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Some Entropy Inequalities for CAR Systems
CAR代数でのLiebのエントロピー不等式とその応用

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1 Main Results

We study equilibrium statistical mechanics of Fermion lattice systems which require a different treatment compared with quantum spin lattice systems due to the non-commutativity of local algebras on disjoint regions.

In this talk, we focus on the entropy and show some difference and similarity on its behavior between Fermion systems and quantum spin systems [2], [9].

The following table shows our main results:

Property	Quantum Spin	Fermion Systems
SSA	○(Lieb-Ruskai)	○(Araki-Moriya)
Triangle	○(Araki-Lieb)	×(Moriya)

2 Entropy and Relative Entropy

2.1 Definitions

We introduce the definitions of density matrix and entropy.

Lemma 2.1. *Let $\mathcal{M}$ be a finite type* **I** *factor.*
(i) *Let φ be a positive linear functional on $\mathcal{M}$. Then there exists a unique $\hat{\rho}_\varphi \in \mathcal{M}_+$ (called adjusted density matrix) satisfying*

$$\varphi(a) = \tau(\hat{\rho}_\varphi a) \quad (\forall a \in \mathcal{M}).$$

(ii) *Let $\mathcal{N}$ be a subfactor of $\mathcal{M}$ and $\varphi_\mathcal{N}$ be the restriction of φ to $\mathcal{N}$. Let $\hat{\rho}_{\varphi_\mathcal{N}} \in \mathcal{N}_+$ be the adjusted density matrix of $\varphi_\mathcal{N}$. Then*

$$\hat{\rho}_{\varphi_\mathcal{N}} = E^\mathcal{M}_\mathcal{N}(\hat{\rho}_\varphi)$$

Remark. The above definition of density matrix is given in terms of the tracial state in contrast to the standard definition using the matrix trace **Tr**. Hence we use the word "adjusted".

Definition 2.2. *Let $\hat{\rho}_\varphi$ be the adjusted density matrix of a positive linear functional φ of a finite type* **I** *factor. Then*

$$\widehat{S}(\varphi) \equiv -\varphi(\log \hat{\rho}_\varphi)$$

is called the adjusted entopy of φ.

Remark. The adjusted density matrix and the adjusted entropy for a type $\mathbf{I}_n$ factor $\mathcal{M}$ with the dimension $\mathbf{Tr}(\mathbf{1}) = n$ are related to the usual ones by the following relations:

$$\hat{\rho}_\varphi = n\rho_\varphi, \; \widehat{S}(\varphi) = S(\varphi) - \varphi(\mathbf{1}) \log n. \tag{2.1}$$

The range of the values of entropy is given by the following well-known lemma.

Lemma 2.3. *If $\mathcal{M}$ is a type $\mathbf{I}_n$ factor and φ is a state of $\mathcal{M}$, then*

$$0 \leq S(\varphi) \leq \log n. \tag{2.2}$$

The equality $S(\varphi) = 0$ holds if and only if φ is a pure state of $\mathcal{M}$. The equality $S(\varphi) = \log n$ holds if and only if φ is the tracial state τ of $\mathcal{M}$.

3 Fermion Lattice Systems

We introduce Fermion lattice systems where there exists one spinless Fermion at each lattice site and they interact with each other. The restriction to spinless particle (i.e., one degree of freedom for each site) is just a matter of simplification of notation. All results and their proofs in the present work goes over to the case of an arbitrary (constant) finite number of degrees of freedom at each lattice site without any essential alteration.

The lattice we consider is ν-dimensional lattice $\mathbb{Z}^\nu$ ($\nu \in \mathbb{N}$, an arbitrary positive integer).

Definition 3.1. *The Fermion $\mathbf{C}^*$-algebra $\mathcal{A}$ is a unital $\mathbf{C}^*$ algebra satisfying the following conditions:*
(1-1) *For each lattice site $i \in \mathbb{Z}^\nu$, there are elements a_i and a_i^* of $\mathcal{A}$ called annihilation and creation operators, respectively, where a_i^* is the adjoint of a_i.*
(1-2) *The following CAR(canonical anticommutation relations) are satisfied for any $i, j \in \mathbb{Z}^\nu$.*

$$\begin{aligned} \{a_i^*, a_j\} &= \delta_{i,j}\mathbf{1} \\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0, \end{aligned} \tag{3.1}$$

where $\{A, B\} = AB + BA$ (anticommutator), $\delta_{i,j} = 1$ for $i = j$, and $\delta_{i,j} = 0$ for $i \neq j$.
(1-3) *Let $\mathcal{A}_\circ$ be the $*$-algebra generated by all a_i and a_i^* ($i \in \mathbb{Z}^\nu$),*

namely the (algebraic) linear span of their monomials $A_1 \cdots A_n$ *where* A_k *is* a_{i_k} *or* $a^*_{i_k}$, $i_k \in \mathbb{Z}^\nu$. *Then* $\mathcal{A}_\circ$ *is dense in* $\mathcal{A}$.
(2) *For each subset* I *of* $\mathbb{Z}^\nu$, *the* $\mathbf{C}^*$*-subalgebra of* $\mathcal{A}$ *generated by* a_i, a^*_i, $i \in \mathrm{I}$, *is denoted by* $\mathcal{A}(\mathrm{I})$ *and called a local algebra for* I. *If the cardinality* $|\mathrm{I}|$ *of the set* I *is finite, then* $\mathcal{A}(\mathrm{I})$ *is refered to as a local algebra or more specifically the local algebra for* I. *For the empty set* $\emptyset$, *we define* $\mathcal{A}(\emptyset) = \mathbb{C}\mathbf{1}$.

Remark For finite I, $\mathcal{A}(\mathrm{I})$ is known to be isomorphic to the tensor product of $|\mathrm{I}|$ copies of the full 2×2 matrix algebra $\mathrm{M}_2(\mathbb{C})$ and hence isomorphic to $\mathrm{M}_{2^{|\mathrm{I}|}}(\mathbb{C})$. Then

$$\mathcal{A}_\circ = \bigcup_{|\mathrm{I}|<\infty} \mathcal{A}(\mathrm{I})$$

has the unique $\mathbf{C}^*$-norm. $\mathcal{A}$ together with its individual elements $\{a_i, a^*_i | i \in \mathbb{Z}^\nu\}$ is uniquely defined up to isomorphism and is isomorphic to the UHF-algebra $\overline{\otimes}_{i\in\mathbb{Z}^\nu}\mathrm{M}_2(\mathbb{C})$, where the bar denotes the norm completion. $\mathcal{A}$ has the unique tracial state τ as the extension of the unique tracial state of $\mathcal{A}(\mathrm{I})$, $|\mathrm{I}| < \infty$.

Definition 3.2. Θ *denotes a unique automorphism of* $\mathcal{A}$ *satisfying*

$$\Theta(a_i) = -a_i,\ \Theta(a^*_i) = -a^*_i, \quad (i \in \mathrm{I}). \tag{3.2}$$

The even and odd parts of $\mathcal{A}$ *are defined as*

$$\mathcal{A}_+ \equiv \{a \in \mathcal{A} \mid \Theta(a) = a\},\ \mathcal{A}_- \equiv \{a \in \mathcal{A} \mid \Theta(a) = -a\}. \tag{3.3}$$

Remark 1. Such Θ exists and is unique because (3.2) preserves CAR. It obviously satisfies

$$\Theta^2 = id. \tag{3.4}$$

Remark 2. For any $a \in \mathcal{A}(\mathrm{I})$,

$$a = a_+ + a_-, \quad a_\pm \equiv \frac{1}{2}\big(a \pm \Theta(a)\big) \tag{3.5}$$

gives the (unique) splitting of a into a sum of $a_+ \in \mathcal{A}(\mathrm{I})_+$ and $a_- \in \mathcal{A}(\mathrm{I})_-$, where the even and odd parts of $\mathcal{A}(\mathrm{I})$ are denoted by $\mathcal{A}(\mathrm{I})_+$ and $\mathcal{A}(\mathrm{I})_-$, respectively.

Remark 3. For any $a \in \mathcal{A}_-$, we have

$$\tau(a) = \tau\big(\Theta(a)\big) = -\tau(a) = 0. \tag{3.6}$$

4 Entropy for Fermion Systems

4.1 SSA for Fermion Systems

We first show the SSA property of entropy for the Fermion case.

Theorem 4.1 (SSA). *For finite subsets* I *and* J *of* $\mathbb{Z}^\nu$, *the following strong subadditivity of* $\widehat{S}$ *holds for any state* ψ *of* $\mathcal{A}$:

$$\widehat{S}(\psi_{\mathrm{I}\cup\mathrm{J}}) - \widehat{S}(\psi_{\mathrm{I}}) - \widehat{S}(\psi_{\mathrm{J}}) + \widehat{S}(\psi_{\mathrm{I}\cap\mathrm{J}}) \le 0,$$

where ψ_{K} *denotes the restriction of* ψ *to* $\mathcal{A}(\mathrm{K})$. $\widehat{S}$ *in the inequality above can be replaced by* S, *namely, the following strong subadditivity of* S *holds for any state* ψ *of* $\mathcal{A}$:

$$S(\psi_{\mathrm{I}\cup\mathrm{J}}) - S(\psi_{\mathrm{I}}) - S(\psi_{\mathrm{J}}) + S(\psi_{\mathrm{I}\cap\mathrm{J}}) \le 0.$$

Remark. As for the proof of SSA for the tensor product systems (quantum spin lattice systems), see the original proof [7] [8].

4.2 Mean Entropy

We then show the existence of mean entropy (von Neumann entropy density) for translation invariant states of $\mathcal{A}$. We have the following result due to the SSA proved in Theorem 4.1 by completely the same method as for quantum spin lattice systems, see e.g. [4]. For $\mathrm{I} \subset \mathbb{Z}^\nu$ and a state ω, we denote the restricrtion of ω to $\mathcal{A}(\mathrm{I})$ $\omega|_{\mathcal{A}(\mathrm{I})}$.

Theorem 4.2 (Mean Entropy). *Let ω be a translation invariant state. The van Hove limit*

$$s(\omega) \equiv \lim_{\mathrm{I}\to\infty} \frac{1}{|\mathrm{I}|} S(\omega_{\mathrm{I}})$$

and

$$s(\omega) = \inf_{s\in\mathbb{N}^\nu} \frac{1}{|R_s|} S(\omega_{R_s}).$$

The mean entropy functional

$$\omega \mapsto s(\omega) \in [0, \log 2]$$

defined on the set of translation invariant states $\mathcal{A}^{*\tau}_{+,1}$, is affine and upper semi-continuous with respect to the weak topology.*

4.3 Entropy Inequalities for Translation Invariant States

The following two results are about the monotone properties of entropy as a functional of the set of finite regions of the lattice.

Theorem 4.3 (Monotonicity 1). *Let ω be a translation invariant state on $\mathcal{A}$ and let R_s and $R_{s'}$ be finite boxes of $\mathbb{Z}^\nu$ such that $R_s \subset R_{s'}$. Then*

$$\frac{1}{|R_s|} S(\omega_{R_s}) \geq \frac{1}{|R_{s'}|} S(\omega_{R_{s'}}).$$

Theorem 4.4 (Monotonicity 2). *Let ω, R_s and $R_{s'}$ be as above. Then*

$$S(\omega_{R_s}) \leq S(\omega_{R_{s'}}).$$

The following basic properties:

- Positivity and finiteness of the entropy of every local system,
- Strong subadditivity,
- Invariance property for states

axiomatically imply the above two theorems, see [5] for the detail.

As for the first theorem (Theorem 4.3), [3] considers more general regions than merely boxes, giving a general criterion of partial ordering sets which satisfy the monotone property of mean entropy. For example, convex octogonal sets in $\mathbb{Z}^2$ produce the monotone decreasing of mean entropy.

4.4 Failure of Triangle Inequality

Let I and J be two disjoint finite regions. For quantum spin systems, the so-called "triangle inequality of entropy" holds for any state ω [1]

$$|S(\omega_{\mathrm{I}}) - S(\omega_{\mathrm{J}})| \leq S(\omega_{\mathrm{I}\cup\mathrm{J}}).$$

In [9], it is shown that the above inequality does not hold for Fermion systems.

Now let I and J be distinct points $\{1\}$ and $\{2\}$ and denote the corresponding Fermion systems by $\mathcal{A}_1^{car}$ and $\mathcal{A}_2^{car}$ (disjoint bipartite Fermion systems). The total system $\mathcal{A}_{1,2}^{car}$ is given by $\mathcal{A}_1^{car} \vee \mathcal{A}_2^{car}$, the algebra algebraically generated by $\mathcal{A}_1^{car}$ and $\mathcal{A}_2^{car}$. We have

Theorem 4.5 (Failure of Traiangle Inequality). *For any positive number $x \in [0, \log 2]$, there exist a pure state φ of $\mathcal{A}_{1,2}^{car}$*

such that

$$|S(\varphi|_{\mathcal{A}_1^{car}}) - S(\varphi|_{\mathcal{A}_2^{car}})| = x$$

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