \underset{\frac{a}{a+c}}{\#} B \geq 1$, then $A^r \underset{\frac{r+d}{r+c}}{\#} B$ is increasing for $r \geq \max(a, -d)$.*

Theorem 3.2. *For a given $c > 0$ define a function $F(r, s)$ by*

$$F(r, s) = A^r \underset{\frac{r}{r+sc}}{\#} B^s \quad \text{for } r > 0,\ s > 0. \tag{12}$$

Then, for $r \geq a > 0$, $s \geq b > 0$ the following hold:

(a) *if A and B are both invertible and $F(a, b) \leq 1$, then $F(r, s) \leq F(a, b)$;*

(b) *if $F(a, b) \geq 1$, then $F(r, s) \geq F(a, b)$.*

Proof. We show only the first statement. From Lemma 3.1 it follows that

$$\begin{aligned} 1 &\geq F(a,b) \geq F(r,b) = A^r \underset{\frac{r}{r+bc}}{\#} B^b = B^b \underset{\frac{bc}{r+bc}}{\#} A^r = B^b \underset{\frac{b}{b+r/c}}{\#} A^r \\ &\geq B^s \underset{\frac{s}{s+r/c}}{\#} A^r = A^r \underset{\frac{r/c}{s+r/c}}{\#} B^s = A^r \underset{\frac{r}{r+sc}}{\#} B^s = F(r,s). \quad \square \end{aligned}$$

By using the above theorem twice, from $F(a, b) \leq 1$ it follows that $F(r_2, s_2) \leq F(r_1, s_1) \leq F(a, b)$ for $r_2 \geq r_1 \geq a$ and for $s_2 \geq s_1 \geq b$.

The case $\lambda = 1/2$ of the following corollary resembles the result shown in [1].

Corollary 3.3. *For a given λ as $0 < \lambda < 1$ the following hold:*

(a) *if* $A \underset{\lambda}{\#} B \leq 1$ *for invertible* A *and* B , *then* $A^r \underset{\lambda}{\#} B^r$ *is decreasing for* $r \geq 1$;

(b) *if* $A \underset{\lambda}{\#} B \geq 1$, *then* $A^r \underset{\lambda}{\#} B^r$ *is increasing for* $r \geq 1$.

Proof. Define c by $\lambda = \frac{1}{1+c}$ and use Theorem 3.2 to get this. □

Now we treat a quadratic equation $B = XAX$ given in the first section. Assume that A and B are invertible. Then the solution is given by $A^{-1}\#B$. By Corollary 3.3 we get:

(a) *if* $A^{-1}\#B \geq 1$ *then the solution* $A^{-r}\#B^r$ *of* $B^r = XA^rX$ *is increasing for* $r \geq 1$;

(b) *if* $A^{-1}\#B \leq 1$ *then* $A^{-r}\#B^r$ *is decreasing for* $r \geq 1$.

The following is the main theorem of this section.

Theorem 3.4. *For real numbers* $c > 0$ *and* d, *define* $F(r,s)$ *by* (12) *and* $G(r,s)$ *by*

$$G(r,s) = A^r \underset{\frac{r+d}{r+sc}}{\#} B^s \quad \text{for } r > 0,\ s > 0 \quad \text{with } 0 \leq \frac{r+d}{r+sc} \leq 1. \tag{13}$$

Let $a > 0$, $b > 0$ *and* $-a \leq d \leq bc$. *Then for* $r_2 \geq r_1 \geq a$ *and for* $s_2 \geq s_1 \geq b$ *the following hold:*

(a) *if* A *and* B *are both invertible and* $F(a,b) \leq 1$, *then* $G(r_2,s_2) \leq G(r_1,s_1)$;

(b) *if* $F(a,b) \geq 1$, *then* $G(r_2,s_2) \geq G(r_1,s_1)$.

The above theorem says that if $F(a,b) \leq 1$, $G(a,b) \leq K$ then $G(r,s) \leq K$ for $r \geq a$, $s \geq b$; moreover, if $F(a,b) = 1$ then $G(r,s)$ is constant, though this directly follows from the definitions of $F(r,s)$ and $G(r,s)$. Notice that $G(r,s) = F(r,s)$ if $d = 0$.

So far, we have seen that $F(a,b) \leq 1$ (or $F(a,b) \geq 1$) has a great influence on $G(r,s)$. Now we give a sufficient condition on $G(r,s)$ in order that $F(a,b) \leq 1$ (or $F(a,b) \geq 1$).

Proposition 3.5. *Let* A *and* B *be invertible. Let* $a > 0$ *and* $c > d > 0$. *Then the following hold:*

$$A^a \underset{\frac{a+d}{a+c}}{\#} B \leq A^{-d} \quad \Rightarrow \quad A^a \underset{\frac{a}{a+c}}{\#} B \leq 1;$$

$$A^a \underset{\frac{a+d}{a+c}}{\#} B \geq A^{-d} \quad \Rightarrow \quad A^a \underset{\frac{a}{a+c}}{\#} B \geq 1.$$

4 Applications

We mentioned after Theorem 2.3 that (9) and (10) are extensions of (1) and (3). However we give a simple proof of (1) to explain how Theorem 3.4 is useful, and we give an extension of (2).

(1): We may assume A and B are invertible. From $A \leq B$ it follows that $A^{-a} \geq B^{-a}$ for every a with $0 < a < 1$. Substitute A^{-1} for A in (12) and (13), and put $c = 1$ and $d = 1$. Then

$$F(a,1) = A^{-a} \underset{\frac{a}{a+1}}{\#} B \geq B^{-a} \underset{\frac{a}{a+1}}{\#} B = 1, \quad G(a,1) = A^{-a} \underset{\frac{a+1}{a+1}}{\#} B = B.$$

Thus by Theorm 3.4

$$G(r,s) = A^{-r} \underset{\frac{r+1}{r+s}}{\#} B^{s}$$

is increasing for $r \geq a$ and for $s \geq 1$; especially, $G(r,s) \geq G(a,1) = B \geq A$. Since a is arbitrary, we have $G(r,s) \geq A$ for $r > 0, s \geq 1$. Replace p for s to get (1). $\square$

Proposition 4.1. *If $A \leq B \leq C$ and if B is invertible, then for $0 \leq t \leq 1$, $t \leq r$, $1 \leq p$ and $1 \leq s$*

$$\begin{aligned} A^{1-t+r} &\leq \{A^{r/2}(B^{-\frac{t}{2}}C^{p}B^{-\frac{t}{2}})^{s}A^{r/2}\}^{\frac{1-t+r}{ps-ts+r}}, \\ \{C^{r/2}(B^{-\frac{t}{2}}A^{p}B^{-\frac{t}{2}})^{s}C^{r/2}\}^{\frac{1-t+r}{ps-ts+r}} &\leq C^{1-t+r}. \end{aligned} \tag{14}$$

Proof. If $t = 0$, (14) reduces to (1). So we assume $0 < t \leq 1$. We may, without loss of generality, assume A is invertible. Put

$$K = B^{-\frac{t}{2}}C^{p}B^{-\frac{t}{2}}.$$

Then (14) is equivalent to

$$A^{1-t} \leq A^{-r} \underset{\frac{r+1-t}{r+ps-ts}}{\#} K^{s} \quad (t \leq r,\ 1 \leq p,\ 1 \leq s).$$

Put

$$F(r,s) = A^{-r} \underset{\frac{r}{r+ps-ts}}{\#} K^{s} \quad \text{and} \quad G(r,s) = A^{-r} \underset{\frac{r+1-t}{r+ps-ts}}{\#} K^{s}.$$

$B^{t} \geq A^{t}$ yields $A^{\frac{t}{2}}B^{-t}A^{\frac{t}{2}} \leq 1$; since $x^{\frac{t}{p}}$ is operator concave (see [5]) we obtain

$$A^{\frac{t}{2}}B^{-\frac{t}{2}}(C^{p})^{\frac{t}{p}}B^{-\frac{t}{2}}A^{\frac{t}{2}} \leq (A^{\frac{t}{2}}B^{-\frac{t}{2}}C^{p}B^{-\frac{t}{2}}A^{\frac{t}{2}})^{\frac{t}{p}},$$

from which it follows that

$$F(t,1) = A^{-t} \#_{\frac{t}{p}} K^1 \geq B^{-\frac{t}{2}} C^t B^{-\frac{t}{2}} \geq 1,$$
$$G(t,1) = A^{-t} \#_{\frac{1}{p}} K^1 \geq B^{-\frac{t}{2}} C B^{-\frac{t}{2}} \geq B^{1-t} \geq A^{1-t}.$$

By virtue of Theorem 3.4, $G(r,s)$ is therefore increasing for $r \geq t$ and for $s \geq 1$; in particular, $G(r,s) \geq A^{1-t}$. Thus we get (14). The second inequality follows from (14) by taking the inverse of it. □

References

[1] T. Ando, On some operator inequalities, *Math. Ann.* 279(1987), 157–159.

[2] M. Fujii, T. Furuta, E. Kamei, Furuta's inequality and its application to Ando's theorem, *Linear Algebra Appl.* 149(1991), 91–96.

[3] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.* 101(1987), 85–88.

[4] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization, *Linear Algebra Appl.* 219(1995), 139–155.

[5] F. Hansen, G. K. Pedersen, Jensen's inequality for operators and Löwner's theorem. *Math. Ann.* 258(1982), 229 - 241.

[6] F. Kubo, T. Ando, Means of positive linear operators, *Math. Ann.* 246(1980), 205–224.

[7] W. Pusz, S. L. Woronowicz, Functional calculus for sesquilinear forms and the purification map, *Rep. Math. Phys.* 8(1975), 159–170.

[8] G. K. Pedersen, M. Takesaki, The operator equation $THT = K$, Proc. Amer. Math. Soc. 36 (1972), 311–312.

[9] S. Sakai, *C^*-algebras and W^*-algebras,* Berlin, Heidelberg, New York, Springer 1971

[10] M. Uchiyama, Some exponential operator inequalities, *Math. Inequal. Appl.* 2(1999), 469–471.

[11] M. Uchiyama, Operator monotone functions which are defined implicitly and operator inequalities, *J. Funct. Anal.* 175(2000), 330–347.

[12] M. Uchiyama, Mixed matrix (operator) inequality, *Linear Algebra Appl.* to appear.