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SYMBOLIC CONSTRUCTION OF MICROLOCAL SOLUTIONS FOR SOME PARTIAL DIFFERENTIAL EQUATIONS OF IRREGULAR TYPE BY WKB ANALYSIS

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1. Introduction

Differential equations with regular singularities are well known, but in the case of having irregular singularities, as is shown in Stokes phenomena, the structure of their solutions are more complicated.

For example, in the regular case, formal solutions always converge. On the contrary, they never converge in the irregular case. But we can make genuine solutions which converge in some sectors.

These properties reflect on the solution complex in microlocal analysis. Many theorems hold only in the case that sheaves considered there are $\mathbb{R}$-constructible because of the complexities of the irregular case.

In spite of these difficulties, we claim that we can construct the solutions as WKB ones in the case of regular type and irregular type at the same time. A few years ago, Professor Kataoka showed the construction of the solutions of some microdifferential equations of regular type (Fuchsian microdifferential equations) using a successive approximation. He estimated the term in the series solutions by his formal norms. Our construction, however, needs WKB analysis by using pseudodifferential operators with some exponentially decreasing growth order.

In this thesis, for some (partial) differential equations, we construct their solutions which are pseudodifferential operators with exponential decreasing growth. To start with, we make the formal solutions by using WKB analysis. Then we estimate them by the symbolic calculus with respect to pseudodifferential operators. Thus we find that they possess exponentially decreasing growth, using the classical Cauchy-Kovalevski theorem in the end.

We consider the next partial differential operator:

$$P(z,\partial_z,\partial_w) := \sum_{j=0}^{m} a_j(z)\partial_w^{j\theta}\partial_z^{m-j} \ , \tag{1.1}$$

where $\partial_w = \partial/\partial w$, $\partial_z = \partial/\partial z$, $a_j(z)$ $(j = 0, 1, \cdots, m)$ are holomorphic in a neighborhood of $z = 0 \in \mathbb{C}$ and

$$0 < \theta \leq 1 \ . \tag{1.2}$$

Fractional derivations are defined as the Riemann-Liouville integral, but we do not write down the detail because we regard ∂_w as a large parameter. In the case that $0 < \theta < 1$, this operator is of irregular type. This fact requires a new class of psedodifferential operators.

2. Definitions of several sheaves in microlocal analysis

We prepare some sheaves for using them in a following section.

In this section, let X be a complex manifold of dimension n and Y a submanifold of codimension d. Then we can define a sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ of holomorphic microfunctions by

$$\mathcal{C}^{\mathbb{R}}_{Y|X} := \mu_Y(\mathcal{O}_X) \otimes or_{Y|X}[d] ,$$

where $\mu_Y(\cdot)$ means a microlocalization functor along Y (see [KS]).

Let us consider the next spaces with local charts :

$$\begin{array}{ccccc} \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d} & \hookrightarrow & \mathbb{R}^d \times \mathbb{C}^{n-d} & \hookrightarrow & \mathbb{C}^d \times \mathbb{C}^{n-d} = \mathbb{C}^n \\ \Cup & & \Cup & & \Cup \\ x = (x', x'') & & (x', z'') & & (z', z'') = z \,. \end{array}$$

Let $M = M' \times M''$, $N = M' \times X''$ and $X = X' \times X''$ be subsets of $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$, $\mathbb{R}^d \times \mathbb{C}^{n-d}$ and $\mathbb{C}^d \times \mathbb{C}^{n-d} = \mathbb{C}^n$ respectively. We set the following :

$$\begin{array}{ccc} \Sigma := T^*_M X \underset{T^*X}{\times} \widetilde{\Sigma} & \longrightarrow & \widetilde{\Sigma} := T^*_N X \underset{T^*X}{\times} X \simeq T^*_{M'} X' \times X'' \\ \Cup & & \Cup \\ (x', \xi', x'') & \longmapsto & (x', \xi', z'') \,. \end{array}$$

Then a sheaf of microfunctions with holomorphic parameters on $\widetilde{\Sigma}$ can be defined by

$$\mathcal{CO}_N := \mu_N(\mathcal{O}_X) \otimes or_{N|X}[d] ,$$

where $\mu_N(\cdot)$ is a microlocalization functor. We often write $\mathcal{C}_d\mathcal{O}_{n-d}$ or $\mathcal{C}_{x'}\mathcal{O}_{z''}$ instead of $\mathcal{CO}_N$.

Set $z^* = (0; \zeta_0) \in \dot{T}^*X$ and

$$V_\varepsilon = \{(z;\zeta) \in \dot{T}^*X; |z| < \varepsilon,\ |\zeta| > \frac{1}{\varepsilon},\ \left|\frac{\zeta}{|\zeta|} - \frac{\zeta_0}{|\zeta_0|}\right| < \varepsilon\} \,.$$

We set the following spaces :

$$\begin{aligned} S(V_\varepsilon) :&= \{p(z,\zeta) \in \mathcal{O}_{T^*X}(V_\varepsilon); |p(z,\zeta)| \le C\exp(h_1|\zeta|) \\ &\qquad \text{for any } h_1 > 0 \text{ and with } C > 0\}, \\ S^{(\rho-0)}(V_\varepsilon) &:= \{p(z,\zeta) \in \mathcal{O}_{T^*X}(V_\varepsilon); |p(z,\zeta)| \le C_1\exp(h_2|\zeta|^\rho) \\ &\qquad \text{for any } h_2 > 0 \text{ and with } C_1 > 0, \text{ where } 0 < \rho \le 1\}, \\ N(V_\varepsilon) &:= \{p(z,\zeta) \in \mathcal{O}_{T^*X}(V_\varepsilon); |p(z,\zeta)| \le C'\exp(-l|\zeta|) \ \text{ with } C', l > 0\}, \\ N^{(\rho)}(V_\varepsilon) &:= \{p(z,\zeta) \in \mathcal{O}_{T^*X}(V_\varepsilon); |p(z,\zeta)| \le C_2\exp(-l_2|\zeta|^\rho) \ \text{ with } C_2, l_2 > 0\}. \end{aligned}$$

Then the next proposition is obtained by Kataoka and Aoki's symbolic calculus (see [A3]).

Proposition 2.1. *There is an isomorphism of vector spaces :*

$$\lim_{\varepsilon \to 0} S(V_\varepsilon)/N(V_\varepsilon) \xrightarrow{\sim} \mathcal{E}^{\mathbb{R}}_{X,z^*} ,$$

where $\mathcal{E}^{\mathbb{R}}_X$ stands for a sheaf of pseudodifferential operators in microlocal analysis.

In a similar manner, we get the following definition.

Definition 2.2. We define a sheaf $\mathcal{E}_X^{\mathbb{R},(\rho-0),d}$ of pseudodifferential operators with exponentially decreasing growth order $(\rho-0)$ by

$$\mathcal{E}_X^{\mathbb{R},(\rho-0),d}{}_{,z^*} := \lim_{\varepsilon\to 0} S^{(\rho-0)}(V_\varepsilon)/N^{(\rho)}(V_\varepsilon) \;. \tag{2.1}$$

Proposition 2.3. *The space $N^{(\rho)}(V_\varepsilon)$ is an ideal in $S^{(\rho-0)}(V_\varepsilon)$.*

Proof. In the definitions above, take $Q(z,\zeta)\in N^{(\rho)}(V_\varepsilon)$ and $P(z,\zeta)\in S^{(\rho-0)}(V_\varepsilon)$. Then the multiplication PQ (as functions) satisfies the property of $N^{(\rho)}(V_\varepsilon)$ if we set $h_2 = l_2/2$. □

3. Construction of solution operators on microfunction

In this section, let X be a manifold $\mathbb{C}_z\times\mathbb{C}_w^n$ and N a submanifold

$$N=\{(z,w)\in X; \mathrm{Im} w = 0\}\simeq N^{\mathbb{R}} \;,$$

where $N^{\mathbb{R}}$ is the underlying real analytic manifold of X. We denote $(z,w;\zeta,\tau)$ by the coordinates of T^*X ;

$$z=x+iy\in\mathbb{C},\quad w=u+iv\in\mathbb{C}^n,\quad \zeta\in\mathbb{C},\quad \tau\in\mathbb{C}^n \;,$$

and (ζ,τ) is identified with the real cotangent vector $\mathrm{Re}(\zeta dz+\tau\cdot dw)$ of $T^*N^{\mathbb{R}}$. Then the sheaf $\mathcal{CO}_N$ on

$$T_N^*X=\{(z,w;\zeta,\tau)\in T^*X; \zeta=0,\ \mathrm{Im}w=0,\ \mathrm{Re}\tau=0\}$$

of microfunctions with a holomorphic parameter z is defined by

$$\mathcal{CO}_N := \{f(z,w)\in\mathcal{C}_{N^{\mathbb{R}}}; \overline{\partial}_z f=0\} \;.$$

Here $\mathcal{C}_{N^{\mathbb{R}}}$ is a sheaf of usual microfuntions on $N^{\mathbb{R}}$ and it is well-known that $\mathcal{CO}_N$ is identified with a sheaf $\mathcal{C}_{N|X}$ of relative microfunctions as $\mathcal{E}_X$-modules.

3.1. **Main result.** Our aim is to construct solutions which are microfunctions $f(z,w)$ with a holomorphic parameter z for the equation

$$P(z,\partial_z,\partial_w)f(z,w) := \left(\sum_{j=0}^{m} a_j(z)\partial_w^{j\theta}\partial_z^{m-j}\right) f(z,w)=0 \;, \tag{3.1}$$

where $\partial_w=\partial/\partial w$, $\partial_z=\partial/\partial z$, $a_j(z)$ $(j=0,1,\cdots,m)$ are holomorphic in a neighborhood of $z=0$ and

$$0<\theta\le 1. \tag{3.2}$$

In the following, we consider ∂_w as a large parameter ξ. Thus we make its solutions on

$$D=\{(z,\xi)\in\mathbb{C}\times\mathbb{C}; 0<|z|<\varepsilon, |\xi|>1/\varepsilon\} \;. \tag{3.3}$$

Here we can set $a_0(z)=1$ because there only exists a quantized contact transformation which preserves a sheaf $\mathcal{CO}_N$ of microfunctions with a holomorphic parameter according to [KtS].

We will construct a microdifferential operator $U(z,\partial_w)$ of some type instead of a solution $f(z,w)$, satisfying

$$f(z,w)=U(z,\partial_w)g(w) \;, \tag{3.4}$$

with an arbitrary microfunction $g(w)$ in a neighborhood of $w = 0$.
In consequence, we get the following theorem.

Theorem 3.1. *We consider the following equation*

$$P(z,\partial_z,\partial_w)U(z,\partial_w) = \left(\sum_{j=0}^{m} a_j(z)\partial_w^{j\theta}\partial_z^{m-j}\right) U(z,\partial_w) = 0 ,$$

where $\partial_z = \partial/\partial z$, $\partial_w = \partial/\partial w$, *each* $a_j(z)$ $(j = 1,2,\cdots,m)$ *is holomorphic in a neighborhood of* $z = 0 \in \mathbb{C}$ *and*

$$0 < \theta \leq 1 .$$

By virtue of regarding ∂_w *as a large parameter* ξ, *at the point in which the solutions* $\eta = \alpha_k(z,\xi)$ $(k = 1,2,\cdots,m)$ *of its total symbol*

$$\sum_{j=0}^{m} a_j(z)\xi^{j\theta}\eta^{m-j} = 0 \tag{3.5}$$

ramify on

$$D = \{(z,\xi) \in \mathbb{C}\times\mathbb{C}\ ; 0 < |z| < \varepsilon,\ |\xi| > 1/\varepsilon\} ,$$

the equation has solutions which are pseudodifferential operators with exponentially decreasing growth order $(\theta - 0)$, *that is, we have*

$$U(z,\partial_w) \in \Gamma\left(D; \mathcal{E}^{\mathbf{R},(\theta-0),d}\right) .$$

Remark 3.2. On the classical Stokes lines, namely,

$$\operatorname{Re}\int^{z} \{\alpha_j(z,\xi) - \alpha_k(z,\xi)\}dz = 0,\quad j,k = 1,2,\cdots,m,\quad j \neq k, \tag{3.6}$$

where $\alpha_j(z,\xi)$ $(j = 1,2,\cdots,m)$ are solutions of (3.5), the equation (3.1) is of hyperbolic type. Hence there exist solutions on these lines except at $z = 0$.

3.2. **Solutions of WKB type.** For the sake of brevity, we consider the next partial differential operator

$$P(z,\partial_z,\partial_w) = \partial_z^m + P_1(z,\partial_w)\partial_z^{m-1} + \cdots + P_m(z,\partial_w) . \tag{3.7}$$

Then the principal symbol of this operator is

$$\sigma_0(P) = (\eta - \alpha_1'(z,\xi))\cdots(\eta - \alpha_m'(z,\xi)) , \tag{3.8}$$

where each $\alpha_j'(z,\xi)$ $(j = 1,\cdots,m)$ is homogeneous of degree 1.
We find the solutions of WKB type :

$$f_k(z,w) = :\sum_{j=0}^{\infty} u_j(z,\xi)\exp\left(\int^{z}\alpha_k'(z,\xi)dz\right) : g(w) ,\quad k = 1,2,\cdots,m \tag{3.9}$$

where $g(w)$ is a microfunction and $: \cdot :$ means its operation on $g(w)$.
By the Weierstrass division theorem, the operator (3.7) is reduced to $(m-1)$-order$\times(\partial_z - \alpha_1(z)\partial_w + \text{lower order})$, namely, the equation we consider becomes

$$(\partial_z - \alpha_1(z)\partial_w + \text{lower})U(z,\partial_w) = 0 . \tag{3.10}$$

In this equation, the lower order term can be written as $Q^{-1}(\partial_z - \alpha(z)\partial_w)Q$, where Q is a microdifferential operator of order 0. Hence we get

$$(\partial_z - \alpha_1(z)\partial_w)Q = Q(\partial_z - \alpha_1(z)\partial_w + A(z, \partial_w, \partial_z)) ,$$

where A is a microdifferential operator of order 0.

Here $Q = Q(z, \partial_w)$ satisfies

$$\frac{\partial Q}{\partial z}(z, \partial_w) = QA(z, \partial_w, \partial_z) .$$

Set $Q = \sum_{j=0}^{\infty} Q_j$, $Q_0 = 1$. Then a successive approximation leads to the relation

$$\frac{\partial Q_{j+1}}{\partial z} = Q_j A .$$

Therefore we have

$$Q = \sum_{j=0}^{\infty} Q_j = 1 + \int_0^z QAdz .$$

The operator Q has a form as follows :

$$Q(z, \partial_w) = \sum_{l=-\infty}^{0} Q_l(z)\partial_w^{-l} ,$$

where Q_{-l} is of order $l!$. Hence Q is a pseudodifferential operator.

Thanks to the formal symbol theory of the symbolic calculus, we get the next proposition.

Proposition 3.3. *For the formal sum*

$$\sum_{j=0}^{\infty} u_j(z, \xi) \exp\left(\int^z \alpha'_k(z, \xi)dz\right), \quad k = 1, 2, \cdots, m,$$

corresponding to $\eta = \alpha'_k(z, \xi)$ $(k = 1, 2, \cdots, m)$ *of simple zeros of the principal symbol* $\sigma_0(P') = (\eta - \alpha'_1(z, \xi)) \cdots (\eta - \alpha'_m(z, \xi))$*, its simple symbol* $U(z, \xi)$ *has the next property :*
for an arbitrary $\varepsilon > 0$*, there exists a positive constant* $C > 0$ *such that*

$$|P(z, \partial_z, \xi)U(z, \xi)| = |F(z, \xi)| \leq C \exp\left(\varepsilon|\xi|^{\theta}\right), \quad \textit{for } |\xi| \gg 1 . \tag{3.11}$$

3.3. **Construction of solutions from formal solutions.** The results of this section are due to [KtF].

We formally have

$$\widetilde{U}(z, \xi) = \xi^{-m\theta} \sum_{k=0}^{\infty} \left(- \sum_{j=0}^{m-1} \xi^{(j-m)\theta} a_j(z) \partial_z^{m-j} \right)^k F(z, \xi) \tag{3.12}$$

as the solution of (3.1), where $a_0(z) = 1$.

This formal sum does not converge in general. But we can make solutions of some type in the following manner.

Let A be a sufficiently large constant. We set $W_k := \{\xi \in (1,\infty); \xi^\theta > Ak\}$ for $k \in \mathbb{N}\backslash\{0\}$. Then we define U_1 instead of $\widetilde{U}$ by the following :

$$U_1(z,\xi) = \xi^{-m\theta}\sum_{k=0}^{\infty}\chi_{W_k}(\xi^\theta)\left(-\sum_{j=0}^{m-1}\xi^{(j-m)\theta}a_j(z)\partial_z^{m-j}\right)^k F(z,\xi)\ , \tag{3.13}$$

where $\chi_{W_k}(\xi^\theta)$ is a characteristic function of W_k, namely,

$$\chi_{W_k}(\xi^\theta) = \begin{cases} 1, & \text{if } \xi^\theta \in W_k \\ 0, & \text{if } \xi^\theta \notin W_k\ . \end{cases}$$

In the following, we set $D_r = \{z \in \mathbb{C}; |z| < r\}$.

Lemma 3.4. *For an arbitrary $\varepsilon > 0$, there exists a positive constant M_ε such that*

$$|U_1(z,\xi)| \le M_\varepsilon \exp\left(\varepsilon|\xi|^\theta\right)\ . \tag{3.14}$$

The function $U_1(z,\xi)$ is not a solution of (3.1) but it gives a sufficient approximation of the solution of (3.1) in the following sense.

By virtue of the construction, we have

$$\begin{aligned}
&P(z,\partial_z,\xi)U_1(z,\xi)\\
&= \sum_{k=0}^{\infty}\chi_{W_k}(\xi^\theta)\left(-\sum_{j=0}^{m-1}\xi^{(j-m)\theta}a_j(z)\partial_z^{m-j}\right)^k F(z,\xi)\\
&\qquad - \sum_{k=0}^{\infty}\chi_{W_k}(\xi^\theta)\left(-\sum_{j=0}^{m-1}\xi^{(j-m)\theta}a_j(z)\partial_z^{m-j}\right)^{k+1} F(z,\xi)\\
&= F(z,\xi) - \sum_{k=0}^{\infty}\chi_{W_{k-1}\backslash W_k}(\xi^\theta)\left(-\sum_{j=0}^{m-1}\xi^{(j-m)\theta}a_j(z)\partial_z^{m-j}\right)^k F(z,\xi)\ .
\end{aligned}$$

Therefore we find

$$P(z,\partial_z,\xi)U_1(z,\xi) = F(z,\xi) - F_0(z,\xi) \tag{3.15}$$

on $D_1 \times (1,\infty)$, where

$$F_0(z,\xi) = \sum_{k=0}^{\infty}\chi_{W_{k-1}\backslash W_k}(\xi^\theta)\left(-\sum_{j=0}^{m-1}\xi^{(j-m)\theta}a_j(z)\partial_z^{m-j}\right)^k F(z,\xi)\ . \tag{3.16}$$

This error function $F_0(z,\xi)$ is holomorphic in D_1 for each fixed $\xi \in (1,\infty)$ since the sum is locally finite on $D_1 \times (1,\infty)$. Then the next lemma is obtained.

Lemma 3.5. *There exist positive constants δ_1 and M_1 such that*

$$|F_0(z,\xi)| \le M_1 \exp\left(-\delta_1|\xi|^\theta\right),\quad z \in D_1\ . \tag{3.17}$$

Next we prove the existence of U_0 in some class, satisfying

$$P(z, \partial_z, \xi) U_0(z, \xi) = F_0(z, \xi) \ , \tag{3.18}$$

by the Cauchy-Kovalevski theorem with a large parameter. Then $U = U_0 + U_1$ satisfies the equation (3.1) with desired properties.

For $r > 0$ and $L \geq 1$, set

$$\Omega_r = \{z \in \mathbb{C}; |z| < r\} \ ,$$
$$\Omega_{L,r} = \{z \in \mathbb{C}; L|z| < r\} \ .$$

Now we consider the Cauchy problem :

$$\begin{cases} P(z, \partial_z, \xi) U_0(z, \xi) = F_0(z, \xi) \ , \\ \partial_z^j U_0(z, \xi)\big|_{z=0} = 0, \quad j < m \ . \end{cases} \tag{3.19}$$

Lemma 3.6. *There exists $L \geq 1$ such that the Cauchy problem* (3.19) *has a unique solution $U_0 \in \mathcal{O}(\Omega_{L,r})$ for any ξ. Moreover, there exist $r_1 > 0$, $M_0 > 0$ and $\delta_2 > 0$ such that*

$$|U_0(z, \xi)| \leq M_0 \exp\left(-\delta_2 |\xi|^\theta\right) \ , \quad z \in \Omega_{L,r_1} . \tag{3.20}$$

Hence we have proved the main theorem.

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