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BOUNDARY VALUES OF CLASSICAL FORMAL SYMBOLS OF PSEUDO-DIFFERENTIAL OPERATORS

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1. Introduction

Let X be a complex manifold $\mathbb{C}_w \times \mathbb{C}_z^n$ and Z, M be its submanifolds

$$Z = \{(w,z) \in X; \operatorname{Im} z = 0\} \simeq Z^{\mathbb{R}} \supset M = \{\operatorname{Im} w = 0, \operatorname{Im} z = 0\}.$$

Here $Z^{\mathbb{R}}$ is the underlying real manifold of Z. We denote by $(w,z;\tau,\zeta)$ the coordinates of T^*X;

$$w = u + iv \in \mathbb{C},\ z = x + iy \in \mathbb{C}^n,\ \tau \in \mathbb{C},\ \zeta = \xi + i\eta \in \mathbb{C}^n.$$

Then the sheaf $\mathcal{CO}_Z$ on

$$T_Z^*X = \{(w,z;\tau,\zeta) \in T^*X; \tau = 0, \operatorname{Im} z = 0, \operatorname{Re}\zeta = 0\}$$

of microfunctions with a holomorphic parameter w is defined by

$$\mathcal{CO}_Z := \{f(u,v,x) \in \mathcal{C}_{Z^{\mathbb{R}}}; \bar{\partial}_w f = 0\}.$$

Here $\mathcal{C}_{Z^{\mathbb{R}}}$ is the sheaf of usual microfunctions on $Z^{\mathbb{R}}$, and it is well-known that $\mathcal{CO}_Z$ is identified with the sheaf $\mathcal{C}_{Z|X}$ of relative microfunctions as $\mathcal{E}_X^{\mathbb{R}}$-modules. Then our main theorem is the following:

Theorem. *Let $U = \sum_{j=-\infty}^{0} U_j(w,z,\zeta)$ be a classical formal symbol of a pseudo-differential operator with order ≤ 0 defined in an $\mathbb{R}$-conic open set*

$$W_r \equiv \Big\{(w,z;*,\zeta) \in T^*X; \operatorname{Im} w > 0, |w| < r, |z| < \kappa,$$
$$|\zeta_j| < \rho|\zeta_n|\ (1 \leq \forall j \leq n-1), |\operatorname{Re}\zeta_n| < \delta \operatorname{Im}\zeta_n\Big\}$$

for some $r, \kappa, \rho, \delta > 0\,(\delta < 1)$. We suppose that $U_j \in \mathcal{O}(W_r)$ $(\forall j \leq 0)$ and that there exists some constants $C, \mu > 0$ satisfying the following inequalities:

$$|U_{-p}(w,z,\zeta)| \leq C^{p+1}p!|\operatorname{Im} w|^{-p-\mu}|\zeta|^{-p} \ \text{ on } W_r\ (\forall p \geq 0).$$

Then for any microfunction $f(x) \in \mathcal{C}_N|_{(0;idx_n)}$, a section $U(w,x,D_x)f(x) \in \Gamma(\{w \in \mathbb{C}; \operatorname{Im} w > 0, |w| < r\} \times \{(0;idx_n)\}; \mathcal{CO}_Z)$ has a microfunction boundary value at $(0,0;idx_n)$ from $\operatorname{Im} w > 0$.

Further, we show by a counter-example that the growth condition above is the best possible in some sense.

2. PRELIMINARIES

We give here the precise meaning concerning boundary values of sections of $\mathcal{CO}_Z$. Let K be a real analytic submanifold of T_Z^*X with codimension 2, and H be a real analytic hypersurface in T_Z^*X passing through K given as follows:

$$\begin{aligned} K &= \{(w, x; i\eta) \in T_Z^*X; w = \psi(x,\eta)\} \\ &\subset H = \{(u+iv, x; i\eta) \in T_Z^*X; \Phi(u,v,x,\eta) = 0\}. \end{aligned}$$

Here $\psi(x,\eta)$ is a complex valued analytic function of (x,η) with homogeneous degree 0 with respect to η, and $\Phi(u,v,x,\eta)$ is a real-valued analytic function of (u,v,x,η) of homogeneous degree 0 with respect to η satisfying the following:

$$\nabla\Phi \neq 0 \text{ on } \Phi = 0, \quad \Phi \circ \psi = 0.$$

It is known that we can choose a holomorphic contact transformation S defined in a neighborhood $\mathring{p} \in K$ such that

$$\begin{cases} S(K) = \{w^* = 0\} \cap T_Z^*X & \subset\ S(H) = \{\operatorname{Im} w^* = 0\} \cap T_Z^*X, \\ \qquad\qquad S(T_Z^*X) & \subset\ T_Z^*X. \end{cases} \tag{2.1}$$

Set σ = the signature of $S^*(d \operatorname{Im} w^*)/d\Phi$, where $S^*(\omega)$ denotes the pull-back of a differential form ω by S. We denote by $\pi : T^*X \to X$ the canonical projection, and by $\mathcal{BO}_Z = \mathcal{CO}_Z|_Z$ the sheaf on Z of hyperfunctions with a holomorphic parameter w.

Definition 2.1. Let $\mathring{p} = (\mathring{w}, \mathring{x}; i\mathring{\eta})$ be a point of K, and $f(w,x)$ be a section of $\mathcal{CO}_Z$ on $\{\Phi > 0\} \cap U$ with an $\mathbb{R}$-conic neighborhood $U \subset T_Z^*X$ of $\mathring{p}$. Then, $f(w,x)$ is said to have a boundary value at $\mathring{p}$ from $\Phi > 0$ if there exist a small neighborhood U' of $\mathring{p}$ and a section $F(w^*, x^*) \in \Gamma(\{\sigma \operatorname{Im} w^* > 0\} \cap \pi(S(U')); \mathcal{BO}_Z)$ satisfying

$$(\mathcal{T}_S^{-1} f)(w^*, x^*) = [F(w^*, x^*)]$$

as sections of $\Gamma(\{\sigma \operatorname{Im} w^* > 0\} \cap S(U'); \mathcal{CO}_Z)$. Here $\mathcal{T}_S$ is a quantization of S.

Though the boundary value $[F(u^* + i\sigma 0, x^*)]$ itself depends on a choice of $\mathcal{T}_S$, this definition neither depends on a choice of S nor $\mathcal{T}_S$ (shown as below).

Remark 2.2. A germ of $\mathcal{CO}_Z$ is represented by a germ of $\mathcal{BO}_Z$. However it is well-known that a section of $\mathcal{CO}_Z$ cannot be represented globally by a section of $\mathcal{BO}_Z$ in general. Indeed, the cohomological boundary value $(\mathcal{T}_S^{-1} f)(u^* + i\sigma 0, x^*)$ defines a second hyperfunction on $\Sigma = \{(w^*, x^*; i\eta^*) \in T_Z^*X; \operatorname{Im} w^* = 0\}$. On the other hand the sheaf $\mathcal{B}_\Sigma^2$ of second hyperfunctions is essentially larger than the sheaf $\mathcal{C}_M|_\Sigma$.

Here $M = \{(w,z) \in X; \operatorname{Im} w = 0, \operatorname{Im} z = 0\}$. Hence the definition above is equivalent to the following:

$$(\mathcal{T}_S^{-1} f)(u^* + i\sigma 0, x^*) \in \mathcal{C}_M|_{S(\mathring{p})}.$$

Further this boundary value is equal to $[F(u^* + i\sigma 0, x^*)]$ as a microfunction of (u^*, x^*) at $S(\mathring{p})$. The uniqueness of this boundary value $[F(u^* + i\sigma 0, x^*)] \in \mathcal{C}_M|_{S(\mathring{p})}$ for a section $(\mathcal{T}_S^{-1} f)(w^*, x^*)$ is justified by Schapira's N-regularity property of $\overline{\partial}_w$-operator. We refer to [2, 3] as for the second microlocal analysis, and to [8] as for the N-regularity of $\overline{\partial}_w$-operator. Further as for a self-contained proof of the equivalent fact, see Proposition 4.1.11 of [5].

3. Boundary values and the main theorem

Lemma 3.1. *Let $f(w)$ be a holomorphic function defined in a neighborhood of $G = \{w \in \mathbb{C}; r_0 < \operatorname{Im} w \leq r_1, |\operatorname{Re} w| < r_1\}$ satisfying an estimate*

$$|f(w)| \leq C|\operatorname{Im} w|^{-\mu} \quad (\forall w \in G)$$

for some constants $C, \mu, r_1 > 0$ and r_0 $(0 \leq r_0 < r_1 \leq 1)$. Choose an positive integer p as $\mu < p \leq \mu + 1$. Then the $(p+1)$-times integration

$$g_p(w) \equiv \int_{ir_1}^{w} \frac{(w - w')^p}{p!} f(w') dw'$$

is holomorphic in a neighborhood of G, continuous up to $\operatorname{Im} w = r_0$, and satisfies

$$|g_p(w)| \leq C(2^p + 1)/p! \ (\forall w \in G).$$

Theorem 3.2. *Let $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ be a classical formal symbol of a pseudo-differential operator with order ≤ 0 defined in an $\mathbb{R}$-conic open set*

$$W_r \equiv \Big\{(w, z; *, \zeta) \in T^*X; \operatorname{Im} w > 0, |w| < r, |z| < \kappa, \\ |\zeta_j| < \rho|\zeta_n| \ (1 \leq \forall j \leq n-1), |\operatorname{Re} \zeta_n| < \delta \operatorname{Im} \zeta_n\Big\} \tag{3.1}$$

for some $r, \kappa, \rho, \delta > 0$ $(\delta < 1)$. We suppose that $U_j \in \mathcal{O}(W_r)$ $(\forall j \leq 0)$ and that there exists some constants $C, \mu > 0$ satisfying the following inequalities:

$$|U_{-p}(w, z, \zeta)| \leq C^{p+1} p! |\operatorname{Im} w|^{-p-\mu} |\zeta|^{-p} \ \text{ on } W_r \ (\forall p \geq 0). \tag{3.2}$$

Then, for a sufficiently large number

$$\lambda > \max\{1, 2560C/r\} \tag{3.3}$$

we have 2 holomorphic functions $E^{(k)}(w,z,z-z^*,s)$ $(k=1,2)$ *defined in*

$$(3.4)\quad \mathcal{W}^{(1)} \equiv \Big\{(w,z,z-z^*,s) \in \mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}; |w|<r/40, \max\{0,-80\,\mathrm{Im}\,w/\lambda r\} < \mathrm{Im}\,s \le \lambda^{-1}, |\mathrm{Re}\,s| < \lambda^{-1}, |z|<\kappa,\ |z_j-z_j^*| > \rho^{-1}|z_n-z_n^*|\ (j=1,\dots,n-1)\Big\},$$

and

$$(3.5)\quad \mathcal{W}^{(2)} \equiv \Big\{(w,z,z-z^*,s) \in \mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}; |\mathrm{Re}\,s| < \lambda^{-1}, |w-ir/80| < r/320,\ -(320\lambda)^{-1} < \mathrm{Im}\,s \le \lambda^{-1}, |z|<\kappa,\ |z_j-z_j^*| > \rho^{-1}|z_n-z_n^*|\ (j=1,\dots,n-1)\Big\}.$$

respectively satisfying the following:

$$(3.6)\quad \sum_{k=1}^{2} E^{(k)}(w,z,z-z^*,s) = \sum_{|\alpha'|\ge 0, p\ge 0} \frac{\alpha'!}{|\alpha'|!}\Big(\prod_{j=1}^{n-1}\frac{(z_n-z_n^*)^{\alpha_j}}{(z_j^*-z_j)^{\alpha_j+1}}\Big) \times \int_{i/\lambda}^{s} \frac{(s-s^*)^{p+\nu+3}}{(p+\nu+3)!}ds^* \int_{\lambda}^{\infty} U_{-p,\alpha'}(w,z,it)\cdot(it)^{p-2}e^{its^*}\frac{dt}{2\pi}$$

on $\mathcal{W}^{(1)}\cap\mathcal{W}^{(2)} \equiv \mathcal{W}^{(3)}$*:*

$$(3.7)\quad \mathcal{W}^{(3)} = \Big\{(w,z,z-z^*,s) \in \mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}; 0<\mathrm{Im}\,s \le \lambda^{-1},\ |\mathrm{Re}\,s| < \lambda^{-1},\ |w-ir/80| < r/320, |z|<\kappa,\ |z_j-z_j^*| > \rho^{-1}|z_n-z_n^*|\ (j=1,\dots,n-1)\Big\}.$$

Here we expand each U_{-p} *in* W_r *as follows:*

$$U_{-p}(w,z,\zeta) = \sum_{\alpha'\ge 0} U_{-p,\alpha'}(w,z,\zeta_n)(\zeta'/\zeta_n)^{\alpha'},$$

with $\zeta'/\zeta_n = (\zeta_1/\zeta_n,\dots,\zeta_{n-1}/\zeta_n)$. *Further* ν *is some positive integer.*

Proof. Each $U_{-p,\alpha'}(w,z,\zeta_n)$ is holomorphic in $W_r^{(1)} \equiv$

$$\{(w,z;\zeta_n) \in \mathbb{C}^{n+1}\times\mathbb{C}; \mathrm{Im}\,w>0, |w|<r, |z|<\kappa, |\mathrm{Re}\,\zeta_n| < \delta\,\mathrm{Im}\,\zeta_n\}$$

and satisfies

$$(3.8)\qquad |U_{-p,\alpha'}(w,z,\zeta_n)| \le p!C^{p+1}\rho^{-|\alpha'|}|\mathrm{Im}\,w|^{-p-\mu}|\zeta_n|^{-p}$$

on $W_r^{(1)}$. By the preceding lemma, we get holomorphic functions $V_{p,\alpha'}(w,z,\zeta_n) \in \mathcal{O}(W_{r/2}^{(1)})$ satisfying

$$(3.9)\qquad \begin{cases} \partial_w^{p+\nu+1}V_{p,\alpha'}(w,z,\zeta_n) = U_{-p,\alpha'}(w,z,\zeta_n), \\ |V_{p,\alpha'}(w,z,\zeta_n)| \le C^{p+1}2^{p+\nu+1}\rho^{-|\alpha'|}|\zeta_n|^{-p} \end{cases}$$

on $W^{(1)}_{r/2}$ for all p, α'. Here ν is the integer satisfying $\mu < \nu \le \mu + 1$. Take a conformal mapping $\tilde{w} = \varphi(w)$:

$$\varphi : \{w \in \mathbb{C}; \operatorname{Im} w > 0, |w| < r/2\} \xrightarrow{\sim} \{\tilde{w} \in \mathbb{C}; |\tilde{w}| < 1\}$$

such that $\varphi(0) = 1$; for example,

$$\begin{aligned}(3.10)\qquad \varphi(w) &= \Big\{i\Big(\frac{r+2w}{r-2w}\Big)^2 + 1\Big\}\Big/\Big\{\Big(\frac{r+2w}{r-2w}\Big)^2 + i\Big\} \\ &= \Phi\Big(-2i(\sqrt{2}-1)w/r\Big)\Phi\Big(2i(\sqrt{2}+1)w/r\Big),\end{aligned}$$

where $\Phi(t) = (1+t)(1-t)^{-1}$. Then by expanding $V_{p,\alpha'}(\varphi^{-1}(\tilde{w}), z, \zeta_n)$ into a power series of $\tilde{w}$, we have expansions

$$(3.11)\quad V_{p,\alpha'}(w,z,\zeta_n) = \sum_{\ell=0}^{\infty} V_{p,\alpha',\ell}(z,\zeta_n)\varphi(w)^\ell \quad (\forall (w,z,\zeta_n) \in W^{(1)}_{r/2}).$$

Here $V_{p,\alpha',\ell}(z,\zeta_n)$'s are holomorphic functions in

$$W^{(2)} \equiv \{\zeta_n \in \mathbb{C}; |z| < \kappa, |\operatorname{Re}\zeta_n| < \delta \operatorname{Im}\zeta_n\}$$

satisfying

$$|V_{p,\alpha',\ell}(z,\zeta_n)| \le 2^\nu (2C)^{p+1}\rho^{-|\alpha'|}|\zeta_n|^{-p}$$

on $W^{(2)}$ for all p, α', ℓ. Therefore we have

$$(3.12)\qquad U_{-p} = \sum_{\alpha',\ell} \partial_w^{p+\nu+1}\{\varphi(w)^\ell\} \cdot (\zeta'/\zeta_n)^{\alpha'} V_{p,\alpha',\ell}(z,\zeta_n).$$

Now for a large positive constant $\lambda > 1$ we introduce 2 kernel functions for $V_{p,\alpha',\ell}(z,\zeta_n)\zeta_n^{p-2}$:

$$(3.13)\qquad A^{(1)}_{p,\alpha',\ell}(z,s) \equiv \int_{\lambda(\ell+1)}^{\infty} V_{p,\alpha',\ell}(z,it)\cdot(it)^{p-2}e^{its}\frac{dt}{2\pi},$$

$$(3.14)\qquad A^{(2)}_{p,\alpha',\ell}(z,s) \equiv \int_{\lambda}^{\lambda(\ell+1)} V_{p,\alpha',\ell}(z,it)\cdot(it)^{p-2}e^{its}\frac{dt}{2\pi},$$

which are holomorphic functions defined in

$$W^{(3)} \equiv \{s \in \mathbb{C}; |z| < \kappa, \operatorname{Im} s > -\delta|\operatorname{Re} s|\}$$

with the estimates:

$$(3.15)\qquad |A^{(1)}_{p,\alpha',\ell}(z,s)| \le 2^\nu(2C)^{p+1}\rho^{-|\alpha'|}e^{-\lambda(\ell+1)\operatorname{Im} s}/\pi$$

on $W^{(3)}$ for all p, α', ℓ. Further $A^{(2)}_{p,\alpha',\ell}(z,s)$ are entire functions satisfying

$$(3.16)\qquad |A^{(2)}_{p,\alpha',\ell}(z,s)| \le 2^\nu(2C)^{p+1}\rho^{-|\alpha'|}e^{\lambda(\ell+1)(-\operatorname{Im} s)_+}/\pi$$

on $\{s \in \mathbb{C}\}$ for all p, α', ℓ. Here $(t)_+ = t$ $(\forall t \ge 0)$, $= 0$ $(\forall t < 0)$. Therefore for each $k = 1, 2$ and each p, α' the series

$$(3.17)\qquad A^{(k)}_{p,\alpha'}(w,z,s) \equiv \sum_{\ell=0}^{\infty} A^{(k)}_{p,\alpha',\ell}(z,s)\varphi(w)^\ell$$

converges locally uniformly in

$$\text{(3.18)}\qquad \begin{cases} \{|\varphi(w)| < e^{\lambda \operatorname{Im} s}\} & (\forall s \in W^{(3)}) \quad \text{for } k=1,\\ \{|\varphi(w)|e^{\lambda(-\operatorname{Im} s)_+} < 1\} & (\forall s \in \mathbb{C}) \quad \text{for } k=2. \end{cases}$$

We note that $\varphi(w)$ at (3.10) is holomorphic in $|w| < (\sqrt{2}-1)r/2$ and

$$\text{(3.19)}\quad \log|\Phi(t)| = \frac{1}{2}\log\Big(\Phi\Big(\frac{2\operatorname{Re} t}{1+|t|^2}\Big)\Big) = \sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\Big(\frac{2\operatorname{Re} t}{1+|t|^2}\Big)^{2\ell+1}$$
$$\le 4(\operatorname{Re} t)_+ - (-\operatorname{Re} t)_+$$

for $\forall t$ ($|t| \le 1/4$). Consequently if $|w| < (\sqrt{2}-1)r/8$, we have

$$\log|\varphi(w)| \le (6\sqrt{2}+10)(-\operatorname{Im} w)_+/r - (10-6\sqrt{2})(\operatorname{Im} w)_+/r.$$

Hence $A^{(1)}_{p,\alpha'}(w,z,s)$'s are holomorphic in

$$\text{(3.20)}\qquad \{(w,z,s) \in \mathbb{C}\times W^{(3)};\, |w| < (\sqrt{2}-1)r/8,\, |z| < \kappa,$$
$$(10+6\sqrt{2})(-\operatorname{Im} w)_+ < \lambda r \operatorname{Im} s/2\}$$
$$\supset\{(w,z,s) \in \mathbb{C}^{n+1}\times\mathbb{C};\, |w| < r/20,\, |z| < \kappa,$$
$$(-\operatorname{Im} w)_+ < \frac{\lambda r}{40}\operatorname{Im} s\} \equiv W^{(4)}_\lambda$$

because $|\varphi(w)|e^{-\lambda \operatorname{Im} s} \le e^{-\lambda \operatorname{Im} s/2} < 1$ on $W^{(4)}_\lambda$. At the same time we have the following estimates:

$$|A^{(1)}_{p,\alpha'}(w,z,s)| \le \frac{2^\nu(2C)^{p+1}}{\pi\rho^{|\alpha'|}(1-e^{-\lambda \operatorname{Im} s/2})} \le \frac{e2^{\nu+1}(2C)^{p+1}}{\pi\lambda\rho^{|\alpha'|}\operatorname{Im} s}$$

on $W^{(4)}_\lambda \cap \{0 < \operatorname{Im} s \le 1/\lambda\}$. Consequently we obtain

$$|\partial_w^{p+\nu+1}A^{(1)}_{p,\alpha'}(w,z,s)| \le (p+\nu+1)!\frac{re2^{\nu+1}(2C)^{p+1}}{80\pi\rho^{|\alpha'|}}\Big(\frac{80}{\lambda r \operatorname{Im} s}\Big)^{p+\nu+2}$$

on

$$\text{(3.21)}\quad W^{(5)}_\lambda \equiv \{(w,z,s) \in \mathbb{C}^{n+1}\times\mathbb{C};\, |w| < r/40,\, |z| < \kappa,$$
$$\max\{0, -80\operatorname{Im} w/(\lambda r)\} < \operatorname{Im} s \le 1/\lambda,\, |\operatorname{Re} s| < \lambda^{-1}\}.$$

Further $A^{(2)}_{p,\alpha'}(w,z,s)$'s are holomorphic in

$$\text{(3.22)}\quad \{(w,z,s) \in \mathbb{C}^{n+1}\times\mathbb{C};\, |w| < (\sqrt{2}-1)r/8,\, |z| < \kappa,$$
$$\lambda(-\operatorname{Im} s)_+ - (10-6\sqrt{2})\operatorname{Im} w/(2r) < 0\}$$
$$\supset\{(w,z,s) \in \mathbb{C}^{n+1}\times\mathbb{C};\, |w| < r/20,\, |z| < \kappa,$$
$$\operatorname{Im} w > 0, -\operatorname{Im} w/(2\lambda r) < \operatorname{Im} s\} \equiv W^{(6)}_\lambda$$

and satisfy the following estimates

$$|A^{(2)}_{p,\alpha'}(w,z,s)| \le \frac{2^\nu(2C)^{p+1}}{\pi\rho^{|\alpha'|}(1-e^{-\operatorname{Im} w/(2r)})} \le \frac{re2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}\operatorname{Im} w}$$

on $W_\lambda^{(6)}$ because $|\varphi(w)|e^{\lambda(-\operatorname{Im} s)_+} \le e^{-\operatorname{Im} w/(2r)} < 1$ on $W_\lambda^{(6)}$. Consequently, setting

$$(3.23) \quad W_\lambda^{(6)} \supset W_\lambda^{(7)} \equiv \{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |z| < \kappa, \\ |w - ir/80| < r/320, \ \operatorname{Im} s > -(320\lambda)^{-1}\},$$

we get the following estimates:

$$(3.24) \qquad |\partial_w^{p+\nu+1} A_{p,\alpha'}^{(2)}(w,z,s)| \le (p+\nu+1)! \frac{re2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}(r/320)^{p+\nu+2}}$$

on $W_\lambda^{(7)}$. Fixing the initial point $s = i/\lambda$, we apply again the preceding lemma to $\partial_w^{p+\nu+1} A_{p,\alpha'}^{(k)}(w,z,s)$ for $k = 1,2$. That is, we have holomorphic functions $E_{p,\alpha'}^{(1)}(w,z,s) \in \mathcal{O}(W_\lambda^{(5)})$ satisfying

$$(3.25) \qquad \begin{cases} \partial_s^{p+\nu+4} E_{p,\alpha'}^{(1)}(w,z,s) = \partial_w^{p+\nu+1} A_{p,\alpha'}^{(1)}(w,z,s), \\ |E_{p,\alpha'}^{(1)}(w,z,s)| \le \dfrac{2^{p+\nu+4} re2^{\nu+1}(2C)^{p+1}}{80\pi\rho^{|\alpha'|}} \Big(\dfrac{80}{\lambda r}\Big)^{p+\nu+2} \end{cases}$$

on $W_\lambda^{(5)}$ for all p, α', and $E_{p,\alpha'}^{(2)}(w,z,s) \in \mathcal{O}(W_\lambda^{(7)} \cap \{|s - i/\lambda| < 2/\lambda\})$ satisfying

$$(3.26) \qquad \begin{cases} \partial_s^{p+\nu+4} E_{p,\alpha'}^{(2)}(w,z,s) = \partial_w^{p+\nu+1} A_{p,\alpha'}^{(2)}(w,z,s), \\ |E_{p,\alpha'}^{(2)}(w,z,s)| \le \dfrac{re2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}(r/320)^{p+\nu+2}} (2/\lambda)^{p+\nu+4} \end{cases}$$

on $W_\lambda^{(7)} \cap \{|s - i/\lambda| < 2/\lambda\}$ for all p, α'. Choose λ (> 1) as

$$(3.27) \qquad 1280C/(\lambda r) \le 1/2.$$

Then we can introduce 2 kernel functions $E^{(k)}(w,z,z-z^*,s)$ for $k = 1,2$ as follows:

$$(3.28) \qquad E^{(k)} \equiv \sum_{|\alpha'| \ge 0, p \ge 0} \frac{\alpha'!}{|\alpha'|!} E_{p,\alpha'}^{(k)}(w,z,s) \prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j+1}}.$$

Here, $E^{(k)}$ are holomorphic in $\mathcal{W}^{(k)}$ at (3.4), (3.5), respectively for $k = 1,2$. On the other hand, from (3.13), (3.14) and (3.11) we obtain the following:

$$(3.29) \quad \sum_{k=1}^{2} \partial_w^{p+\nu+1} A_{p,\alpha'}^{(k)}(w,z,s) = \partial_w^{p+\nu+1} \Big(\sum_{\ell=0}^{\infty} \sum_{k=1}^{2} A_{p,\alpha',\ell}^{(k)}(z,s)\varphi(w)^\ell \Big) \\ = \partial_w^{p+\nu+1} \Big(\int_\lambda^\infty V_{p,\alpha'}(w,z,it) \cdot (it)^{p-2} e^{its} \frac{dt}{2\pi} \Big) \\ = \int_\lambda^\infty U_{-p,\alpha'}(w,z,it) \cdot (it)^{p-2} e^{its} \frac{dt}{2\pi}$$

for any $(w,z,s) \in W_\lambda^{(5)} \cap W_\lambda^{(7)} \cap \{|s - i\lambda^{-1}| < 2\lambda^{-1}\}$

$$=\{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w - ir/80| < r/320, \quad |z| < \kappa,\ 0 < \operatorname{Im} s \le \lambda^{-1},\ |\operatorname{Re} s| < \lambda^{-1}\}.$$

Hence we have our conclusion (3.6). □

By using the expressions of the kernel functions obtained in the preceding theorem, we prove that $U(w,x,D_x)f(x)$ has a boundary value at $w = 0$ from $\operatorname{Im} w > 0$ for any microfunction $f(x)$. To do so, we introduce actions of $E^{(*)}(w,z,z-z^*,s)$ on holomorphic functions $F(z)$ similar to the Bony-Schapira actions of microdifferential operators on holomorphic functions [3] (also see [4] concerning the action of $\mathcal{E}_X^{\mathbb{R}}$).

Definition 3.3. We inherit the notation from the preceding theorem. Let $F(z)$ be a holomorphic function defined in

(3.30)
$$\Omega \equiv \{z \in \mathbb{C}^n; |z'| < r', |\operatorname{Re} z_n| < (3\lambda)^{-1} + r', r' > \operatorname{Im} z_n > k|\operatorname{Im} z'|\} \cup \textstyle\bigcup_{\sigma=\pm 1}\{z \in \mathbb{C}^n; |z'| < r', |z_n - \sigma/(3\lambda)| < r'\}$$

for positive small constants r' ($r' < 1/(3\lambda)$), and k ($< \rho/(2n)$). Let $E(w,z,z-z^*,s)$ be a holomorphic kernel function defined in $\mathcal{W}^{(1)}$ at (3.4). For a sufficiently small $\varepsilon > 0$, we define a holomorphic function $(E * F)_{\lambda,\varepsilon}(w,z)$ depending on λ, ε by

$$(3.31)\quad \int_{i\varepsilon}^{z_n} dz_n^* \int_\gamma dz^{*\prime} \int_{-1/(3\lambda)}^{1/(3\lambda)} E(w,z,z-z^*, z_n^* - z_n^{**}) F(z^{*\prime}, z_n^{**}) dz_n^{**}.$$

Here the path for z_n^* is the line segment

$$z_n^*(t) = z_n + t(i\varepsilon - z_n) \quad (0 \le t \le 1)$$

combining z_n with $i\varepsilon$, $\gamma = \{z_j^* = z_j + R(z_n, z_n^*(t))e^{i\theta_j}\ (0 \le \theta_j \le 2\pi); j = 1, \dots, n-1\}$ with some $R(z_n, z_n^*(t)) > \rho^{-1}|z_n - z_n^*(t)| = t\rho^{-1}|z_n - i\varepsilon|$. Further the path for z_n^{**} is the line graph passing through

$$-(3\lambda)^{-1},\ -(3\lambda)^{-1} + ih \operatorname{Im} z_n^*(t),\ (3\lambda)^{-1} + ih \operatorname{Im} z_n^*(t),\ (3\lambda)^{-1}$$

for a constant h ($1/2 < h < 1$). That is,

$$z_n^{**}(\theta;t) = \begin{cases} -1/(3\lambda) + 3ih\theta \operatorname{Im} z_n^*(t) & (0 \le \theta \le 1/3), \\ (2\theta - 1)/\lambda + ih \operatorname{Im} z_n^*(t) & (1/3 \le \theta \le 2/3), \\ 1/(3\lambda) + 3ih(1-\theta) \operatorname{Im} z_n^*(t) & (2/3 \le \theta \le 1) \end{cases}$$

Indeed this integral is well-defined if $\operatorname{Im} w > 0, |w| < r/40, |z| < \kappa, |\operatorname{Re} z_n| < (3\lambda)^{-1}, 0 < \operatorname{Im} z_n \le \varepsilon < r'$ and the following sets are contained in Ω :

$$\Big\{(z_1 + t\omega_1, \dots, z_{n-1} + t\omega_{n-1}, q + ih(\operatorname{Im} z_n + t(\varepsilon - \operatorname{Im} z_n))); 0 \le t \le 1,$$
$$|q| \le (3\lambda)^{-1}\ (q \in \mathbb{R}), |\omega_1| \le \rho^{-1}|z_n - i\varepsilon|, \dots, |\omega_{n-1}| \le \rho^{-1}|z_n - i\varepsilon|\Big\},$$

$$\Big\{(z_1+t\omega_1,\dots,z_{n-1}+t\omega_{n-1},\pm(3\lambda)^{-1}+iq(\operatorname{Im} z_n+t(\varepsilon-\operatorname{Im} z_n)));$$
$$0\le t\le 1, 0\le q\le h, |\omega_1|\le \rho^{-1}|z_n-i\varepsilon|,\dots,|\omega_{n-1}|\le \rho^{-1}|z_n-i\varepsilon|\Big\}.$$

The former set is contained in Ω if $\varepsilon < r'$ and

$$|\operatorname{Re} z_n| < (2h-1)(\varepsilon-\operatorname{Im} z_n),\ \operatorname{Im} z_n > (k/h)|\operatorname{Im} z'|,$$
$$|z'|+(n/\rho)|z_n-i\varepsilon| < r'.$$

The latter set is contained in Ω if $|\operatorname{Im} z_n|\le\varepsilon < r'$ and

$$|z'|+(n/\rho)|z_n-i\varepsilon| < r'.$$

Hence we obtain the following lemma:

Lemma 3.4. *Let ε (>0) be smaller than* $\min\{\kappa/2,(1+2n/\rho)^{-1}r'\}$. *Then $(E*F)_{\lambda,\varepsilon}(w,z)$ is holomorphic in*

$$\Big\{\operatorname{Im} w>0, |w|<r/40,\ |z'|<\varepsilon, |z_n|<\varepsilon,$$
$$\operatorname{Im} z_n > k|\operatorname{Im} z'|, |\operatorname{Re} z_n| < \varepsilon-\operatorname{Im} z_n\Big\} \tag{3.32}$$

Further let $E'(w,z,z-z^*,s)$ be a holomorphic kernel function defined in $\mathcal{W}^{(2)}$ at (3.5). *Then $(E'*F)_{\lambda,\varepsilon}(w,z)$ is holomorphic in a neighborhood of*

$$\Big\{|w-ir/80|<r/320,\ z=0\Big\}.$$

Proof. We have only to prove the latter statement. In this case we modify the paths of integrations as follows:

$$z_n^*(t)=z_n+t(i\varepsilon-z_n)\quad (0\le t\le 1),$$
$$z_n^{**}(\theta;t)=\begin{cases}-1/(3\lambda)+3i\theta\psi(t) & (0\le\theta\le 1/3),\\ (2\theta-1)/\lambda+i\psi(t) & (1/3\le\theta\le 2/3),\\ 1/(3\lambda)+3i(1-\theta)\psi(t) & (2/3\le\theta\le 1),\end{cases}$$

where $\psi(t)=\max\{\operatorname{Im} z_n^*(t),\epsilon\}$ with some small $\epsilon>0$. If we choose $\epsilon<\min\{(640\lambda)^{-1},\varepsilon\}$, for any $z_n=iy_n$ $(y_n\in(-\epsilon,\varepsilon))$ and $t\in[0,1]$ we have

$$\psi(t)=\max\{y_n+t(\varepsilon-y_n),\epsilon\} > \frac{t}{2}(\varepsilon-y_n)\ge\frac{nk}{\rho}t(\varepsilon-y_n).$$

Therefore $(E'*F)_{\lambda,\varepsilon}(w,z)$ is holomorphic in a neighborhood of

$$\Big\{|w-ir/80|<r/320,\ z'=0, \operatorname{Re} z_n=0, -\epsilon<\operatorname{Im} z_n<\varepsilon\Big\}.$$

This completes the proof. □

The following is our main result. K. Uchikoshi [9] used a similar method (Bronshtein's method) of considering boundary values of holomorphic pseudodifferential operators for constructing fundamental solutions of weakly hyperbolic microdifferential operators. However the situations are different from each other, and the proofs and results are completely independent.

The proof of this theorem is a little long and the most part of the proof is devoted to the proof of the compatibility of actions of pseudo-differential operators. So we omit the proof, which will be given in another paper.

Theorem 3.5. *Let* $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ *be the classical formal symbol of the pseudo-differential operator treated in Theorem 3.2. Then for any microfunction* $f(x) \in \mathcal{C}_N|_{(0;idx_n)}$, *a section* $U(w, x, D_x)f(x) \in \Gamma(\{w \in \mathbb{C}; \operatorname{Im} w > 0, |w| < r\} \times \{(0; idx_n)\}; \mathcal{C}\mathcal{O}_Z)$ *has a boundary value at* $(0, 0; idx_n)$ *from* $\operatorname{Im} w > 0$ *in the sense of Definition 2.1.*

Remark 3.6. The growth order condition (3.2) for the lower order terms of $\sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ is the best possible in the following sense: For any constant k $(1 < k < 2)$ there exists a classical formal symbol $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ satisfying the following (1)–(3):

(1) $U_j \in \mathcal{O}(W_r)$ $(\forall j \le 0)$.

(2) For some constants $C, \mu > 0$ we have

$$|U_{-p}(w, z, \zeta)| \le C^{p+1} p! |\operatorname{Im} w|^{-kp-\mu} |\zeta|^{-p} \text{ on } W_r \ (\forall p \ge 0).$$

(3) $U(w, x, D_x)\delta(x_n)$ does not have a boundary value at $(0, 0; idx_n)$ in microfunctions of $(\operatorname{Re} w, x)$ from $\operatorname{Im} w > 0$.

Indeed, we can give an explicit example as follows:

$$U_{-p}(w, z, \zeta) \equiv p! \{-i(w/i)^k\}^{-p-1} \zeta_n^{-p-1}, \tag{3.33}$$

where $p = 0, 1, 2, \ldots$ and $|\arg(w/i)| < \pi/2$. It is easy to see that the above conditions 1, 2 are satisfied and that

$$\begin{aligned} U(w, x, D_x)\delta(x_n) &= \Big[\sum_{p=0}^{\infty} -\frac{1}{2\pi i} \{-i(w/i)^k\}^{-p-1} z_n^p \log z_n\Big] \\ &= \Big[\frac{1}{2\pi i} \cdot \frac{\log z_n}{z_n + i(w/i)^k}\Big]. \end{aligned}$$

Here the equalities are valid for sections of $\mathcal{C}\mathcal{O}_Z$ over $\{\operatorname{Im} w > \varepsilon\} \times \{(0; idx_n)\}$ with any small positive ε. Then by the following lemma we get the condition (3) above for U.

The following example is a variant of the example in [7] of second hyperfunctions:

Lemma 3.7. *Let k be a constant satisfying* $1 < k < 2$. *Then the microfunction* $f(w, x) = \dfrac{\log(x + i0)}{x + i(w/i)^k}$ *extends to* $\{(w, x; i\eta dx) \in \mathbb{C} \times$

$T^*_{\mathbb{R}}\mathbb{C}; \operatorname{Im} w > 0, \eta > 0\}$ *as a microfunction with holomorphic parameter* w. *However* $f(w,x)$ *never has a microfunction boundary value at* $(0,0;idx)$ *from* $\operatorname{Im} w > 0$.

Proof. Consider a holomorphic function

$$F_1(w,z) = \frac{\log z - \log\{(w/i)^k/i\}}{z + i(w/i)^k}$$

defined in $\{(w,z) \in \mathbb{C}^2; \operatorname{Im} z > 0, \pi(1-k^{-1})/2 < \arg w < \pi\}$, where $-\pi < \arg\{(w/i)^k/i\} < \pi(k-1)/2 < \pi/2$. Further set

$$F_2(w,z) = F_1(w,z) + \frac{-2\pi i}{z + i(w/i)^k}.$$

Then $F_2(w,z) \in \mathcal{O}(\{(w,z) \in \mathbb{C}^2; \operatorname{Im} z > 0, 0 < \arg w < \pi(1+k^{-1})/2\})$. Hence the extension of $f(w,x)$ is given by

$$f(w,x) = \begin{cases} [F_1(w, x+i0)] & (\pi(1-k^{-1})/2 < \arg w < \pi), \\ [F_2(w, x+i0)] & (0 < \arg w < \pi(1+k^{-1})/2) \end{cases}$$

as microfunctions with holomorphic parameter w. We suppose here that $f(w,x)$ has a microfunction boundary value at $(0,0;idx)$ from $\operatorname{Im} w > 0$. Therefore, we have a holomorphic function $G(w,z)$ defined in $\{\operatorname{Im} w > 0, \operatorname{Im} z > 0, |w| + |z| < \varepsilon\}$ with some $\varepsilon > 0$ satisfying

$$f(w,x) = [G(w, x+i0)]$$

as sections of microfunctions with holomorphic parameter w in

$$\{(w,x;i\eta dx) \in \mathbb{C} \times T^*_{\mathbb{R}}\mathbb{C}; |w| + |x| < \varepsilon, \operatorname{Im} w > 0, \eta > 0\}.$$

Since

$$\begin{aligned} 0 &= \{x + i(w/i)^k\}\Big(f(w,x) - [G(w,x+i0)]\Big) \\ &= [\log(x+i0) - \{x + i(w/i)^k\}G(w, x+i0)], \end{aligned}$$

we conclude that

$$\begin{aligned} A(w,z) &\equiv \log z - \{z + i(w/i)^k\}G(w,z) \\ &\in \mathcal{O}(\{\operatorname{Im} w > 0, \operatorname{Im} z > 0, |w| + |z| < \varepsilon\}) \end{aligned}$$

extends holomorphically to $\{|w| + |z| < \varepsilon', \operatorname{Im} z = 0, \operatorname{Im} w > 0\}$ with some smaller $\varepsilon' > 0$. Therefore by Kashiwara's theorem on the local version of Bochner's tube theorem we can extend $A(w,z)$ to a holomorphic function $\tilde{A}(w,z)$ in

$$\Omega = \{|w| + |z| < \varepsilon'', |\operatorname{Im} z| < \varepsilon'' \operatorname{Im} w\}$$

for some smaller $\varepsilon'' > 0$. Set

$$P(r,\theta) = (re^{i\theta}, r^k e^{i(k\theta - \pi(k+1)/2)}) \in \{(w,z) \in \mathbb{C}^2; z + i(w/i)^k = 0\}$$

for $r > 0, 0 < \theta < \pi$. We note that $P(r, \theta) \in \Omega$ for any $\theta \in (0, \pi)$ with any sufficiently small $r > 0$ because $k > 1$. Therefore $H(w) \equiv \tilde{A}(w, -i(w/i)^k)$ is a holomorphic function in

$$W = \{w \in \mathbb{C}; 0 < \arg w < \pi, 0 < |w| < \varphi(\arg w)\}$$

with a positive valued continuous function $\varphi(\theta)$ on $(0, \pi)$. On the other hand, we have that

$$H(re^{i\theta}) = A(re^{i\theta}, r^k e^{ik\{\theta - \pi(1+k^{-1})/2\}}) = k \log r + ik\{\theta - \pi(1 + k^{-1})/2\}$$

for $\pi(1 + k^{-1})/2 < \theta < \pi$, and that

$$H(re^{i\theta}) = A(re^{i\theta}, r^k e^{ik\{\theta + \pi(3k^{-1}-1)/2\}}) = k \log r + ik\{\theta + \pi(3k^{-1} - 1)/2\}$$

for $0 < \theta < \pi(1 - k^{-1})/2$. That is, $H(w) - k \log w \in \mathcal{O}(W)$ coincides with 2 different constants $-\pi(k+1)i/2$ and $-\pi(k-3)i/2$ in the above domains, respectively. This contradicts with the connectedness of W. Thus $f(w, x)$ never have a microfunction boundary value at $(0, 0; idx)$ from $\operatorname{Im} w > 0$. $\square$

References

[1] T. Aoki, K. Kataoka and S. Yamazaki, Construction of kernel functions of pseudodifferential operators of infinite order. In *Actual problems in Mathematical Analysis , Proceedings of the conference dedicated to the seventieth birthday of Professor Yu. F. Korobeinik"*, GinGo Publisher, Rostov on Don, 2000, 28–40.

[2] J.-M. Bony, Extensions du théorème de Holmgren. Séminaire Goulaouic-Schwartz (1975/1976) *Équations aux dérivées partielles et analyse fonctionnelle,* Exp. No. **17** (1976).

[3] J.-M. Bony and P. Schapira, Propagation des singularités analytiques pour les solutions des équations aux dérivées partielles. Ann. Inst. Fourier, Grenoble, **26**-1(1976), 81–140.

[4] M. Kashiwara and T. Kawai, On holonomic systems of microdifferential equations. III. Systems with regular singularities. Publ. Res. Inst. Math. Sci., **17**-3 (1981), 813–979.

[5] K. Kataoka, On the theory of Radon transformations of hyperfunctions. J. Fac. Sci. Univ. Tokyo Sect. IA Math., **28**-2(1981), 331–413.

[6] K. Kataoka, Microlocal analysis of boundary value problems with regular or fractional power singularities. In *Structure of solutions of differential equations (Katata/Kyoto, 1995)*, World Sci. Publishing, River Edge, NJ, 1996, 215–225.

[7] K. Kataoka, Y. Okada, and N. Tose, Decomposition of second microlocal analytic singularities. In *D-modules and microlocal geometry (Lisbon, 1990)*, de Gruyter, Berlin, 1993, 163–171.

[8] P. Schapira, Propagation au bord et réflexion des singularités analytiques des solutions des équations aux dérivées partielles. In *Séminaire Goulaouic-Schwartz 1975/1976: Équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 6*, Centre Math., École Polytech., Palaiseau, 1976.

[9] K. Uchikoshi, Construction of the solutions of microhyperbolic pseudodifferential equations. J. Math. Soc. Japan, **40**-2(1988), 289–318.