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Necessary Conditions for the Gevrey–Well–Posedness of Schrödinger Type Equations

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1 Introduction

We study necessary conditions under which the following Cauchy problem of Schrödinger type,

$$Lu = \Big(i\partial_t + \Delta + \sum_{j=1}^{n} b_j(x)\partial_{x_j} + c(x)\Big)u = f(t,x), \quad u(0,x) = \varphi(x), \tag{1.1}$$

is well–posed in Gevrey spaces G^s, $1 < s < \infty$. Here $G^s = \varinjlim_{\varrho>0} G^s_\varrho$, and G^s_ϱ is the Hilbert space $G^s_\varrho = \{v \in L^2(\mathbb{R}^n)\colon \|v\|_{s,\varrho} = \left\|\exp(\varrho\langle\xi\rangle^{1/s})\hat{v}(\xi)\right\|_{L^2} < \infty\}$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\hat{v}$ is the usual Fourier transform of v.

Definition 1.1. We say that the Cauchy problem for the operator L is *forward G^s well–posed* if for every $T > 0$ and every $\varrho_0 > 0$ there are constants $C = C(T, \varrho_0)$ and $\varrho > 0$ such that for every $\varphi \in G^s_{\varrho_0}$, $f \in C([0,T], G^s_{\varrho_0})$ there is a unique solution $u \in C([0,T], G^s_\varrho)$ to (1.1) with

$$\|u(t,\cdot)\|_{s,\varrho} \le C\,\|\varphi\|_{s,\varrho_0} + C\int_0^t \|f(\tau,\cdot)\|_{s,\varrho_0}\,d\tau, \quad 0 \le t \le T.$$

If the coefficients b_j are purely imaginary valued, then *a priori* estimates of a solution u to (1.1) in the spaces L^2, H^∞, and G^s_ϱ can be easily derived, and the well–posedness of this Cauchy problem follows by standard arguments. The situation is more delicate when $\Re b_j \not\equiv 0$. For example, the Cauchy problem for the operator

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$L = i\partial_t + \partial_x^2 + \partial_x$ is neither well–posed in L^2 nor in G^s, $1 < s < \infty$, as can be shown by an explicit representation of the solution, see also [12]. Generally, well–posedness requires a certain decay of $\Re b_j(x)$ at infinity.
Therefore, we propose the following condition:

Condition 1. There is a constant $M = M(d_0)$ such that

$$\sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}}\left|\int_0^\sigma \sum_{j=1}^n \Re b_j(x+2\theta\omega)\omega_j\,d\theta\right| \le M(1+|\sigma|)^{d_0}, \quad \forall\sigma\in\mathbb{R}.$$

We assume that the coefficients b_j and c belong to Gevrey spaces $G^{s_b}_{L^\infty}$, $G^s_{L^\infty}$:

$$\begin{aligned} &\|\partial_x^\alpha b_j(\cdot)\|_{L^\infty} \le C^{1+|\alpha|}\alpha!^{s_b}, \quad \forall\alpha, \\ &\|\partial_x^\alpha c(\cdot)\|_{L^\infty} \le C^{1+|\alpha|}\alpha!^{s}, \quad \forall\alpha. \end{aligned} \tag{1.2}$$

Theorem 1. *Let (1.2) be satisfied, and let d_0 be a number with $d_0 > 3/(s+1)$ and $d_0 > 2/(s+1-s_b)$. If Condition 1 is violated, then the Cauchy problem for the operator L is* not *G^s well-posed.*

Sufficient conditions for the G^s well–posedness of the Cauchy problem for the operator $L = i\partial_t+\triangle+\sum_{j=1}^n b_j(t,x)\partial_{x_j}+c(t,x)$ were given in [2], namely $\Re b_j(t,x) = o(\langle x\rangle^{1/s-1})$. In case of the model operator $L = i\partial_t + \triangle + \langle x\rangle^{d-1}\partial_x$ with $x\in\mathbb{R}^1$, and $0 < d < 1$, the Cauchy problem is therefore well–posed if $d < 1/s$. On the other hand, Theorem 1 implies ill–posedness for $d > 3/(s+1)$.
This gap can be closed if we suppose that the coefficients b_j decay not too rapidly:

Condition 2. There are $x_0 \in \mathbb{R}^n$, $\omega_0 \in S^{n-1}$, and $\varepsilon_0 > 0$, $c_0 > 0$ such that

$$-\sum_{j=1}^n \Re b_j(x+\tau\omega')\omega_j \ge 2c_0\langle\tau\rangle^{d_0-1},$$

for all $\tau \ge 0$, $|x-x_0| < \varepsilon_0$, and all ω, $\omega' \in S^{n-1}$ with $|\omega-\omega_0| < \varepsilon_0$, $|\omega'-\omega_0| < \varepsilon_0$.

Theorem 2. *Suppose (1.2) with $s_b < s$ and Condition 2. Then $d_0 \le 1/s$ is necessary for the G^s well-posedness.*

A necessary condition for H^∞ well–posedness was given in [7]:

$$\sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}}\left|\int_0^\sigma \sum_{j=1}^n \Re b_j(x+2\theta\omega)\omega_j\,d\theta\right| \le M\log(1+|\sigma|)+N, \quad \forall\sigma\in\mathbb{R}.$$

This condition is sufficient in the case of one space dimension; and it is sufficient in the cases of two or more space dimensions if one supposes certain relations on derivatives of the coefficients b_j, see [8].
The investigation of an operator with variable coefficients in the principal part, $L = i\partial_t + \sum_{j,k} a_{jk}(x)\partial_{x_j}\partial_{x_k} + \sum_j b_j(x)\partial_{x_j} + c(x)$, where $a(x,\xi) = \sum_{j,k} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2$, $c_0 > 0$, requires the introduction of the bicharacteristic strip $(X,P) = (X,P)(t,x,p)$, which is the solution to the Hamilton–Jacobi equations,

$$\partial_t X_j = \partial_{P_j} a(X,P), \quad \partial_t P_j = -\partial_{X_j} a(X,P), \quad (X,P)(0,x,p) = (x,p).$$

Then a necessary condition for the H^∞ well-posedness is

$$\sup_{x,\omega} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(\theta,x,\omega))P_j(\theta,x,\omega)\,d\theta \right| \leq M \log(1+|\sigma|) + N, \quad \forall \sigma \in \mathbb{R},$$

under some additional condition. For details, see [6].
Sufficient and necessary conditions for H^s well-posedness were discussed in [3], [4] and [13]. These conditions are similar to the conditions for H^∞ well-posedness if a loss of regularity is allowed, otherwise similar to the conditions of L^2 well-posedness.
In [9] and [11], the following necessary condition for L^2 well-posedness was shown:

$$\sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}} \left| \int_0^\sigma \sum_{j=1}^n \Re b_j(X(\theta,x,\omega))P_j(\theta,x,\omega)\,d\theta \right| \leq M, \quad \forall \sigma \in \mathbb{R}.$$

This condition is also sufficient, see [10].
Schrödinger type equations with a lower order term of order strictly less than 1 were investigated in [1]; and sufficient conditions for G^s well-posedness were proved.
Theorem 1 and Theorem 2 will be discussed simultaneously; and the both cases will be called Case I and Case II, respectively. The following lemma, which gives us an integrated estimate of $\Re b_j$ from below, is quite essential.

Lemma 1.1. *Assume that $0 < d_0 < 1$ and that Condition 1 is violated. Then, for each $k \in \mathbb{N}$, there are $x_k \in \mathbb{R}^n$, $\sigma_k \in \mathbb{R}_+$, $\omega_k \in S^{n-1}$ with the property that*

$$-\int_0^{\sigma_k} \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,d\theta = k(1+\sigma_k)^{d_0},$$

$$-\int_0^{\sigma} \sum_{j=1}^n \Re b_j(x_k + 2\theta\omega_k)\omega_{k,j}\,d\theta \geq k d_0 \sigma(1+\sigma_k)^{d_0-1}, \quad 0 \leq \sigma \leq \sigma_k,$$

where σ_k tends to infinity for $k \to \infty$.

This lemma gives us a sequence $\{\sigma_k\}_k$ tending to infinity in Case I. In Case II, we choose this sequence arbitrarily, but still tending to infinity. Now we fix special initial data, $\varphi_k(x) = \varphi(x - x_k)$ (in Case I), and $\varphi_k(x) = \varphi(x - x_0)$ (in Case II), where $\varphi \in G^s_{\varrho_0}$ is given by $\hat{\varphi}(\xi) = \langle\xi\rangle^{-(n+1)/2}\exp(-\varrho_0\langle\xi\rangle^{1/s})$. Assuming that (1.1) is G^s well-posed, there is a unique solution $u_k \in C^1([0,T], G^s_\varrho)$ of

$$Lu_k = 0, \quad u_k(0,x) = \varphi_k(x). \tag{1.3}$$

Next we define a seminorm $E_k(t)$ for the function $u_k(t,\cdot)$.
Let $h = h(x) \in G^{s_0}$ (with $s_0 > 1$ very close to 1) be a function with

$$h(x) = \begin{cases} 0 & : |x| \geq 1, \\ 1 & : |x| \leq 1/2, \end{cases} \qquad 0 \leq h(x) \leq 1.$$

We choose the symbols

$$w_k(t,x,\xi) = h\left(\frac{x - x_k - 2t\sigma_k^{\delta_3}\omega_k}{\sigma_k^{-\delta_1}}\right) h\left(\frac{\xi - \sigma_k^{\delta_3}\omega_k}{\sigma_k^{\delta_2}}\right), \qquad \text{(Case I)},$$
$$w_k(t,x,\xi) = h\left(\frac{x - x_0 - 2t\sigma_k\omega_0}{\varepsilon\langle 2t\sigma_k\rangle}\right) h\left(\frac{\xi - \sigma_k\omega_0}{\sigma_k^{\delta_2}}\right), \qquad \text{(Case II)},$$

where $0 < \varepsilon \ll \varepsilon_0$, $\delta_1 = 1 - d_0$, and δ_2, δ_3 are certain positive constants. For multiindizes $\alpha, \beta \in \mathbb{N}^n$, we specify

$$w_k^{(\alpha\beta)}(t,x,\xi) = \partial_y^\alpha h(y)\partial_\eta^\beta(\eta)\Big|_{y=\sigma_k^{\delta_1}(x-x_k-2t\sigma_k^{\delta_3}\omega_k),\ \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k^{\delta_3}\omega_k)},$$
$$w_k^{(\alpha\beta)}(t,x,\xi) = \partial_y^\alpha h(\varepsilon^{-1}y)\partial_\eta^\beta(\eta)\Big|_{y=(2t\sigma_k)^{-1}(x-x_0-2t\sigma_k\omega_0),\ \eta=\sigma_k^{-\delta_2}(\xi-\sigma_k\omega_0)},$$

in Case I, Case II, respectively. These cut–off symbols are supported near the bicharacteristic strip. With some positive constant κ, we set $\mathbb{N} \ni N_0 = \lfloor\sigma_k^{\kappa/s_1}\rfloor$, choose $s_1 > s_0$, and define the seminorm

$$E_k(t) = \sum_{|\alpha|\leq N_0, |\beta|\leq N_0-2} (\alpha!\beta!)^{-s_1}\left\|W_k^{(\alpha\beta)}(t,x,D_x)u_k(t,x)\right\|_{L^2(\mathbb{R}^n_x)}.$$

The ill–posedness of the Cauchy problem can be proved by estimates of $E_k(t)$ from above and below which contradict for large σ_k if we choose δ_1, δ_2, δ_3, κ, ε suitably. For reasons of space, we omit the tedious calculations, which can be found in [5], and only sketch the proof.

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It is easy to estimate E_k from above: the symbols $w_k^{(\alpha\beta)}$ belong to the Hörmander class $S^0_{0,0}$, then the Calderon–Vaillancourt theorem and the presumed well–posedness of the Cauchy problem give

$$E_k(t) \leq C\sigma_k^C \|\varphi\|_{s,\varrho_0}.$$

To get an estimate from below, we write

$$v_k^{(\alpha\beta)}(t,x) = W_k^{(\alpha\beta)}(t,x,D_x)u_k(t,x),$$
$$B(x,D_x) = -\sum_{j=1}^{n} \Re b_j(x) D_{x_j},$$

and can deduce that

$$\begin{aligned}
&\left\|v_k^{(\alpha\beta)}\right\|_{L^2} \partial_t \left\|v_k^{(\alpha\beta)}\right\|_{L^2} = \Re\left(\partial_t v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)\\
&= \Re\left(-i[L, W_k^{(\alpha\beta)}]u_k, v_k^{(\alpha\beta)}\right) + \Re\left(i \bigtriangleup v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)\\
&\quad + \sum_{j=1}^{n} \Re\left(ib_j\partial_{x_j} v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) + \Re\left(icv_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)\\
&\geq -\left\|[L, W_k^{(\alpha\beta)}]u_k\right\|_{L^2} \left\|v_k^{(\alpha\beta)}\right\|_{L^2} + \Re\left(B(x,D_x)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right) - C\left\|v_k^{(\alpha\beta)}\right\|^2_{L^2}.
\end{aligned}$$

Now we need an estimate of $\left\|[L, W_k^{(\alpha\beta)}]u_k\right\|_{L^2}$ from above, and an estimate of $\Re\left(B(x,D_x)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)$ from below.

The symbol of $[L, W_k^{(\alpha\beta)}]$ can be written as an asymptotic expansion, up to some remainder, and $\left\|[L, W_k^{(\alpha\beta)}]u_k\right\|_{L^2}$ can be estimated by certain norms $\left\|v_k^{(\alpha+\gamma,\beta+\delta)}\right\|_{L^2}$ plus some remainder which becomes negligible for $\sigma_k \to \infty$.

The term $\Re\left(B(x,D_x)v_k^{(\alpha\beta)}, v_k^{(\alpha\beta)}\right)$ can be estimated using Condition 2 and Garding's inequality, or Lemma 1.1 and Gronwall's Lemma.

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