数理解析研究所講究録 1259 巻 2002 年 100-109

Relations between two operator inequalities and their applications to paranormal operators

東京理科大・理　柳田 昌宏 (Masahiro Yanagida)
Department of Applied Mathematics,
Tokyo University of Science

神奈川大・工　山崎 丈明 (Takeaki Yamazaki)
Department of Mathematics,
Kanagawa University

1 Introduction

This report is based on the following preprint:

T.Yamazaki and M.Yanagida, *Relations between two operator inequalities and their applications to paranormal operators*, preprint.

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The following Theorem F is well known as a recent development on order preserving operator inequalities.

Theorem F (Furuta inequality [11]).

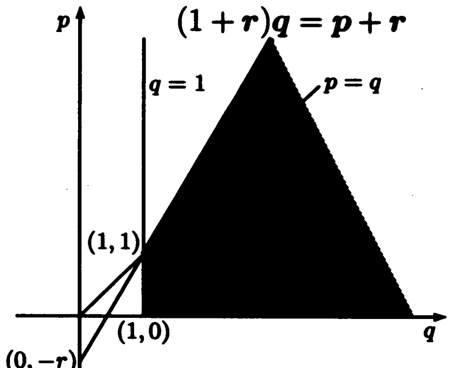
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

Theorem F yields the famous Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$*" by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18], and also an elementary one page proof in [12]. It was shown in [19] that the domain drawn for p, q and r in the Figure is the best possible for Theorem F.

For positive invertible operators A and B, the order defined by $\log A \geq \log B$ is called the chaotic order. The chaotic order is weaker than the usual order since $\log t$ is an operator monotone function. The following result is a characterization of the chaotic order which is an application of Theorem F.

Theorem 1.A **([7][13]).** *For positive invertible operators A and B, the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

(iii) $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

The case $p = r$ of Theorem 1.A was shown in [4]. An alternative proof of Theorem 1.A was shown in [8], and also a breathtakingly simple proof in [21]. It was attempted in [22] to remove the invertibility of operators in Theorem 1.A.

Recently, Ito-Yamazaki [17] showed the following result on the relations between the two inequalities in Theorem 1.A.

Theorem 1.B **([17]).** *Let A and B be positive operators. Then for each $p > 0$ and $r \geq 0$, the following assertions hold:*

(i) *If* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, *then* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(ii) *If* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and* $N(A) \subseteq N(B)$, *then* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

It turns out by the following Lemma F that the two inequalities in Theorem 1.B are equivalent in case A and B are invertible.

Lemma F **([14]).** *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

In fact, for each $p \geq 0$ and $r \geq 0$,

$$\begin{aligned} A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} &\Longleftrightarrow A^p \geq A^{\frac{p}{2}}B^{\frac{r}{2}}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{-r}{p+r}}B^{\frac{r}{2}}A^{\frac{p}{2}} \qquad \text{by Lemma F} \\ &\Longleftrightarrow B^{-r} \geq (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{-r}{p+r}} \\ &\Longleftrightarrow (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r. \end{aligned}$$

2 Relations between two operator inequalities

As a parallel result to Theorem 1.B, we obtain the following result.

Theorem 2.1. *Let A and B be positive operators. Then for each $p > 0$, $r \geq 0$ and $\lambda > 0$, the following assertions hold:*

(i) *If* $\dfrac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r$, *then* $A^p \geq \dfrac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I}$.

(ii) *If* $A^p \geq \dfrac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I}$ *and* $N(A) \subseteq N(B)$, *then* $\dfrac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r$.

We remark that the two inequalities in Theorem 2.1 are equivalent in case A and B are invertible. In fact, for each $p \geq 0$, $r \geq 0$ and $\lambda > 0$,

$$\begin{aligned} A^p \geq \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I} &\iff A^p \geq \frac{(p+r)\lambda^p}{rI + p\lambda^{p+r}A^{\frac{-p}{2}}B^{-r}A^{\frac{-p}{2}}} \\ &\iff \frac{rI + p\lambda^{p+r}A^{\frac{-p}{2}}B^{-r}A^{\frac{-p}{2}}}{(p+r)\lambda^p} \geq A^{-p} \\ &\iff \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \geq B^r. \end{aligned}$$

We also remark that the inequalities in Theorem 2.1 are weaker than those in Theorem 1.B. In fact, by the arithmetic-geometric-harmonic mean inequality,

$$\begin{aligned} (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} &= \left(\frac{B^{\frac{r}{2}}A^pB^{\frac{r}{2}}}{\lambda^p}\right)^{\frac{r}{p+r}} (\lambda^r)^{\frac{p}{p+r}} \\ &\leq \frac{r}{p+r}\frac{B^{\frac{r}{2}}A^pB^{\frac{r}{2}}}{\lambda^p} + \frac{p}{p+r}\lambda^r I = \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \end{aligned}$$

and

$$\begin{aligned} (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} &= \left(\frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\lambda^r}\right)^{\frac{p}{p+r}} (\lambda^p)^{\frac{r}{p+r}} \\ &\geq \left\{\frac{p}{p+r}\left(\frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{\lambda^r}\right)^{-1} + \frac{r}{p+r}(\lambda^p I)^{-1}\right\}^{-1} = \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I} \end{aligned}$$

hold for each positive invertible operators A and B, $p \geq 0$, $r \geq 0$ and $\lambda > 0$. Hence Theorem 2.1 can be understood as a parallel result to Theorem 1.B.

In order to give a proof of Theorem 2.1, we use the following lemma.

Lemma 2.A ([17]). *Let A be a positive operator. Then*

$$\lim_{\varepsilon \to +0} A^{\frac{1}{2}}(A+\varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon \to +0}(A+\varepsilon I)^{-1}A = P_{N(A)^{\perp}}$$

holds, where $P_{\mathcal{M}}$ is the projection onto a closed subspace $\mathcal{M}$.

Proof of Theorem 2.1.

Proof of (i). By the assumption,

$$A^{\frac{p}{2}}B^{\frac{r}{2}}(B^r+\varepsilon I)^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\left(\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p}+\varepsilon I\right)^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}$$

holds for any $\varepsilon > 0$. By tending $\varepsilon \to +0$ and Lemma 2.A, we have

$$A^p \geq A^{\frac{p}{2}}P_{N(B)^{\perp}}A^{\frac{p}{2}} \geq A^{\frac{p}{2}}B^{\frac{r}{2}}\left(\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p}\right)^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}} = \frac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}$$

since

$$\begin{aligned}A^{\frac{p}{2}}B^{\frac{r}{2}}(rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I)^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}} &= U|A^{\frac{p}{2}}B^{\frac{r}{2}}|(r|A^{\frac{p}{2}}B^{\frac{r}{2}}|^2+p\lambda^{p+r}I)^{-1}|A^{\frac{p}{2}}B^{\frac{r}{2}}|U^*\\ &= U|A^{\frac{p}{2}}B^{\frac{r}{2}}|^2U^*(r|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+p\lambda^{p+r}I)^{-1}\\ &= |B^{\frac{r}{2}}A^{\frac{p}{2}}|^2(r|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+p\lambda^{p+r}I)^{-1}\\ &= \frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I},\end{aligned}$$

where $A^{\frac{p}{2}}B^{\frac{r}{2}} = U|A^{\frac{p}{2}}B^{\frac{r}{2}}|$ is the polar decomposition of $A^{\frac{p}{2}}B^{\frac{r}{2}}$.

Proof of (ii). By the assumption,

$$B^{\frac{r}{2}}A^{\frac{p}{2}}\left(\frac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}+\varepsilon I\right)^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}} \geq B^{\frac{r}{2}}A^{\frac{p}{2}}(A^p+\varepsilon I)^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}}$$

holds for any $\varepsilon > 0$. By tending $\varepsilon \to +0$ and Lemma 2.A, we have

$$\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p} \geq \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}P_{N(A^{\frac{p}{2}}B^{\frac{r}{2}})^{\perp}}}{(p+r)\lambda^p} \geq B^{\frac{r}{2}}P_{N(A)^{\perp}}B^{\frac{r}{2}} \geq B^r$$

since $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)^{\perp}} \geq P_{N(B)^{\perp}}$ and

$$\begin{aligned}&\lim_{\varepsilon\to+0} B^{\frac{r}{2}}A^{\frac{p}{2}}\left(\frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}+\frac{\varepsilon I}{(p+r)\lambda^p}\right)^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}}\\ &= \lim_{\varepsilon\to+0} a(\varepsilon)B^{\frac{r}{2}}A^{\frac{p}{2}}\left(\frac{A^{\frac{p}{2}}B^rA^{\frac{p}{2}}+b(\varepsilon)I}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}\right)^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}}\\ &= \lim_{\varepsilon\to+0} a(\varepsilon)V\frac{|B^{\frac{r}{2}}A^{\frac{p}{2}}|(|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+b(\varepsilon)I)^{-1}|B^{\frac{r}{2}}A^{\frac{p}{2}}|}{(r|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+p\lambda^{p+r}I)^{-1}}V^*\\ &= V\frac{P_{N(B^{\frac{r}{2}}A^{\frac{p}{2}})^{\perp}}}{(r|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+p\lambda^{p+r}I)^{-1}}V^* \quad \text{by } a(0)=1,\ \lim_{\varepsilon\to+0} b(\varepsilon)=0 \text{ and Lemma 2.A}\\ &= V(r|B^{\frac{r}{2}}A^{\frac{p}{2}}|^2+p\lambda^{p+r}P_{N(B^{\frac{r}{2}}A^{\frac{p}{2}})^{\perp}})V^*\\ &= r|A^{\frac{p}{2}}B^{\frac{r}{2}}|^2+p\lambda^{p+r}P_{N(A^{\frac{p}{2}}B^{\frac{r}{2}})^{\perp}}\\ &= rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}P_{N(A^{\frac{p}{2}}B^{\frac{r}{2}})^{\perp}},\end{aligned}$$

where $B^{\frac{r}{2}}A^{\frac{p}{2}} = V|B^{\frac{r}{2}}A^{\frac{p}{2}}|$ is the polar decomposition of $B^{\frac{r}{2}}A^{\frac{p}{2}}$, $a(\varepsilon) = \frac{(p+r)\lambda^p}{(p+r)\lambda^p+\varepsilon r}$ ar
$b(\varepsilon) = \frac{\varepsilon p\lambda^{p+r}}{(p+r)\lambda^p+\varepsilon r}$. Therefore the proof is complete.

3 Classes of non-normal operators

In the following sections, we shall show applications of Theorem 2.1 to non-normal operators. To begin with, we introduce several classes of non-normal operators.

Definition ([2][9][10][15][16][23]). Let $p > 0$ and $r > 0$.

(i) T is p-hyponormal $\Longleftrightarrow (T^*T)^p \geq (TT^*)^p$.

(ii) T is log-hyponormal $\Longleftrightarrow T$ is invertible and $\log T^*T \geq \log TT^*$.

(iii) T is hyponormal $\Longleftrightarrow T^*T \geq TT^* \Longleftrightarrow T$ is 1-hyponormal.

(iv) T belongs to class $\mathrm{A}(p,r) \Longleftrightarrow (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r}$.

(v) T belongs to class A $\Longleftrightarrow |T^2| \geq |T|^2 \Longleftrightarrow T$ belongs to class $\mathrm{A}(1,1)$.

(vi) T is w-hyponormal $\Longleftrightarrow |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \Longleftrightarrow T$ belongs to class $\mathrm{A}(\frac{1}{2},\frac{1}{2})$ ([17]).

(vii) T is absolute-(p,r)-paranormal $\Longleftrightarrow \||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r}$ for all $\|x\| = 1$.

(viii) T is paranormal $\Longleftrightarrow \|T^2x\| \geq \|Tx\|^2$ for all $\|x\| = 1$
$\Longleftrightarrow T$ is absolute-$(1,1)$-paranormal.

Inclusion relations among these classes are as follows and can be expressed as the diagram on the next page.

Theorem 3.A ([9][17][23]).

(i) *T is p-hyponormal for some $p > 0$ or log-hyponormal*
$\Longrightarrow$ *T belongs to class $A(p,r)$ for all $p > 0$ and $r > 0$.*

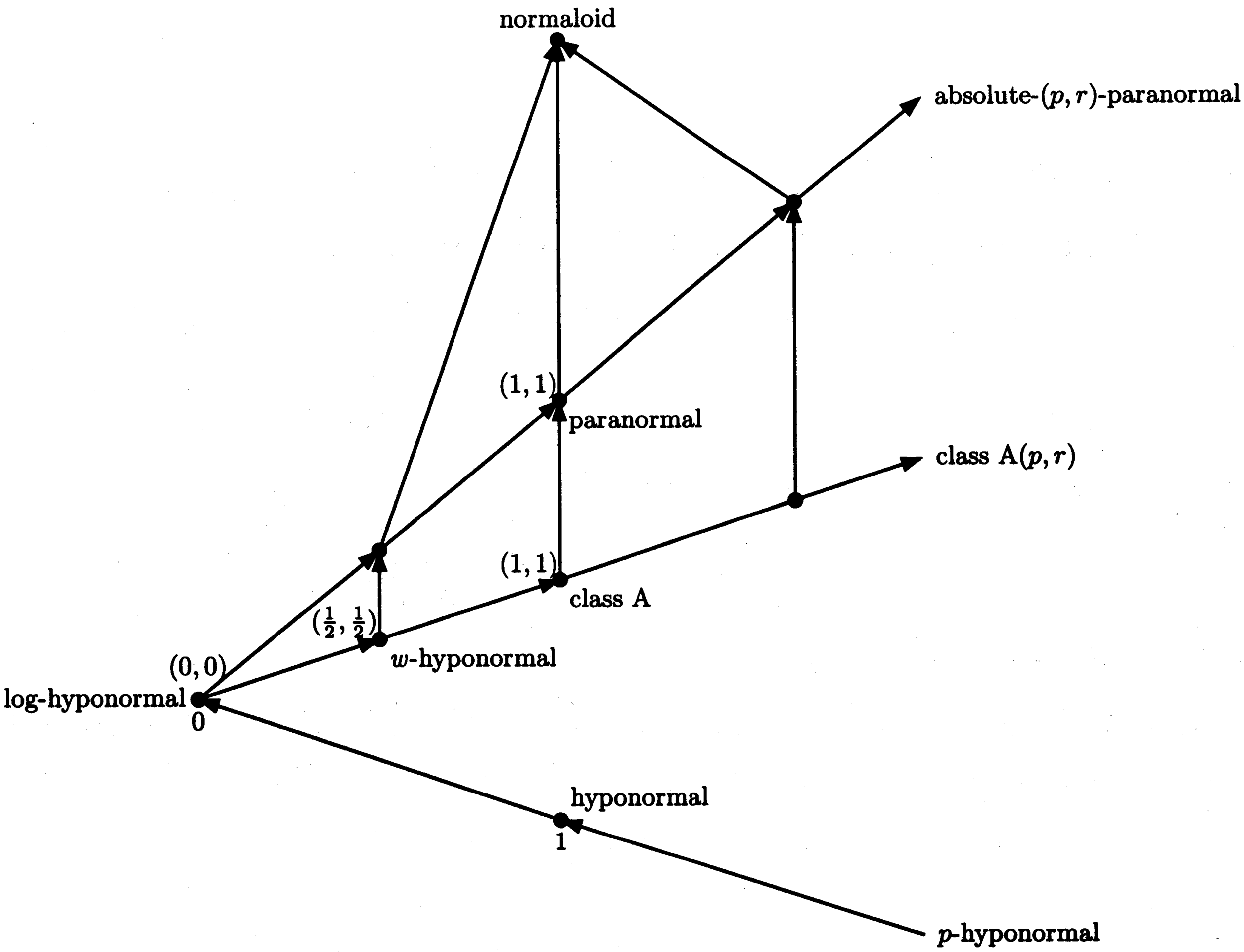
(ii) *For each $p > 0$ and $r > 0$,*
T belongs to class $A(p,r) \Longrightarrow T$ is absolute-(p,r)-paranormal.

(iii) *T is absolute-(p,r)-paranormal for some $p > 0$ and $r > 0$*
$\Longrightarrow$ *T is normaloid (i.e., $\|T\| = r(T)$).*

(iv) *T is log-hyponormal*
$\Longleftrightarrow$ *T is invertible and absolute-(p,p)-paranormal for all $p > 0$*
$\Longleftrightarrow$ *T is invertible and absolute-(p,r)-paranormal for all $p > 0$ and $r > 0$.*

(v) *For each $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$,*
T belongs to class $A(p_1,r_1) \Longrightarrow T$ belongs to class $A(p_2,r_2)$.

(vi) *For each $0 < p_1 \leq p_2$ and $0 < r_1 \leq r_2$,*
T is absolute-(p_1,r_1)-paranormal $\Longrightarrow T$ is absolute-(p_2,r_2)-paranormal.



4 Normality conditions via paranormality

Recently, Ito-Yamazaki [17] showed the following result on the normality of class A(p,r) operators.

Theorem 4.A ([17]). *Let $p_1 > 0$, $p_2 > 0$, $r_1 > 0$ and $r_2 > 0$. If T belongs to class $A(p_1,r_1)$ and T^* belongs to class $A(p_2,r_2)$, then T is normal.*

On the other hand, Ando [3] showed the following result on the normality of paranormal operators under the condition $N(T) = N(T^*)$.

Theorem 4.B ([3]). *If T and T^* are paranormal with $N(T) = N(T^*)$, then T is normal.*

We obtain the following result as an application of Theorem 2.1.

Theorem 4.1. *Let $p_1 > 0$, $p_2 > 0$, $r_1 > 0$ and $r_2 > 0$. If T is absolute-(p_1,r_1)-paranormal and T^* is absolute-(p_2,r_2)-paranormal, then T is normal.*

Theorem 4.1 is an extension of Theorem 4.A by (ii) of Theorem 3.A. Theorem 4.1 is also an extension of Theorem 4.B since the following result can be obtained as a simple corollary of Theorem 4.1 by putting $p_1 = p_2 = r_1 = r_2 = 1$. We remark that Corollary 4.2 requires no kernel conditions.

Corollary 4.2. *If T and T^* are paranormal, then T is normal.*

In order to give a proof of Theorem 4.1, we prepare the following results.

Theorem 4.C ([23]). *Let $p > 0$ and $r > 0$. T is absolute-(p, r)-paranormal if and only if*

$$r|T^*|^r|T|^{2p}|T^*|^r - (p+r)\lambda^p|T^*|^{2r} + p\lambda^{p+r}I \geq 0 \quad \textit{for all } \lambda > 0.$$

Theorem 4.D ([3]). *Let A and B be positive operators. If*

$$\frac{A^2+\lambda^2 I}{2\lambda} \geq B \qquad \textit{and} \qquad B \geq \frac{2\lambda A^2}{A^2+\lambda^2 I}$$

hold for all $\lambda > 0$, then $A = B$.

Proof of Theorem 4.1. Put $k = \max\{p_1, p_2, r_1, r_2\}$. If T is absolute-(p_1, r_1)-paranormal, then T is absolute-(k, k)-paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T^*|^k|T|^{2k}|T^*|^k - 2k\lambda^k|T^*|^{2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$

This is equivalent to

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}{2\lambda^k} \geq |T^*|^{2k} \quad \text{and} \quad |T|^{2k} \geq \frac{2\lambda^k|T|^k|T^*|^{2k}|T|^k}{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}. \tag{4.1}$$

On the other hand, if T^* is absolute-(p_2, r_2)-paranormal, then T^* is absolute-(k, k)-paranormal by (vi) of Theorem 3.A. By Theorem 4.C, we have

$$k|T|^k|T^*|^{2k}|T|^k - 2k\lambda^k|T|^{2k} + k\lambda^{2k}I \geq 0 \quad \text{for all } \lambda > 0.$$

This is equivalent to

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{2k},$$

so that by (i) of Theorem 2.1, we have

$$\frac{|T|^k|T^*|^{2k}|T|^k + \lambda^{2k}I}{2\lambda^k} \geq |T|^{2k} \quad \text{and} \quad |T^*|^{2k} \geq \frac{2\lambda^k|T^*|^k|T|^{2k}|T^*|^k}{|T^*|^k|T|^{2k}|T^*|^k + \lambda^{2k}I}. \tag{4.2}$$

Hence $(|T^*|^k|T|^{2k}|T^*|^k)^{\frac{1}{2}} = |T^*|^{2k}$ and $(|T|^k|T^*|^{2k}|T|^k)^{\frac{1}{2}} = |T|^{2k}$ by (4.1), (4.2) and Theorem 4.D, that is, T and T^* belong to class A(k, k). Therefore T is normal by Theorem

5 Normality conditions via Aluthge transformation

Let T be an operator whose polar decomposition is $T = U|T|$. Then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation of T. Aluthge transformation was firstly introduced in [1] and has been studied by many researchers.

Chō-Huruya-Kim [5] showed the following result on the normality of w-hyponormal operators via Aluthge transformation.

Theorem 5.A ([5]). *If T is w-hyponormal and $\tilde{T}$ is normal, then T is also normal.*

We remark that Theorem 5.A can be considered as an extension of the following result since every log-hyponormal operator is w-hyponormal by (i) of Theorem 3.A and $T_t = U|T|^{2t}$ is log-hyponormal for any $t > 0$ if $T = U|T|$ is log-hyponormal.

Theorem 5.B ([20]). *If $T = U|T|$ is log-hyponormal and $\tilde{T}_t = |T|^t U|T|^t$ is normal for some $t > 0$, then T is also normal.*

As an application of Theorem 2.1, we obtain the following result which is an extension of Theorem 5.A since every w-hyponormal operator is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal by (ii) of Theorem 3.A.

Theorem 5.1. *If T is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal and $(\tilde{T})^*$ is hyponormal, then T is normal.*

Proof. If T is absolute-$(\frac{1}{2}, \frac{1}{2})$-paranormal, then

$$\frac{|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}} + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T^*| \tag{5.1}$$

holds for all $\lambda > 0$ by Theorem 4.C. Applying (i) of Theorem 2.1 to (5.1), we have

$$|T| \geq \frac{2\lambda^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}}{|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}} + \lambda I}. \tag{5.2}$$

Let $T = U|T|$ be the polar decomposition of T. Then by (5.1) and (5.2),

$$\begin{aligned} \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} &= \frac{U^*|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}}U + \lambda I}{2\lambda^{\frac{1}{2}}} \geq U^*\left(\frac{|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}} + \lambda I}{2\lambda^{\frac{1}{2}}}\right)U \\ &\geq U^*|T^*|U = |T| \geq \frac{2\lambda^{\frac{1}{2}}|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}}}{|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}} + \lambda I} = \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I}. \end{aligned} \tag{5.3}$$

Since $f(t) = \dfrac{t+\lambda}{2\lambda^{\frac{1}{2}}}$ and $g(t) = \dfrac{2\lambda^{\frac{1}{2}}t}{t+\lambda}$ are operator monotone,

$$\frac{|(\tilde{T})^*|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq \frac{|\tilde{T}|^2 + \lambda I}{2\lambda^{\frac{1}{2}}} \geq |T| \quad \text{and} \quad |T| \geq \frac{2\lambda^{\frac{1}{2}}|(\tilde{T})^*|^2}{|(\tilde{T})^*|^2 + \lambda I} \geq \frac{2\lambda^{\frac{1}{2}}|\tilde{T}|^2}{|\tilde{T}|^2 + \lambda I} \tag{5.4}$$

hold by (5.3) and the hyponormality of $(\tilde{T})^*$. By (5.4) and Theorem 4.D, we have $|\tilde{T}| = |T| = |(\tilde{T})^*|$, that is, T is w-hyponormal and $\tilde{T}$ is normal. Hence T is normal by Theorem 5.A. □

References

[1] A.Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[2] A.Aluthge and D.Wang, *w-Hyponormal operators*, Integral Equations Operator Theory **36** (2000), 1–10.

[3] T.Ando, *Operators with norm condition*, Acta Sci. Math. (Szeged) **33** (1972), 169–178.

[4] T.Ando, *On some operator inequalities*, Math. Ann. **279** (1987), 157–159.

[5] M.Chō, T.Huruya and Y.O.Kim, *A note on w-hyponormal operators*, to appear in J. Inequal. Appl.

[6] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory **23** (1990), 67–72.

[7] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161–169.

[8] M.Fujii, J.F.Jiang and E.Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc. **125** (1997), 3655–3658.

[9] M.Fujii, D.Jung, S.H.Lee, M.Y.Lee and R.Nakamoto *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japon. **51** (2000), 395–402.

[10] T.Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.

[11] T.Furuta, $A \geq B \geq 0$ *assures* $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ *for* $r \geq 0$, $p \geq 0$, $q \geq 1$ *with* $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85–88.

[12] T.Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 126.

[13] T.Furuta, *Applications of order preserving operator inequalities*, Operator Theory Adv. Appl. **59** (1992), 180–190.

[14] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[15] T.Furuta, M.Ito and T.Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), 389–403.

[16] V.Istrăţescu, T.Saito and T.Yoshino, *On a class of operators*, Tohoku Math. J. **18** (1966), 410–413.

[17] M.Ito and T.Yamazaki, *Relations between two inequalities* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *and* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and their applications*, to appear in Integral Equations Operator Theory.

[18] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883–886.

[19] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141–146.

[20] K.Tanahashi, *On log-hyponormal operators*, Integral Equations Operator Theory **34** (1999), 364–372.

[21] M.Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl. **2** (1999), 469–471.

[22] M.Uchiyama, *Inequalities for semibounded operators and their applications to log-hyponormal operators*, preprint.

[23] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, Sci. Math. **3** (2000), 23–32.