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Analytic smoothing effects for a class of dispersive equations

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1 Introduction and the main results

There are many reserches about smoothing effects for dispersive equations. We can find many results about smoothing effects for the Schödinger equation. However the study for general class of dispersive operators is not enough when we compare the results for a general class of dispersive operators with that of Schödinger operators. One of our aim is to know the dependence of the order of operators. Our class of operators we define later include not only Schödinger operators but also linearized KdV operators.

First let us describe our problem.

Let m be an integer greater than or equal to 2. Let $P(y, D_y)$ be a linear differential operator of order m in $\mathbb{R}^n$,

$$P(y,D_y)=\sum_{|\alpha|\leq m} c_\alpha(y)D_y^\alpha. \tag{1. 1}$$

We assume that $P(y, D_y)$ has analytic coefficients in $\mathbb{R}^n$ and a real principal symbol. And we assume that $P(y, D_y)$ is the real principal type (in strong sense). That is, for all $(y,\eta)\in T^*\mathbb{R}^n\backslash 0$ there exists an integer j with $1\leq j\leq n$ such that we have $\partial_{\eta_j}p(y,\eta)\neq 0$, where $p(y,\eta)=\sum_{|\alpha|=m}c_\alpha(y)\eta^\alpha$ be the principal symbol of $P(y, D_y)$.

Let us consider the initial value problem

$$\begin{cases} D_tu+P(y,D_y)u=0, \\ u|_{t=0}=u_0(y). \end{cases} \tag{1. 2}$$

We can study more gereral situations, however we consider the simpler case that the space dimension n equals to 1 in this note.

$$P(y,D_y)u=\sum_{0\leq l\leq m} c_l(y)D_y^l, \tag{1. 3}$$

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where $c_m(y) = 1$ and $c_l(y)$ are analytic in $\mathbf{R}$, that is, the coefficient of the principal part i constant.

Moreover we shall make the following assumption.
One can find positive $C_0 > 0, R_0 > 0, K_0 > 0$, and $\sigma_0 \in (0,1)$ such that for $y \in \mathbf{R}$ with $|y| > R$ and $k \in \mathbf{N} \cup \{0\}$,

$$\sum_{0 \le l \le m-1} |D_y^k c_l(y)| \le C_0 \frac{K_0^k k!}{|y|^{1+\sigma_0+k}}. \tag{1. 4}$$

Let $\rho = (y, \eta) \in T^*\mathbf{R}\backslash 0$, and let $(Y(s; y, \eta), \Theta(s; y, \eta))$ be the solution to the equation,

$$\begin{cases} \frac{d}{ds} Y(s) & = \frac{\partial p}{\partial \eta}(Y(s), \Theta(s)), \quad Y(0) = y, \\ \frac{d}{ds}\Theta(s) & = -\frac{\partial p}{\partial y}(Y(s), \Theta(s)), \quad \Theta(0) = \eta. \end{cases} \tag{1. 5}$$

In our case $p(y, \eta) = p(\eta) = \eta^m$. Therefore

$$\begin{cases} Y(s) & = y + ms\eta^{m-1}, \\ \Theta(s) & = \Theta(0) = \eta. \end{cases} \tag{1. 6}$$

We remark that for $\eta \neq 0$.

$$\lim_{s \to +\infty} |Y(s, y, \eta)| = +\infty.$$

The nontrapping condition is satisfied.

Let $u(t, \cdot) \in C(\mathbf{R}, L^2(\mathbf{R}))$ be the solution of the initial value problem (1.2).
Let us introduce a space of the initial data,

$$\Gamma_{\rho_0}^+ = \{Y(s; y_0, \eta_0) \in \mathbf{R}; s \ge 0\} \tag{1. 7}$$

$$X_{\rho_0}^+ = \{v \in L^2(\mathbf{R}); \exists \varepsilon_0 > 0, \exists \delta_0 > 0, e^{\delta_0 |y|^{\frac{1}{m-1}}} v(y) \in L^2(\Gamma_{\rho_0}^+)\}. \tag{1. 8}$$

The next theorem is the main result of this paper. This is one of the expression for the microlocal smoothing effect.

Theorem 1.1 *Let $P(y, D_y)$ be defined in* (1.3) *satisfying* (1.4) *and $\rho_0 = (y_0, \eta_0) \in T^*\mathbf{R}\backslash 0$. Let $u_0 \in L^2(\mathbf{R})$ be in $X_{\rho_0}^+$. Then for all $t < 0$ ρ_0 does not belong to the analytic wave front set $WF_A[u(t, \cdot)]$ of the solution $u(t, \cdot)$ for* (1.2).

We give a simple application of this theorem.
Let $u(t, y)$ be the solution to the next initial value problem,

$$\begin{cases} D_t u + D_y^m u = 0, \\ u|_{t=0} = u_0(y). \end{cases} \tag{1. 9}$$

Corollary 1.1 *Let the initial data $u_0 \in L^2(\mathbb{R})$ satisfy that there exists a positive constant δ_0 such that*

$$\int_0^\infty e^{2\delta_0|y|^{\frac{1}{m-1}}}|u_0(y)|^2 dy < \infty, \tag{1. 10}$$

in the case m is odd, or

$$\int_{-\infty}^\infty e^{2\delta_0|y|^{\frac{1}{m-1}}}|u_0(y)|^2 dy < \infty, \tag{1. 11}$$

in the case m is odd.
Then the solution to the initial value problem (1.9) *becomes analytic with respect to the spase variable y for $t < 0$.*

Our approach is based on FBI transform which was used by Robbiano and Zuily in [15]. The reader can see the details about this problem and historical results in [15] and [17].

2 Analytic wave front set and FBI transform

In this section we define FBI transform and analytic wave front set. The reader had better refer to [15] and [20].

Let $\rho_0 = (y_0, \eta_0) \in T^*\mathbf{R}\backslash 0$. Let $\varphi(x, y)$ be a holomorphic function in a neighborhood $U_0 \times V_{y_0}$ of $(0, y_0)$ in $\mathbf{C} \times \mathbf{C}$ which satisfies

$$\frac{\partial \varphi}{\partial y}(0, y) = -\eta_0, \tag{2. 1}$$

$$\operatorname{Im} \frac{\partial^2 \varphi}{\partial y^2}(0, y) > 0, \tag{2. 2}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y}(0, y) \neq 0. \tag{2. 3}$$

For above $\varphi(x, y)$ we can define,

$$\Phi(x) = \max_{y \in V_{y_0}}(-\operatorname{Im} \varphi(x, y)), \tag{2. 4}$$

for $x \in U_0$.

Let $a(x, y, \lambda) = \sum_{k \geq 0} a_k(x, y)\lambda^{-k}$ be a analytic symbol of order zero, elliptic in a neighborhood of $(0, y_0)$. Let $\chi \in C_0^\infty$ be a cutoff function with support in a neighberhood of y_0, $0 \leq \chi \leq 1$, and $\chi \equiv 1$ near y_0.

The FBI transform of a distribution $u \in \mathcal{D}'(\mathbf{R})$ is defined by

$$Tu(x, \lambda) = \langle \chi(\cdot)u, e^{i\lambda\varphi(x,\cdot)}a(x, \cdot, \lambda)\rangle, \quad \lambda > 1. \tag{2. 5}$$

According to [20] we can characterize the analytic wave front set of $u \in \mathcal{D}'(\mathbf{R})$ by using FBI transform. Next (2.6) and (2.7) are equivalent,

$$\rho_0 \notin WF_A[u]. \tag{2. 6}$$

$$\begin{aligned} &\exists C > 0, \exists \mu > 0, \exists \lambda_0 \geq 1 \text{ such that} \\ &e^{-\lambda\Phi(x)}|Tu(x,\lambda)| \leq Ce^{-\mu\lambda}, \quad \text{for } \forall x \in U_0, \forall \lambda \geq \lambda_0. \end{aligned} \tag{2. 7}$$

Assume that $u(t,\cdot)$ is a element of a family of distribution on $\mathbf{R}$ depending of a real parameter t. Let $t_0 \in \mathbf{R}$. We shall say that a point $\rho_0 \in T^*\mathbf{R}\backslash 0$ does not belong to the locally uniform analytic wave front set $\widetilde{WF_A}[u(t_0,\cdot)]$ if there exist an FBI transform T, positive constants $C, \mu, \lambda_0, \varepsilon$, and a neighborhood U_0 of 0 such that

$$\begin{aligned} &e^{-\lambda\Phi(x)}|Tu(t,x,\lambda)| \leq Ce^{-\mu\lambda}, \\ &\text{for} \quad \forall x \in U_0, \quad \forall \lambda \geq \lambda_0, \quad \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon). \end{aligned} \tag{2. 8}$$

3 Idea of proof

We make use of FBI transform in order to make a change our operator $P(y, D_y)$ into more simpler one. This idea has already been known by Egorov's theorem in the theory of Fourier integral operators. Thanks to this theorem we can transform any first order real principal type operator into D_x by Fourier integral operators.

Since our operator is of order $m \geq 2$, so we make use of FBI transform instead of Fourier integral operators in order to transform the operator of order m into the first order operator D_x. The parameter λ in the FBI transform means $|\xi|$ in some sense. To balance the order of two operators we introduce the parameter λ. In fact we can realize the next relationship by introducing a suitable FBI transform F,

$$F\frac{1}{\lambda^m}P(y, D_y) = \frac{1}{\lambda}D_xF, \quad \text{(mod analytic)}. \tag{3. 1}$$

This is the main idea of this approach.

Let us give a sketch of proof related to the assumption for the initial data.
We introduce a FBI type transformation.
Let $u(t,z)$ is the solution of the initial value problem,

$$\begin{cases} [D_t + P(z, D_z)]u(t,z) = 0, \\ u|_{t=0} = u_0(z). \end{cases} \tag{3. 2}$$

Let $\chi \in C_0^\infty(\mathbf{R})$ with $0 \leq \chi \leq 1$ and

$$\chi(r) = \begin{cases} 1, & |r| \leq \frac{1}{2}\varepsilon_0, \\ 0, & |r| \geq \varepsilon_0. \end{cases}$$

We introduce

$$Su(t,x,\lambda)=\int_{\mathbb{R}} e^{i\lambda\varphi(x,z)} f(x,z,\lambda)\chi\left(\frac{z-Y(\operatorname{Re} x)}{1+|x|}\right)u(t,z)dz. \tag{3. 3}$$

where $Y(s)$ is given in (1.5).

By operating the operator S to (3.2) we have

$$\left(\frac{\partial}{\partial t}+\lambda^{m-1}\frac{\partial}{\partial x}\right)Su(t,x,\lambda)=i\lambda^m\left(\frac{1}{\lambda}D_x Su-\frac{1}{\lambda^m}SPu\right). \tag{3. 4}$$

We define

$$\begin{aligned} I(t,x,\lambda) &= \frac{1}{\lambda}D_x Su-\frac{1}{\lambda^m}SPu \\ &= \int_{\mathbb{R}}\left(\frac{1}{\lambda}D_x-\frac{1}{\lambda^m}{}^tP(z,D_z)\right)(e^{i\lambda\varphi}f\chi)u(t,z)dz, \end{aligned} \tag{3. 5}$$

where ${}^tP(z,D_z)w=\sum_{l=0}^m(-D_z)^l(a_l(z)w(z))=(-D_z)^m w+\sum_{l=0}^{m-1}b_l(z)D_z^l$. The coefficients also satisfy the condition (1.4).

We define

$$J(x,z,\lambda)=\left(\frac{1}{\lambda}D_x-\frac{1}{\lambda^m}{}^tP(z,D_z)\right)(e^{i\lambda\varphi}f). \tag{3. 6}$$

If $J(x,z,\lambda)$ is small enough in some sense, then we have only to consider the equation

$$\left(\frac{\partial}{\partial t}+\lambda^{m-1}\frac{\partial}{\partial x}\right)Su(t,x,\lambda)=0. \tag{3. 7}$$

This equation is easily solved

$$Su(t,x,\lambda)=Su(0,x-\lambda^{m-1}t,\lambda).$$

Since x is near 0, we have $|z|\geq\frac{1}{4}|\eta_0|^{m-1}|t_0|\lambda^{m-1}$ on the support of χ.

Then we have

$$\begin{aligned} &|Su(0,x-\lambda^{m-1}t,\lambda)| \\ &=\left|\int_{\mathbb{R}} e^{i\lambda\varphi(x-\lambda^{m-1}t,z)}f(x-\lambda^{m-1}t,z,\lambda)\right. \\ &\quad\left.\chi\left(\frac{z-Y(\operatorname{Re} x-\lambda^{m-1}t)}{1+|x-\lambda^{m-1}t|}\right)e^{-\frac{\delta_0}{4}|z|^{\frac{1}{m-1}}}e^{\frac{\delta_0}{4}|z|^{\frac{1}{m-1}}}u_0(z)dz\right| \\ &\leq Ce^{\lambda\Phi(x)-\frac{1}{16}|\eta_0||t_0|^{\frac{1}{m-1}}\delta_0\lambda}\int_{\mathbb{R}}|\chi(\cdots)e^{\frac{\delta_0}{4}|z|^{\frac{1}{m-1}}}u_0(z)|dz \\ &\leq Ce^{\lambda\Phi(x)-\frac{1}{16}|\eta_0||t_0|^{\frac{1}{m-1}}\delta_0\lambda}||e^{\delta_0|z|^{\frac{1}{m-1}}}u_0||_{L^2(\Gamma_{\varepsilon_0})}. \end{aligned} \tag{3. 8}$$

This implies

$$\begin{aligned} &e^{-\lambda\Phi(x)}|Su(t,x,\lambda)|\leq Ce^{-\mu\lambda}, \\ &\text{for}\quad \forall x\in U_0,\quad \forall\lambda\geq\lambda_0,\quad \forall t\in(t_0-\varepsilon,t_0+\varepsilon). \end{aligned} \tag{3. 9}$$

Since we can get same properties for FBI transform with a usual cutoff function as the operator S introduced in (3.3), we can prove Theorem 1.1.

The smallness of $J(x, z, \lambda)$ is important for the above approach. In fact we can globally constract a parametrix along the bicharacteristics.

Lemma 3.1 *There exist* $\varepsilon_1 > 0$, *and a holomorphic function* $\varphi(x, z)$ *in the set*

(3. 10) $$E = \{(x, z) \in \mathbf{C} \times \mathbf{C}; \operatorname{Re} x \geq -\varepsilon_1, |\operatorname{Im} x| < \varepsilon_1, |z - Y(x; y_0, \eta_0)| < \varepsilon_1(1 + |x|)\},$$

such that

(3. 11) $$\frac{\partial\varphi}{\partial x}(x, z) = p_m(z, -\frac{\partial\varphi}{\partial z}(x, z)) \quad \text{in } E,$$

(3. 12) $$\frac{\partial\varphi}{\partial z}(0, y_0) = -\eta_0,$$

(3. 13) $$\operatorname{Im} \frac{\partial^2\varphi}{\partial z^2}(0, y_0) > 0,$$

(3. 14) $$\frac{\partial^2\varphi}{\partial x \partial z}(0, y_0) \neq 0.$$

Lemma 3.2 *There exists an analytic symbol* f *of order zero defined in* E *such that*

(3. 15) $$|J(x, z, \lambda)| \leq Ce^{\lambda\Phi(x) - \mu_0\lambda}(1 + |x|)^{N_0},$$

where $\mu_0 > 0$ *and* N_0 *is a integer.*

In order to prove Lemma 3.1 and Lemma 3.2 we have to solve the eikonal equation and the transport equation. Since we make λ large, we introduce the set E which is global along the bicharacteristic. The way to constract phase and amplitude functions is discussed in [17].

References

[1] H. Chihara, Gain of regularity for semilinear Schrödinger equations, Math. Ann. **315**, (1999), 529-567.

[2] H. Chihara, Gain of analyticity for semilinear Schrödinger equations, preprint.

[3] W. Craig, T. Kappeler and W. Strauss, Miclolocal dispersive smoothing for the Schrödinger equation, Comm. Pure. Appl. Math. **48**, (1995), 769-860.

[4] J-M. Delort, F.B.I. transformation, Lecture Notes in Math. Studies No.122. 1992, Springer-Verlag.

[5] S. Doi, Commutator algebra and abstract smoothing effect, J. Funct. Anal. **168**, (1999), 428-469.

[6] S. Doi, Smoothing effects for Schödinger evolution equation and global behavior of geodestic flow, Math Ann. **318**, (2000), 355-389.

[7] N. Hayashi and K. Kato, Regularity in time of solutions to nonlinear Schrodinger equations, J. Funct. Anal. **128** (1995), no. 2, 253-277.

[8] N. Hayashi and S. Saitoh, Analyticity and smoothing effect for the Schrodinger equation, Ann. Inst. H. Poincare Phys. Theor. **52**, (1990), no. 2, 163-173.

[9] L. Hörmander, The Analysis of Linear Partial Differential Operators, Springer Verlag, 1983-1985.

[10] L. T'Joen, Microlocal smoothing properties for plates equation, preprint.

[11] K. Kajitani and S. Wakabayashi, Analytically smoothing effect for Schrödinger type equations with variable coefficients, in Direct and inverse problems of mathematical physics, Int. Soc. Anal. Appl. Comput., 5, Kluwer Acad. Publ., Dordrecht, (2000), 185-219.

[12] K. Kajitani, Smoothing effect in Gevrey classes for Schrodinger equations II, Workshop on Partial Differential Equations (Ferrara, 1999). Ann. Univ. Ferrara Sez. VII, 173–186.

[13] Y. Morimoto, L. Robbiano, and C. Zuily, Remark on the analytic smoothing for the Schrödinger equation, Indiana Math. J., to appear.

[14] T. Ōkaji, Propergation of wave packets and smoothing properties of solutions to Schrödinger equations with unbounded potential, preprint.

[15] L. Robbiano, and C. Zuily, Miclolocal analytic smoothing effect for the Schrödinger equation, Duke Math. Journal, **100**, no.1, (1999), 93-129.

[16] L. Robbiano, and C. Zuily, Effet régularisant analytique microlocal pour l'équation de Schrödinger : le cas des données oscillantes, Comm. on P.D.E. **25**, (2000), 1891-1906.

[17] H. Takuwa, Analytic smoothing effects for a class of dispersive equations, preprint.

[18] S. Tarama, On the wellposed Cauchy problem for some dispersive equations, J. Math. Soc. Japan **47**, no. 1, (1995), 143-158.

[19] S. Tarama, Analytic solutions of the Korteweg-de Vries equation, preprint.

[20] J. Sjöstrand, Singularités analytiques microlocales, Astérisques, **95** (1982).