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ティープリッツ作用素が正規または解析的になる為の条件

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For a bounded measurable function $\varphi \in L^\infty$ on the unit circle, Toeplitz operator T_φ is defined by $T_\varphi f = P\varphi f$ for $f \in H^2$, where P is the orthogonal projection from L^2 onto H^2. If $\varphi \in H^\infty$, then we say that T_φ is analytic. Halmos askes whether every subnormal Toeplitz operator is either analytic or nomal. And concerning this problem Amemiya-Itô-Wong prove that every quasi-nomal Toeplitz operator is only normal or a scalar multiple of an isometry. We shall give here a condition that the Toeplitz operator T_φ is normal or analytic.

The following results are well known.

Proposition 1. ([2]) If $\mathcal{M}$ is a non-zero invariant subspace of T_z, then there exists an isometric Toeplitz operator T_g such that $\mathcal{M} = T_g H^2$.

Proposition 2. ([3]) $A \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z{}^* A T_z = A$. And, in particular, $A \in \mathcal{B}(H^2)$ is an analytic Toeplitz operator if and only if $T_z A = A T_z$.

Proposition 3. ([3]) $T_\varphi T_\psi$ is a Toeplitz operator if and only if $\overline{\varphi}$ or $\psi \in H^\infty$. In this case, $T_\varphi T_\psi = T_{\varphi\psi}$.

Proposition 4. ([4]) If φ is a non-constant function in L^∞, then $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_\varphi{}^*)} = \emptyset$ where $\sigma_p(T_\varphi)$ denotes the point spectrum of T_φ.

Lemma 1. For any $\varphi \notin H^\infty$, T_φ has no such type of invariant subspace as $T_g H^2$ for some non-constant inner function g.

Proof. For some non-constant inner function g, if $T_\varphi T_g H^2 \subseteq T_g H^2$, then there exists a $C \in \mathcal{B}(H^2)$ such that $T_{\varphi g} = T_g C$ because $T_\varphi T_g = T_{\varphi g}$ by Proposition 3. Since g is inner, $C = T_g{}^* T_{\varphi g} = T_\varphi$ and $T_{\varphi g} = T_g T_\varphi$ and hence $\varphi \in H^\infty$ by Proposition 3 because $\overline{g} \notin H^\infty$.

Lemma 2. For $\varphi \in H^\infty$, if $(T_\varphi{}^* T_\varphi)^2 = T_\varphi{}^{*2} T_\varphi{}^2$, then φ is a scalar multiple of an inner function.

Proof. By Proposition 3 and by the assumption,

$$T_{\overline{\varphi}\varphi}{}^2 = (T_\varphi{}^*T_\varphi)^2 = T_\varphi{}^{*2}T_\varphi{}^2 = T_{\overline{\varphi}^2}T_{\varphi^2} = T_{\overline{\varphi}^2\varphi^2} = T_{|\varphi|^4}$$

and $\overline{\varphi}\varphi \in H^\infty$ and hence $|\varphi|$ is constant. Therefore φ is a scalar multiple of an inner function.

For $\varphi \in L^\infty$, let $X_\varphi = T_\varphi T_z - T_z T_\varphi$ and let $Y_\varphi = T_z{}^*T_\varphi{}^*T_\varphi T_z - T_\varphi{}^*T_\varphi$. Then

$$\begin{aligned} X_\varphi = O &\rightleftharpoons \varphi \in H^\infty \text{ by Proposition 2,} \\ Y_\varphi = O &\rightleftharpoons T_\varphi{}^*T_\varphi \text{ is a Toeplitz operator by Proposition 2} \\ &\rightleftharpoons \varphi \in H^\infty \quad \text{by Proposition 3,} \\ \text{and} \quad Y_\varphi &= T_z{}^*T_\varphi{}^*(T_zT_\varphi + X_\varphi) - T_\varphi{}^*T_\varphi = T_z{}^*T_\varphi{}^*X_\varphi. \end{aligned}$$

Since $Y_\varphi = T_z{}^*T_\varphi{}^*(I - T_zT_z{}^*)T_\varphi T_z$ and since $(I - T_zT_z{}^*)H^2 = \vee\{1\}$, Y_φ is an at most rank one positive operator and $Y_\varphi T_z{}^*T_\varphi{}^*1 = \|Y_\varphi\|T_z{}^*T_\varphi{}^*1$.

And since, for any $f \in H^2$, $\|X_\varphi f\|_2{}^2 = \|(I - T_zT_z{}^*)T_\varphi T_z f\|_2{}^2 = \langle Y_\varphi f,\ f\rangle = \|Y_\varphi{}^{\frac{1}{2}} f\|_2{}^2$, $\mathcal{N}_{X_\varphi} = \mathcal{N}_{Y_\varphi}$ and $X_\varphi{}^*H^2 = Y_\varphi H^2 = \vee\{T_z{}^*T_\varphi{}^*1\}$ and hence

$$\begin{aligned} H^2 =& \{f \in H^2 :\ Y_\varphi f = o\} \oplus \{f \in H^2 :\ Y_\varphi f = \|Y_\varphi\| f\} \\ =& \mathcal{N}_{X_\varphi} \oplus \vee\{T_z{}^*T_\varphi{}^*1\} \end{aligned} \tag{$\sharp$}$$

and also we have $X_\varphi H^2 \subseteq \mathcal{N}_{T_z{}^*} = \vee\{1\}$.

Lemma 3. If $\{o\} \neq \mathcal{N}_{T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*} \neq H^2$, then $Y_\varphi - Y_{\overline{\varphi}} \neq O$ and $(Y_\varphi - Y_{\overline{\varphi}})H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$.

Proof. If $Y_\varphi - Y_{\overline{\varphi}} = O$, then $T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*$ is a Hermitian Toeplitz operator by Proposition 2 because $Y_\varphi - Y_{\overline{\varphi}} = T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_z - (T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)$. Let $T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^* = T_\psi$. Then the assumption implies $\psi \neq o$ and $0 \in \sigma_p(T_\psi)$. This contradicts Proposition 4. And since, for any $f \in H^2$,

$$(Y_\varphi - Y_{\overline{\varphi}})f = \langle f,\ T_z{}^*T_\varphi{}^*1\rangle \|Y_\varphi\| T_z{}^*T_\varphi{}^*1 - \langle f,\ T_z{}^*T_\varphi 1\rangle \|Y_{\overline{\varphi}}\| T_z{}^*T_\varphi 1,$$

we have $(Y_\varphi - Y_{\overline{\varphi}})H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$.

Theorem. If T_φ satisfies the following conditions ; (i) $(T_\varphi{}^*T_\varphi)^2 = T_\varphi{}^{*2}T_\varphi{}^2$, (ii) $\{o\} \neq \mathcal{N}_{T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*}$, (iii) Every eigen-space of $T_\varphi{}^*T_\varphi$ is invariant under $T_\varphi{}^*$ and

(iv) $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly dependent, then T_φ is normal or a scalar multiple of an isometry.

Proof. By Lemma 2, we have only to prove that there is no non-normal, non-analytic Toeplitz operator which satisfies the conditions (i), (ii), (iii) and (iv).

Let T_φ be non-normal and non-analytic. Since $T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_\varphi = O$ by (i),

$$\begin{aligned}&T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})T_\varphi = T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)T_zT_\varphi\\ &=(T_z{}^*T_\varphi{}^* - X_\varphi{}^*)(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)(T_\varphi T_z - X_\varphi)\\ &= - T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi - X_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)(T_\varphi T_z - X_\varphi) \qquad (1)\end{aligned}$$

and $T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi H^2 \subseteq X_\varphi{}^*H^2 + T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})H^2$ and hence, by Lemma 3,

$$T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 = \alpha T_z{}^*T_\varphi{}^*1 + \beta T_\varphi{}^*T_z{}^*T_\varphi{}^*1 + \gamma T_\varphi{}^*T_z{}^*T_\varphi 1 \quad \text{for some} \quad \alpha,\beta,\ \gamma \in \mathbb{C} \qquad (2)$$

because the conditions of Lemma 3 are satisfied by (ii) and by the non-normality of T_φ. And since

$$\begin{aligned}T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 &=(T_\varphi{}^*T_z{}^* + X_\varphi{}^*)T_\varphi 1 - (T_\varphi T_z{}^* + X_{\overline{\varphi}}{}^*)T_\varphi{}^*1\\ &=T_\varphi{}^*T_z{}^*T_\varphi 1 + aT_z{}^*T_\varphi{}^*1 - T_\varphi T_z{}^*T_\varphi{}^*1 + bT_z{}^*T_\varphi 1\\ &\quad \text{for some} \quad a,\ b \in \mathbb{C},\end{aligned}$$

$$\begin{aligned}&T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1\\ &=(T_\varphi{}^*T_z{}^* + X_\varphi{}^*)(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1\\ &=T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 + X_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1\\ &=T_\varphi{}^*T_z{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)1 + cT_z{}^*T_\varphi{}^*1 \quad \text{for some} \quad c \in \mathbb{C}\\ &=T_\varphi{}^*(T_\varphi{}^*T_z{}^*T_\varphi 1 + aT_z{}^*T_\varphi{}^*1 - T_\varphi T_z{}^*T_\varphi{}^*1 + bT_z{}^*T_\varphi 1) + cT_z{}^*T_\varphi{}^*1\\ &=T_\varphi{}^{*2}T_z{}^*T_\varphi 1 + aT_\varphi{}^*T_z{}^*T_\varphi{}^*1 - T_\varphi{}^*T_\varphi T_z{}^*T_\varphi{}^*1 + bT_\varphi{}^*T_z{}^*T_\varphi 1 + cT_z{}^*T_\varphi{}^*1\end{aligned}$$

and, by (2),

$$\begin{aligned}&T_\varphi{}^*T_\varphi T_z{}^*T_\varphi{}^*1\\ &=T_\varphi{}^{*2}T_z{}^*T_\varphi 1 + (c-\alpha)T_z{}^*T_\varphi{}^*1 + (a-\beta)T_\varphi{}^*T_z{}^*T_\varphi{}^*1 + (b-\gamma)T_\varphi{}^*T_z{}^*T_\varphi 1. \qquad (3)\end{aligned}$$

Since, by (1),

$$T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})T_\varphi H^2 \subseteq T_z{}^*T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*)X_\varphi H^2 + X_\varphi{}^* H^2 \tag{4}$$

and since $T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})T_\varphi H^2 \subseteq T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})H^2$,

$$\begin{aligned} &T_\varphi{}^*T_z{}^*T_\varphi{}^*1 = \lambda_1 T_z{}^*T_\varphi{}^*1 \\ \text{and}\quad &T_\varphi{}^*T_z{}^*T_\varphi 1 = \lambda_2 T_z{}^*T_\varphi{}^*1 \\ &\text{for some } \lambda_1,\ \lambda_2 \in \mathbb{C} \end{aligned} \tag{5}$$

by (iv), Lemma 3 and (2). And hence, by (3),

$$T_\varphi{}^*T_\varphi(T_z{}^*T_\varphi{}^*1) = \{\lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2\} T_z{}^*T_\varphi{}^*1. \tag{6}$$

Let $r = \lambda_2\lambda_1 + (c-\alpha) + (a-\beta)\lambda_1 + (b-\gamma)\lambda_2$ and $\mathcal{M} = \{f \in H^2 \ :\ T_\varphi{}^*T_\varphi f = rf\}$. Since, for any $f \in \mathcal{M}$,

$$\begin{aligned} (T_\varphi{}^*T_\varphi - rI)T_z{}^*f =& T_\varphi{}^*(T_z{}^*T_\varphi - X_{\overline{\varphi}}{}^*)f - rT_z{}^*f \\ =& (T_z{}^*T_\varphi{}^* - X_\varphi{}^*)T_\varphi f - T_\varphi{}^*X_{\overline{\varphi}}{}^*f - rT_z{}^*f \\ =& -X_\varphi{}^*T_\varphi f - T_\varphi{}^*X_{\overline{\varphi}}{}^*f \\ =& -a_1T_z{}^*T_\varphi{}^*1 - T_\varphi{}^*(b_1T_z{}^*T_\varphi 1) \quad \text{for some} \quad a_1,\ b_1 \in \mathbb{C} \\ =& -(a_1 + b_1\lambda_2)T_z{}^*T_\varphi{}^*1 \quad \text{by (5)} \end{aligned}$$

and since $T_z{}^*T_\varphi{}^*1 \in \mathcal{M}$ by (6), $(T_\varphi{}^*T_\varphi - rI)^2T_z{}^*f = o$ and $(T_\varphi{}^*T_\varphi - rI)T_z{}^*f = o$ because $\|(T_\varphi{}^*T_\varphi - rI)T_z{}^*f\|_2{}^2 = \langle (T_\varphi{}^*T_\varphi - rI)^2T_z{}^*f,\ T_z{}^*f\rangle = 0$ and hence $\mathcal{M}$ is invariant under $T_z{}^*$. Since T_φ is non-analytic by the assumption, $T_z{}^*T_\varphi{}^*1 \neq o$ by (♮) and by Proposition 2 and $\mathcal{M} \neq H^2$ by Proposition 3 and hence $\mathcal{M}$ is non-trivial. Therefore $\mathcal{M}^\perp = T_gH^2$ for some non-constant inner function g by Proposition 1. Since $\mathcal{M}$ is invariant under $T_\varphi{}^*$ by (iii), T_gH^2 is invariant under T_φ and $\varphi \in H^\infty$ by Lemma 1. This contradicts the assumption that T_φ is non-analytic.

Corollary. ([1]) Every quasi-normal T_φ (i.e., T_φ commutes with $T_\varphi{}^*T_\varphi$) is only normal or a scalar multiple of an isometry.

Proof. It is clear that every quasi-normal T_φ satisfies the conditions (i), (ii) and (iii). And, by Theorem, we have only to show that quasi-normal T_φ satisfies the condition (iv).

If $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly independent, then

$$(Y_\varphi - Y_{\overline{\varphi}})T_\varphi H^2 = \vee\{T_z{}^*T_\varphi{}^*1,\ T_z{}^*T_\varphi 1\}$$

because, for any $f \in H^2$,

$$\begin{aligned}(Y_\varphi - Y_{\overline{\varphi}})T_\varphi f &= \langle T_\varphi f,\ T_z{}^*T_\varphi{}^*1\rangle \|Y_\varphi\| T_z{}^*T_\varphi{}^*1 - \langle T_\varphi f,\ T_z{}^*T_\varphi 1\rangle \|Y_{\overline{\varphi}}\| T_z{}^*T_\varphi 1 \\ &= \langle f,\ T_\varphi{}^*T_z{}^*T_\varphi{}^*1\rangle \|Y_\varphi\| T_z{}^*T_\varphi{}^*1 - \langle f,\ T_\varphi{}^*T_z{}^*T_\varphi 1\rangle \|Y_{\overline{\varphi}}\| T_z{}^*T_\varphi 1.\end{aligned}$$

And since $T_\varphi{}^*(Y_\varphi - Y_{\overline{\varphi}})T_\varphi H^2 \subseteq X_\varphi{}^*H^2$ by (4) in the proof of Theorem because $T_\varphi{}^*(T_\varphi{}^*T_\varphi - T_\varphi T_\varphi{}^*) = O$ by the quasi-normality of T_φ,

$$\begin{aligned} & T_\varphi{}^*T_z{}^*T_\varphi{}^*1 = \lambda_1 T_z{}^*T_\varphi{}^*1 \\ \text{and}\quad & T_\varphi{}^*T_z{}^*T_\varphi 1 = \lambda_2 T_z{}^*T_\varphi{}^*1 \quad \text{for some} \quad \lambda_1,\ \lambda_2 \in \mathbb{C}\end{aligned}$$

and this contradicts the assumption that $T_\varphi{}^*T_z{}^*T_\varphi{}^*1$ and $T_\varphi{}^*T_z{}^*T_\varphi 1$ are linearly independent.

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