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Fourier ultra-hyperfunctions as the boundary values of smooth solutions of heat equations

上智大学理工学研究科 諏訪 将範
Sophia University, Masanori Suwa

1 Introduction

1987, Matsuzawa characterized the space of hyperfunctions with compact support K (denote by $\mathcal{A}'(K)$) as the boundary value of C^{∞}-solutions of the heat equations:

Theorem 1.1 (Matsuzawa [4]). *Let* $u\in\mathcal{A}'(K)$ *and* $U(x,t):=<u_{y},E(x-y,t)>$, $E(x,t)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{x^{2}}{4t}}$ (heat kernel). *Then* $U(x,t)\in C^{\infty}(\mathbb{R}_{+}^{n+1})$, $\mathbb{R}_{+}^{n+1}:=\{(x,t);x\in\mathbb{R}^{n},t>0\}$ *and* $U(\cdot,t)\in\mathcal{A}(\mathbb{C}^{n})$, $\mathcal{A}(\mathbb{C}^{n})$ *is the space of entire functions, for each* $t>0$. *Furthermore* $U(x,t)$ *satisfies the heat equation:*

$$\left(\frac{\partial}{\partial t}-\triangle\right)U(x,t) = 0 \quad in\ \mathbb{R}_{+}^{n+1}, \tag{1}$$

where $\triangle=\sum_{j=1}^{n}\frac{\partial^{2}}{\partial x_{j}^{2}}$ (Laplacian).

For every $\varepsilon>0$, *we have*

$$|U(x,t)| \leq C_{\varepsilon}e^{\frac{\varepsilon}{t}} \quad in\ \mathbb{R}_{+}^{n+1}. \tag{2}$$

We have for any $\delta>0$,

$$U(\cdot,t)\rightarrow 0 \tag{3}$$

uniformly in $\{x\in\mathbb{R}^{n};dis(x,K)\geq\delta\}$ *as* $t\rightarrow 0_{+}$.

We have $U(\cdot,t)\rightarrow u$ *in* $\mathcal{A}'(K)$ *as* $t\rightarrow 0_{+}$, *i.e.*

$$u(\varphi) = \lim_{t\rightarrow 0_{+}}\int U(x,t)\chi(x)\varphi(x)dx, \quad \varphi\in\mathcal{A}(\mathbb{C}^{n}), \tag{4}$$

for any $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ *such that* $\chi = 1$ *in a neighborhood of* K.

Conversely, every C^∞*-function* $U(x,t)$ *defined in* $\mathbb{R}_+^{n+1}$ *satisfying the conditions* (1), (2) *and* (3) *can be expressed in the form* $U(x,t) =< u_y, E(x-y,t) >$ *with unique element* $u \in \mathcal{A}'(K)$.

Furthermore 1989, Matsuzawa characterized the space of distributions with compact support K (denote by $\mathcal{E}'_K$) and the space of ultradistributions with compact support K (denote by $\mathcal{E}_K^{\{s\}'}$, $\mathcal{E}_K^{(s)'}$, $s > 1$) by the same way (for details of definitions of ultradistributions, we refer the reader to [3]):

Theorem 1.2 (Matsuzawa [5]). *Let* $u \in \mathcal{E}_K^{\{s\}'}$ $(\mathcal{E}_K^{(s)'})$ *with* $s > 1$ *and* $U(x,t) =< u_y, E(x-y,t) >$. *Then* $U(x,t) \in C^\infty(\mathbb{R}_+^{n+1})$. *Furthermore* $U(x,t)$ *satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) \;=\; 0 \quad in\ \mathbb{R}_+^{n+1}. \tag{5}$$

For every $\varepsilon > 0$, $\delta > 0$, *there exists a positive constant* $C_{\varepsilon,\delta}$ *such that*

$$|U(x,t)| \;\le\; C_{\varepsilon,\delta} e^{\left(\frac{\varepsilon}{t}\right)^{\frac{1}{2s-1}} - \frac{dis(x,K_\delta)^2}{8t}} \quad in\ \mathbb{R}_+^{n+1}, \tag{6}$$

where $K_\delta := \{x \in \mathbb{R}^n; dis(x,K) \le \delta\}$.

We have $U(x,t) \to u$ *as* $t \to 0_+$ *in* $\mathcal{E}^{\{s\}'}(\mathbb{R}^n)$ (resp. $\mathcal{E}^{(s)'}(\mathbb{R}^n)$).

Conversely, every C^∞*-function* $U(x,t)$ *defined in* $\mathbb{R}_+^{n+1}$ *satisfying conditions* (5) and (6) *can be expressed in the form* $U(x,t) =< u_y, E(x-y,t) >$ *with unique element* $u \in \mathcal{E}_K^{\{s\}'}$ (resp. $\mathcal{E}_K^{(s)'}$) .

Theorem 1.3 (Matsuzawa [5]). *Let* $u \in \mathcal{E}'_K$ *and* $U(x,t) =< u_y, E(x-y,t) >$. *Then* $U(x,t) \in C^\infty(\mathbb{R}_+^{n+1})$. *Furthermore* $U(x,t)$ *satisfies the following conditions:*

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) \;=\; 0 \quad in\ \mathbb{R}_+^{n+1}. \tag{7}$$

There exists a nonnegative integer $N = N(u)$ *such that*

$$|U(x,t)| \;\le\; C_\delta t^{-N} e^{-\frac{dis(x,K_\delta)^2}{8t}} \quad in\ \mathbb{R}_+^{n+1}, \tag{8}$$

We have $U(x,t) \to u$ *as* $t \to 0_+$ *in* $\mathcal{E}'_K$.

Conversely, every C^∞*-function* $U(x,t)$ *defined in* $\mathbb{R}^{n+1}_+$ *satisfying conditions* (7) and (8) *can be expressed in the form* $U(x,t) =< u_y, E(x-y,t) >$ *with unique element* $u \in \mathcal{E}'_K$.

1993, K.W.Kim, S.-Y.Chung and D.Kim characterized the space of Fourier hyperfunctions with compact support K in $\mathbb{D}^n$ (denote by $\mathcal{F}'(K)$) by the same way ([2]):

Theorem 1.4 (K.W.Kim, S.-Y.Chung and D.Kim, [2]). *Let* $u \in \mathcal{F}'(K)$ *and* $U(x,t) =< u_y, E(x-y,t) >$. *Then* $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ *and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0 \quad in\ \mathbb{R}^{n+1}_+. \tag{9}$$

For any $\varepsilon > 0$, *there is a constant* $C \geq 0$ *such that*

$$|U(x,t)| \leq Ce^{\frac{\varepsilon}{t}+\varepsilon t+\varepsilon|x|-\frac{dis(x,K_\delta \cap \mathbb{R}^n)^2}{8t}} \quad in\ \mathbb{R}^{n+1}_+. \tag{10}$$

We have $U(x,t) \to u$ *as* $t \to 0_+$ *in* $\mathcal{F}'(K)$.

Conversely, every C^∞*-function* $U(x,t)$ *defined in* $\mathbb{R}^{n+1}_+$ *satisfying conditions* (9) *and* (10) *can be expressed in the form* $U(x,t) =< u_y, E(x-y,t) >$ *with unique element* $u \in \mathcal{F}'(K)$.

For $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions, the following result is known:

Theorem 1.5 (Matsuzawa [6]). *Let* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $U(x,t) =< u_y, E(x-y,t) >$. *Then* $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$ *and satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0 \quad in\ \mathbb{R}^{n+1}_+. \tag{11}$$

There exists constants $C \geq 0$, $\nu \geq 0$ *and* $k \geq 0$ *such that*

$$|U(x,t)| \leq Ct^{-\nu}(1+|x|)^k \quad in\ \mathbb{R}^{n+1}_+. \tag{12}$$

We have $U(x,t) \to u$ *as* $t \to 0_+$ *in* $\mathcal{S}'(\mathbb{R}^n)$.

Conversely, every C^∞*-function* $U(x,t)$ *defined in* $\mathbb{R}^{n+1}_+$ *satisfying conditions* (11) *and* (12) *can be expressed in the form* $U(x,t) =< u_y, E(x-y,t) >$ *with unique element* $u \in \mathcal{S}'(\mathbb{R}^n)$.

Besides many authors research generalized function by the same way. For example, 1999, M.Budinčević, Z.L.-Crvenković and D.Perošić characterized the spaces of Beurling and Roumieu type tempered ultradistributions (for details, we refer the reader to [1]).

2 Main theorem

Now, we can obtain the same result for Fourier ultra-hyperfunctions. First, we give some notations:

Notations

$$\begin{aligned}
\mathbb{C}^n &= \mathbb{R}^n + \imath\mathbb{R}^n. \\
z &= x + \imath y, \quad \zeta = \xi + \imath\eta. \\
z &= (z_1, z_2, \cdots, z_n), \quad z_j = x_j + \imath y_j, \quad j = 1, 2, \cdots, n. \\
\zeta &= (\zeta_1, \zeta_2, \cdots, \zeta_n), \quad \zeta_j = \xi_j + \imath\eta_j, \quad j = 1, 2, \cdots, n. \\
<\zeta, z> &= \sum_{j=1}^{n} \zeta_j z_j. \quad \text{In particular, } z^2 =< z, z > . \\
E(z,t) &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{z^2}{4t}}, \quad z \in \mathbb{C}^n, \quad t > 0.
\end{aligned}$$

Let K be a convex compact set in $\mathbb{R}^n$. Then we define supporting function $h_K(x)$ by

$$h_K(x) = \sup_{\xi \in K} < \xi, x >.$$

We denote "*complex Laplacian*" by Δ:

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}.$$

Let L be a closed set in $\mathbb{C}^n$ and $\mathring{L}$ be interior of L. We denote by $\mathcal{H}(\mathring{L})$ the spaces of holomorphic functions on $\mathring{L}$ and by $\mathcal{C}(L)$ the spaces of continuous functions on L.

Definition 2.1. Let K and K' be convex compact sets in $\mathbb{R}^n$. Then we define $Q_b(\mathbb{R}^n + \imath K, K')$ as follows:

$$Q_b(\mathbb{R}^n + \imath K, K') \\ := \{f \in \mathcal{H}(\mathbb{R}^n + \imath \overset{\circ}{K}) \cap \mathcal{C}(\mathbb{R}^n + \imath K) : \sup_{z \in \mathbb{R}^n + \imath K} |f(z) e^{h_{K'}(x)}| < +\infty\}.$$

Definition 2.2. We define the space Q_0 as follows:

$$Q_0 := \varprojlim_{K, K' \subset\subset \mathbb{R}^n} Q_b(\mathbb{R}^n + \imath K, K'),$$

where $\varprojlim$ means projective limit.

Definition 2.3. We denote by Q'_0 the dual space of Q_0. The element of Q'_0 is called Fourier ultra-hyperfunctions.

For details of Fourier ultra-hyperfunctions, we refer the reader to [7].

The following theorem is a main result:

Theorem 2.4. *Let $T \in Q'_0$ and $U(z,t) =< T_\zeta, E(z-\zeta, t) >$. Then $U(z,t)$ is an entire function of z and C^∞-function of t, $t > 0$ satisfying the following conditions:*

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(z,t) = 0, \tag{13}$$

$$U(z,t) \longrightarrow T, \ (t \to 0_+), \ \text{in} \ Q'_0 \tag{14}$$

$\exists R \geq 0$, $\exists b \geq 0$, $\exists C \geq 0$, *s.t.*

$$|U(z,t)| \leq C e^{\frac{1}{4t}\sum_{j=1}^n (b+|y_j|)^2 + R\sum_{j=1}^n |x_j| + nR^2 t}. \tag{15}$$

Conversely, for a function $U(z,t)$, $t > 0$, entire function of z, C^∞-function of t, satisfying (13) *and* (15), *there exists unique $T \in Q'_0$ such that $< T_\zeta, E(z-\zeta,t) >= U(z,t)$.*

For details we refer the reader to [8].

At present:

Recently we obtained the same results for tempered distributions with support in a proper convex cone. This paper will be soon appeared.

Reference

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