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# Structure of the solutions to Fuchsian systems

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**Abstract**: To a certain Volevič system of homogeneous singular partial differential equations in a complex domain, called a Fuchsian system, holomorphic solutions which have singularities only on the initial surface are considered. All the solutions are constructed and parametrized in a good way, without any assumptions on the characteristic exponents.

## 1 Introduction

We consider a system of linear partial differential operators

$$P = tD_tI_m - A(t,x;D_x), \qquad (t,x) \in \boldsymbol{C} \times \boldsymbol{C}^n, \tag{1.1}$$

where $I_m$ is the $m \times m$ unit matrix, and

$$A = \big(A_{i,j}(t,x;D_x)\big)_{1\le i,j\le m}\ , \qquad A_{i,j} = \sum_{\alpha:\text{finite}} a_{i,j;\alpha}(t,x)D_x^{\alpha}\ ,$$

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$a_{i,j;\alpha}$ are holomorphic in a neighborhood of the origin $(0,0) \in \boldsymbol{C}^{n+1}$. We use $D_t = \dfrac{\partial}{\partial t}$, $D_x = (D_1, \dots, D_n)$, $D_j = \dfrac{\partial}{\partial x_j}$, without dividing by $\sqrt{-1}$.

$P$ is called a *Fuchsian system* if $P$ satisfies the following two conditions.

(A-1) There exists $n_j \in \boldsymbol{N} := \{0, 1, 2, \dots\}$ such that

$$\operatorname{ord}_{D_x} A_{i,j}(t, x; D_x) \leq n_i - n_j + 1 \ .$$

(A-2) $A(0, x; D_x) =: A_0(x)$ is independent of $D_x$.

The condition (A-1) is equivalent to each of the following condition ([4], [5]).

(A-1)' $$\max_{\substack{1 \leq p \leq m \\ 1 \leq i_1 < \cdots < i_p \leq m}} \left( \frac{1}{p} \max_{\pi \in S_p} \sum_{k=1}^{p} \operatorname{ord}_{D_x} A_{i_k, i_{\pi(k)}} \right) =: \rho(A) \leq 1$$

($\rho(A)$ is called the *matrix order* of $A$.)

When this condition is satisfied, the system $D_t I_m - A(t, x; D_x)$ is called a *kowalevskian system in Volevič's sense* ([4]).

The polynomial

$$\mathcal{C}(x; \lambda) := \det(\lambda I_m - A_0(x))$$

of $\lambda$ is called the *indicial polynomial* of $P$, and a root $\lambda$ of $\mathcal{C}(x; \lambda) = 0$ is called a *characteristic exponent* or a *characteristic index* of $P$ at $x$.

The second author ([6], [7]) has shown the following fundamental theorems corresponding to the Cauchy-Kowalevsky theorem and the Holmgren theorem. Let $\mathcal{O}_{(0,0)}$ denote the germ space of holomorphic functions at $(0,0) \in \boldsymbol{C} \times \boldsymbol{C}^n$.

**Theorem 1.1 ([6, Theorem 1.2.10]).** *If $\mathcal{C}(0; j) \neq 0$ $(j \in \boldsymbol{N})$, then for every $\vec{f} \in (\mathcal{O}_{(0,0)})^m$, there exists a unique $\vec{u} \in (\mathcal{O}_{(0,0)})^m$ such that $P\vec{u} = \vec{f}(t, x)$.*

**Theorem 1.2 ([7, Theorem 2]).** *Let $\Omega$ be an open neighborhood of $0 \in \boldsymbol{R}^n$ and $T > 0$. Let $L \in \boldsymbol{R}$ satisfy that if $\mathcal{C}(x; \lambda) = 0$ $(x \in \Omega)$, then $\operatorname{Re} \lambda < L$. If $\vec{u}(t) = \vec{u}(t, x) \in C^1((0, T], \mathcal{D}'(\Omega))^m$ satisfies $P\vec{u} = \vec{0}$ in $(0, T) \times \Omega$, and if $t^{-L}\vec{u} \in C^0([0, T], \mathcal{D}'(\Omega))^m$, then $\vec{u} = \vec{0}$ near $(0,0)$ in $(0, T) \times \Omega$. Here, $\mathcal{D}'(\Omega)$ denotes the space of Schwartz distributions on $\Omega$.*

Now, we introduce the following notation.

$$\mathcal{O}(\Omega) := \{\text{holomorphic functions on } \Omega\} ,$$

$$B_R := \{x \in \boldsymbol{C}^n : |x| < R\} , \qquad \Delta_T := \{t \in \boldsymbol{C} : |t| < T\} \quad (T > 0) ,$$

$$\mathcal{O}_0 := \bigcup_{R>0} \mathcal{O}(B_R) , \qquad \mathcal{O}_{(0,0)} := \bigcup_{R>0,T>0} \mathcal{O}(\Delta_T \times B_R) ,$$

$$S_{\infty,T} := \mathcal{R}(\Delta_T \setminus \{0\}) \quad (\text{the universal covering of } \Delta_T \setminus \{0\}) ,$$

$$S_{\theta,T} := \{t \in S_{\infty,T} : |\arg t| \le \theta\} , \qquad \widetilde{\mathcal{O}} := \bigcup_{T>0,R>0} \mathcal{O}(S_{\infty,T} \times B_R) .$$

Now, we consider solutions of $P\vec{u} = \vec{0}$ which are singular only at $t = 0$, that is, $\vec{u} \in (\widetilde{\mathcal{O}})^m$. Under the assumption that the characteristic exponents $\lambda_j(x)$ $(j = 1, 2, \ldots, m)$ of $P$ do *not* differ by integers, that is, $\lambda_i(0) - \lambda_j(0) \notin \boldsymbol{Z}$ $(i \neq j)$, the structure of the kernel $\mathrm{Ker}_{(\widetilde{\mathcal{O}})^m} P$ of the map $P : (\widetilde{\mathcal{O}})^m \to (\widetilde{\mathcal{O}})^m$ has been studied by the second author([6]).

Our purpose of this talk is to construct a solution map, that is, a linear isomorphism

$$(\mathcal{O}_0)^m \xrightarrow{\sim} \mathrm{Ker}_{(\widetilde{\mathcal{O}})^m} P := \{\vec{u} \in (\widetilde{\mathcal{O}})^m : P\vec{u} = \vec{0}\} , \tag{1.2}$$

rather explicitly, with *no* assumptions on the characteristic exponents (Theorem 2.2).

In the case of *single* Fuchsian partial differential equations, the first author([2]) have constructed a good solution map. These single equations can be reduced to our Fuchsian systems as follows.

*Remark* 1.3. Let $P'$ be a single Fuchsian partial differential operator with weight 0 ([1], [6], [2], etc.); that is, $P' = (tD_t)^m + \sum_{j=1}^m P'_j(t, x; D_x)(tD_t)^{m-j}$, $\mathrm{ord}_{D_x} P'_j \le j$, and $P'_j(0, x; D_x) =: a_j(x)$ is a function of $x$. Then, by $u_j = (tD_t)^{j-1}u$ $(1 \le j \le m)$, the equation $Pu = f$ is reduced to

$$\left( tD_tI_m - \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -P'_m & -P'_{m-1} & -P'_{m-2} & \dots & -P'_1 \end{pmatrix} \right) \vec{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix} .$$

Since this system satisfies (A-1) with $n_j = j$ and (A-2), it is a Fuchsian system. Further, this system has the same indicial polynomial $\mathcal{C}(x;\lambda)$ as $P'$, where the indicial polynomial of $P'$ is defined by

$$\mathcal{C}[P'](x;\lambda) := \lambda^m + \sum_{j=1}^{m} a_j(x)\lambda^{m-j} = \left[t^{-\lambda}P'(t^\lambda)\right]|_{t=0} .$$

## 2 Construction of the solution map

Let $\mu_l$ $(l = 1,\dots,d)$ be all the distinct roots of $\mathcal{C}(0;\lambda) = 0$, and let $r_l$ be the multiplicity of $\mu_l$. There exists $Q(x) \in GL_m(\mathcal{O}_0)$ such that

- $Q(x)^{-1}A_0(x)Q(x) = A_1(x) \otimes \dots \otimes A_d(x) := \begin{pmatrix} A_1(x) & O & \dots & O \\ O & A_2(x) & O & \vdots \\ \vdots & O & \ddots & \vdots \\ O & \dots & O & A_d(x) \end{pmatrix},$
- $A_l \in M_{r_l}(\mathcal{O}_0)$ $(l = 1,\dots,d)$,
- $\det\bigl(\lambda I_{r_l} - A_l(0)\bigr) = (\lambda - \mu_l)^{r_l}$ $(l = 1,\dots,d)$.

Corresponding to the blocks of $Q(x)^{-1}A_0(x)Q(x)$, we denote the $l$-th block of $\vec{u}$ by $\vec{u}^{\flat(l)} \in \boldsymbol{C}^{r_l}$, that is, $\vec{u} = \begin{pmatrix} \vec{u}^{\flat(1)} \\ \vdots \\ \vec{u}^{\flat(d)} \end{pmatrix}$. Conversely, for an $r_l$-vector $\vec{v} \in \boldsymbol{C}^{r_l}$, we denote by $\vec{v}^{\sharp(l)} \in C^m$ the $m$-vector

$$\vec{v}^{\sharp(l)} = \begin{pmatrix} 0 \\ \vdots \\ \vec{v} \\ \vdots \\ 0 \end{pmatrix} \langle l \text{ th block}$$

with the entries $\vec{v}$ in the $l$-th block and the entries 0 in the other blocks.

Set

$$\Lambda_P := \{ \mu_l - j \in \boldsymbol{C} : 1 \le l \le d, j \in \boldsymbol{N} \} \ . \tag{2.1}$$

Take $\epsilon \ge 0$ as $\mathrm{Re}\, \mu_l - \epsilon \notin \boldsymbol{Z}$ for all $l$. For each $l$, take $L_l \in \boldsymbol{Z}$ as $L_l + \epsilon < \mathrm{Re}\, \mu_l < L_l + \epsilon + 1$.

**Lemma 2.1.** (1) *For each $l$, there exists a domain $D_l$ in $\boldsymbol{C}$ enclosed by a simple closed curve $\Gamma_l$ such that*

(a) $\mu_l \in D_l \quad (1 \le l \le d)$,

(b) $\overline{D_l} \cap \overline{D_{l'}} = \emptyset \quad (l \neq l')$, *where $\overline{D}$ denotes the closure of $D$.*

(c) $\overline{D_l} \cap \Lambda_P = \{ \mu_l \}$ *for every $l$.*

(d) $\overline{D_l} \subset \{ \lambda \in \boldsymbol{C} : L_l + \epsilon < \mathrm{Re}\, \lambda < L_l + \epsilon + 1 \}$ *for every $l$.*

(2) *There exists $R_0 > 0$ such that*

(e) $\mathcal{C}(x; \lambda + j) \neq 0$ *for every $x \in B_{R_0}$, every $\lambda \in \bigcup_{l=1}^d \Gamma_l$, and every $j \in \boldsymbol{N}$.*

The main result is

**Theorem 2.2.** *For every $l$ and every $\vec{\varphi_l} \in (\mathcal{O}_0)^{r_l}$, there exists a unique $\vec{V} = \vec{V}[l, \vec{\varphi_l}](t, x; \lambda) \in \mathcal{O}(\{(0,0)\} \times (\bigcup_{l=1}^d \Gamma_l))^m$ such that*

$$P(t^\lambda \vec{V}) = t^\lambda Q(x) \vec{\varphi_l}^{\sharp(l)}(x) \tag{2.2}$$

*in a neighborhood of* $\{(0,0)\} \times \left(\bigcup_{l=1}^{d} \Gamma_l\right)$.

*Set* $\vec{u_l}[\vec{\varphi_l}](t,x) := \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\Gamma_l} t^{\lambda} \vec{V}[l, \vec{\varphi_l}](t,x;\lambda)\, d\lambda$. *Then, the map*

$$(\mathcal{O}_0)^m \ni \begin{pmatrix} \vec{\varphi_1} \\ \vdots \\ \vec{\varphi_d} \end{pmatrix} \overset{\sim}{\longmapsto} \sum_{l=1}^{d} \vec{u_l}[\vec{\varphi_l}] \in \mathrm{Ker}_{(\tilde{\mathcal{O}})^m} P \tag{2.3}$$

*is a linear isomorphism.*

# 3 Expansion of the solutions

Expand the operator $A$ and the vector $\vec{V}$ as follows.

$$A(t,x;D_x) = A_0(x) + \sum_{l=1}^{\infty} t^l B_l(x;D_x) \ ,$$
$$\vec{V}[l, \vec{\varphi_l}](t,x;\lambda) = \sum_{j=0}^{\infty} t^j \vec{V_j}(x;\lambda) \ .$$

Then, the equation (2.2) for $\vec{V}$ is equivalent to

$$(\lambda I_m - A_0(x))\vec{V_0}(x;\lambda) = Q(x)\vec{\varphi_l}(x) \ , \tag{3.1}$$

$$((\lambda + j)I_m - A_0(x))\vec{V_j}(x;\lambda) = \sum_{l=1}^{j} B_l(x;D_x)\overrightarrow{V_{j-l}}(x;\lambda) \quad (j \geq 1) \ . \tag{3.2}$$

From these equations, we can determine $\vec{V_j}$ by Lemma 2.1 (e), and we get an expansion of $\vec{u_l}[\vec{\varphi_l}]$ as follows.

$$\vec{u_l}[\vec{\varphi_l}](t,x) = \sum_{j=0}^{\infty} t^j \overrightarrow{u_{l,j}}(t,x) \ ,$$
$$\overrightarrow{u_{l,j}}(t,x) := \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} t^{\lambda} \vec{V_j}(x;\lambda)\, d\lambda \ .$$

Especially, the leading term of $\vec{u_l}[\vec{\varphi_l}]$ is

$$\overrightarrow{u_{l,0}}(t,x) = t^{A_0(x)} Q(x) \vec{\varphi_l}^{\sharp(l)}(x) = Q(x)\{t^{A_l(x)}\vec{\varphi_l}(x)\}^{\sharp(l)} \ . \tag{3.3}$$

# 4 Sketch of the proof of the existence of $\vec{V}$

We change the letter $\lambda$ to $\zeta$. Then, the system $P(t^{\zeta}\vec{V}) = t^{\zeta}Q(x)\vec{\varphi_l}^{\sharp(l)}(x)$ is equivalent to another system

$$\widetilde{P}\vec{V} := \left((tD_t+\zeta)I_m - A(t,x;D_x)\right)\vec{V} = Q(x)\vec{\varphi_l}^{\sharp(l)}(x) \ .$$

This is also a Fuchsian system in $(t,x,\zeta)$. Note that we consider $(x,\zeta)$ as the space variables. Further, the indicial polynomial of $\widetilde{P}$ is

$$\mathcal{C}[\widetilde{P}](x,\zeta;\lambda) = \mathcal{C}[P](x;\lambda+\zeta) \ .$$

Since $\mathcal{C}[\widetilde{P}](0,\zeta;j) \not\equiv 0 \ \ (\zeta \in \Gamma_l,\ j \in \boldsymbol{N})$ by Lemma 2.1 (e), we can use Theorem 1.1 to this new system. Thus, there exists a unique $\vec{V} = \vec{V}[l,\vec{\varphi_l}](t,x;\zeta) \in \mathcal{O}(\{(0,0)\} \times \Gamma_l)^m$ such that $\widetilde{P}\vec{V} = Q(x)\vec{\varphi_l}^{\sharp(l)}(x)$.

# 5 Function spaces to estimate the order

**Definition 5.1.** ([2, Definition 5.1]) For $a \in \boldsymbol{R}$, we set

$$W^{(a)} := \bigcup_{R>0,\ T>0} \Big\{ \phi \in \mathcal{O}(S_{\infty,T} \times B_R) : \\ \sup_{|x|<R} |\phi(t,x)| \to 0 \ \ (\text{as } t \to 0 \text{ in } S_{\theta,T}) \text{ for every } \theta > 0 \Big\}$$

**Lemma 5.2.** ([2, Lemma 5.2]) (1) $a' < a \implies W^{(a)} \subset W^{(a')}$.
(2) $t \times W^{(a)} \subset W^{(a+1)}$, $\quad \partial_t(W^{(a)}) \subset W^{(a-1)}$.
(3) *If $B(t,x;D_x)$ is a partial differential operator in $x$ with $\mathcal{O}_{(0,0)}$ coefficients, then $B(t,x;D_x)(W^{(a)}) \subset W^{(a)}$.*

# 6 Keys to the proof of the theorem

The **first key** is the temperedness of the solutions in $(\widetilde{\mathcal{O}})^m$.

**Proposition 6.1.** *There exists $a \in \boldsymbol{R}$ such that if $\vec{u} \in (\tilde{\mathcal{O}})^m$ and $P\vec{u} = \vec{0}$, then $\vec{u} \in (W^{(a)})^m$.*

The **second key** is an estimate of the remainder terms of our solutions $\vec{u_l}[\vec{\varphi_l}](t,x)$.

**Lemma 6.2.** *For $\vec{\varphi_l} \in (\mathcal{O}_0)^{r_l}$, we have*

$$\vec{u_l}[\vec{\varphi_l}](t,x) = Q(x)\{t^{A_l(x)}\vec{\varphi_l}(x)\}^{\sharp(l)} + t\cdot \vec{r_l}[\vec{\varphi_l}](t,x) \ ,$$

*and $\vec{r_l}[\vec{\varphi_l}] \in (W^{(L_l+\epsilon)})^m$. Note that $t^{A_l(x)}\vec{\varphi_l}(x) \in (W^{(L_l+\epsilon)})^{r_l}$ and $\vec{u_l}[\vec{\varphi_l}] \in (W^{(L_l+\epsilon)})^m$.*

The **third key** is the two facts on the Euler system $(tD_t - A_0(x))\vec{u} = \vec{f}(t,x)$ with holomorphic parameters $x$.

**Lemma 6.3.** *If $\vec{u} \in (\tilde{\mathcal{O}})^m$ and $(tD_tI_m - A_0(x))\vec{u} = \vec{0}$, then there exists $\vec{\varphi_l} \in (\mathcal{O}_0)^{r_l}$ $(1 \le l \le d)$ such that*

$$\vec{u} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\vec{\varphi_l}(x)\}^{\sharp(l)} = Q(x)\begin{pmatrix} t^{A_1(x)}\vec{\varphi_1}(x) \\ \vdots \\ t^{A_d(x)}\vec{\varphi_d}(x) \end{pmatrix} .$$

*Further, if $L \in \boldsymbol{Z}$ and $\vec{u} \in \widetilde{W}^{(L+\epsilon)}(\theta,R)^m$, then $\vec{\varphi_l} = 0$ for all $l$ such that $L_l < L$.*

**Proposition 6.4.** *For any $L \in \boldsymbol{Z}$ and any $\vec{g} \in W^{(L+\epsilon)}$, there exists $\vec{v} \in W^{(L+\epsilon)}$ such that $(tD_tI_m - A_0(x))\vec{v} = \vec{g}(t,x)$.*

If a root $\lambda(x)$ of $\mathcal{C}(x;\lambda) = 0$ touches the line $\mathrm{Re}\,\lambda = L+\epsilon$ in $\lambda$-plane, then this proposition does not hold, as the simplest example $tD_tv = \dfrac{1}{\log t}$ shows ($m = 1$, $L+\epsilon = 0$, no parameter $x$). This proposition is the reason why we took $\epsilon$.

# 7 Proof of the injectivity of the solution map

Assume that $\vec{\varphi_l} \in (\mathcal{O}_0)^{r_l}$ $(1 \le l \le d)$, $\sum_{l=1}^{d} \vec{u_l}[\vec{\varphi_l}] = \vec{0}$, and that there exists $l$ such that $\vec{\varphi_l} \neq \vec{0}$. Take $l_0$ as $L_{l_0} = \min\{\, L_l : \vec{\varphi_l} \neq \vec{0}\,\}$.

For each $l$ with $\overrightarrow{\varphi_l} \neq \overrightarrow{0}$, consider $(\overrightarrow{v_l})^{\flat(l_0)}$ : the $l_0$-th block of $\overrightarrow{v_l} := Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}]$. Then, we have by Lemma 6.2

$$(\overrightarrow{v_{l_0}})^{\flat(l_0)} = t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \ .$$

On the other hande, if $l \neq l_0$, then $L_l \geq L_{l_0}$ and hence

$$(\overrightarrow{v_l})^{\flat(l_0)} \in (W^{(L_l+1+\epsilon)})^{r_{l_0}} \subset (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \ .$$

Thus,

$$\overrightarrow{0} = \sum_{l=1}^{d}(Q^{-1}\overrightarrow{u_l}[\overrightarrow{\varphi_l}])^{\flat(l_0)} = t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) + (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}} \ .$$

Namely, $t^{A_{l_0}(x)}\overrightarrow{\varphi_{l_0}}(x) \in (W^{(L_{l_0}+1+\epsilon)})^{r_{l_0}}$. It is easy to show that this implies $\overrightarrow{\varphi_{l_0}} = \overrightarrow{0}$, which contradicts the definition of $l_0$.

# 8 Proof of the surjectivity of the solution map

Let $\overrightarrow{u} \in (\tilde{\mathcal{O}})^m$ and $P\overrightarrow{u} = \overrightarrow{0}$. Decompose $A(t,x;D_x) = A_0(x) + tB(t,x;D_x)$.

(I) By Proposition 6.1, there exists $L \in \mathbf{Z}$ such that $\underline{\overrightarrow{u} \in (W^{(L+\epsilon)})^m}$.

By Lemma 5.2, we have $tB(\overrightarrow{u}) \in (W^{(L+1+\epsilon)})^m$.

(II) By Proposition 6.4, there exists $\overrightarrow{v} \in (W^{(L+1+\epsilon)})^m$ such that $\big(tD_tI_m - A_0(x)\big)\overrightarrow{v} = tB(\overrightarrow{u}) = \big(tD_tI_m - A_0(x)\big)\overrightarrow{u}$.

(III) Since $\big(tD_tI_m - A_0(x)\big)(\overrightarrow{u} - \overrightarrow{v}) = \overrightarrow{0}$ and $\overrightarrow{u} - \overrightarrow{v} \in (W^{(L+\epsilon)})^m$, there exists $\overrightarrow{\varphi_l}[1] \in (\mathcal{O}_0)^{r_l}$ such that $\overrightarrow{\varphi_l}[1] = \overrightarrow{0}$ if $L_l < L$, and that

$$\overrightarrow{u} - \overrightarrow{v} = \sum_{l=1}^{d} Q(x)\{t^{A_l(x)}\overrightarrow{\varphi_l}[1](x)\}^{\sharp(l)} \ .$$

by Lemma 6.3.

(IV) Set

$$\overrightarrow{u}[1] := \overrightarrow{u} - \sum_{l=1}^{d} \overrightarrow{u_l}[\overrightarrow{\varphi_l}[1]] \in (W^{(L+1+\epsilon)})^m \ .$$

Then, we have $P(\overrightarrow{u}[1]) = \overrightarrow{0}$, $\overrightarrow{u}[1] \in (W^{(L+1+\epsilon)})^m$.

Now, we can return to the step (I) taking $\vec{u}[1]$ instead of $\vec{u}$, with order $L+1+\epsilon$ instead of $L+\epsilon$.

Repeating such arguments, we have $\vec{\varphi_l}[j] \in (\mathcal{O}_0)^{r_l}$ $(j=2,3,\dots)$ such that $\vec{\varphi_l}[j] = \vec{0}$ if $L_l < L+j-1$, and that

$$\vec{u}[j] := \vec{u}[j-1] - \sum_{l=1}^{d} \vec{u_l}[\vec{\varphi_l}[j]] \left( = \vec{u} - \sum_{k=1}^{j}\sum_{l=1}^{d} \vec{u_l}[\vec{\varphi_l}[k]] \right) \in (W^{(L+j+\epsilon)})^m ,$$

and $P(\vec{u}[j]) = \vec{0}$.

By Theorem 1.2, we have $\vec{u}[M] = \vec{0}$ for sufficiently large $M$. Thus, we get $\vec{\varphi_l} := \sum_{k=1}^{M} \vec{\varphi_l}[k] \in (\mathcal{O}_0)^{r_l}$.

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