数理解析研究所講究録 1262 巻 2002 年 30-51

SPIN REPRESENTATIONS AND CENTRALIZER ALGEBRAS FOR $Spin(2n)$

KAZUHIKO KOIKE
DEPT. OF MATH. AOYAMAGAKUIN UNIVERSITY
青山学院大学理工学部 小池和彦

1. INTRODUCTION

In this exposition, we deal with the even spin case.

Let G be $Spin(2n,\mathbb{C})$ or $Pin(2n,\mathbb{C})$, namely the double covering group of $SO(2n)$ or $O(2n)$.

Then every irreducible representation of G, not coming from the representations of SO or O can be realized in the tensor space $\Delta^{(\pm)}\bigotimes\overset{k}{\bigotimes}V$ for some k, where $\Delta^{(\pm)}$ is the fundamental spin representation of G and $V=\mathbb{C}^n$ is the natural representation of $O(n)$.

In this case we mainly deal with the $Pin(2n)$ centralizer algebra

$$\mathbf{CP_k}=\mathrm{Hom}_{Pin(2n)}(\Delta\bigotimes\overset{k}{\bigotimes}V,\Delta\bigotimes\overset{k}{\bigotimes}V)$$

and define two kinds of basis of this algebra for $k\leqq n$, just as in the case of $Spin(2n+1)$. The argument goes well for this algebra similarly as that of $Spin(2n+1)$. Finally we consider the associater of the irreducible representations $\Delta\bigotimes\bigotimes^k V$ and $\Delta\bigotimes\bigotimes^k V\bigotimes\det$, where 'det ' denotes the linear representation of $Pin(2n)$ induced from the natural homomorphism $Pin(2n)\longrightarrow O(2n)\xrightarrow{\det}\{\pm1\}$.

This associater is given by the endomorphism $(A\otimes\mathrm{id})\in\mathrm{End}(\Delta\bigotimes\bigotimes^k V)$, where A is the associater for Δ and is given by the degree operator $A[i_1,i_2,i_3,\ldots,i_r]=(-1)^r[i_1,i_2,i_3,\ldots,i_r]$. This associater commutes with the action of $Spin(2n)$ and if it is restricted to the irreducible representation $[\Delta,\delta]$ of $Pin(2n)$, it becomes an associater of $[\Delta,\delta]\cong[\Delta,\delta]\bigotimes\det$ and its $\{\pm1\}$-eigenspaces give the irreducible representation s $(1/2+\delta)^{\pm}_{Spin(2n)}$ of $Spin(2n)$ respectively. So we can pick up the irreducible representation of $Spin(2n)$.

2. A SUMMARY OF REPRESENTATION OF $Spin(2n)$ AND $Pin(2n)$

We first fix notations. As the defining symmetric matrix of $O(2n)$, we take an anti-diagonal matrix $S=(\delta_{i,2n+1-i})$ of size $2n$.

The set P^+ of the dominant integral weights for Lie algebra $\mathfrak{so}(2n,\mathbb{R})$ is given by

$$P^+=\{\lambda_1\epsilon_1+\lambda_2\epsilon_2+\ldots+\lambda_n\epsilon_n;\lambda_1\geq\lambda_2\geq\ldots\geq|\lambda_n|\geq0\}$$

,where all the λ_i's are integers or half-integers (namely $1/2+\mathbb{Z}$) simultaneously.

Let $\lambda_{Spin(2n)}$ denote the irreducible representation (or character) with $\lambda \in P^+$.

If all the λ_i's are integers, the corresponding irreducible representation comes from that of $SO(2n)$ and we also write $\lambda_{SO(2n)}$ instead of $\lambda_{Spin(2n)}$.

Then $(1)_{SO(2n)}$ is the vector representation (the natural representation) and $(1^i)_{SO(2n)}$ is the irreducible representation $\bigwedge^i V$ of $\mathfrak{so}(2n)(i = 1, \ldots, n-1)$.

For $SO(2n)$, $\bigwedge^n V$ is not irreducible and $\bigwedge^n V = (1^n)_{SO(2n)} + (1^{n-1}, -1)_{SO(2n)}$. As usual, we denote this representation $\bigwedge^i V$ by e_i for $i = 1, 2, \ldots, n-1$ and $e_n^+ = (1^n)_{SO(2n)}$ and $e_n^- = (1^{n-1}, -1)_{SO(2n)}$.

We note that $\bigwedge^i V \cong \bigwedge^{2n-i} V$. For a partition $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ we denote the irreducible representations by

$$(1/2+\delta)^+_{Spin(2n)} = (1/2+\delta_1, 1/2+\delta_2, \ldots, 1/2+\delta_n)_{Spin(2n)}$$

and

$$(1/2+\delta)^-_{Spin(2n)} = (1/2+\delta_1, 1/2+\delta_2, \ldots, 1/2+\delta_{n-1}, -1/2-\delta_n)_{Spin(2n)}.$$

We put simply $\Delta^+ = (1/2+\emptyset)^+_{Spin(2n)}$ and $\Delta^- = (1/2+\emptyset)^-_{Spin(2n)}$. We also put $\Delta = \Delta^+ + \Delta^-$ and $\Delta' = \Delta^+ - \Delta^-$.

For $\lambda = (1/2+\delta_1, 1/2+\delta_2, \ldots, 1/2+\delta_n) \in P^+$, we introduce the sum character and the difference character for $Spin(2n)$ and denote them by

$$[\Delta, \delta]_{Spin(2n)} = \lambda^{(+)}_{Spin(2n)} = (1/2+\delta)^+_{Spin(2n)} + (1/2+\delta)^-_{Spin(2n)}$$

and

$$[\Delta', \delta]_{Spin(2n)} = \lambda^{(-)}_{Spin(2n)} = (1/2+\delta)^+_{Spin(2n)} - (1/2+\delta)^-_{Spin(2n)}.$$

Then $[\Delta, \delta]_{Spin(2n)}$ becomes the irreducible character of $Pin(2n)$.

Similarly for a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, we denote the sum character and the difference character for $SO(2n)$ by

$$\mu^{(+)}_{SO(2n)} = (\mu_1, \mu_2, \ldots, \mu_n)_{SO(2n)} + (\mu_1, \ldots, \mu_{n-1}, -\mu_n)_{SO(2n)}$$

and

$$\mu^{(-)}_{SO(2n)} = (\mu_1, \mu_2, \ldots, \mu_n)_{SO(2n)} - (\mu_1, \ldots, \mu_{n-1}, -\mu_n)_{SO(2n)}.$$

For a partition μ with $\ell(\mu) < n$, $\mu_{SO(2n)}$ becomes an irreducible character of $O(2n)$, so we sometimes write $\mu_{O(2n)}$ for $\mu_{SO(2n)}$ in this case.

If $\ell(\mu) = n$, $\mu^{(+)}_{SO(2n)}$ is the irreducible character of $O(2n)$ and we sometimes write $\mu_{O(2n)}$ for $\mu^{(+)}_{SO(2n)}$.

For $\epsilon \in \{\pm 1\}$, if $\epsilon = +1$, we put $\Delta^\epsilon = \Delta^+$, $e_n^\epsilon = e_n^+$, $(1/2+\delta)^\epsilon_{Spin(2n)} = (1/2+\delta)^+_{Spin(2n)}$, etc. and if $\epsilon = -1$, $\Delta^\epsilon = \Delta^-$, $e_n^\epsilon = e_n^-$, $(1/2+\delta)^\epsilon_{Spin(2n)} = (1/2+\delta)^-_{Spin(2n)}$, etc..

Theorem 2.1. *For $\epsilon, \epsilon_1, \epsilon_2 \in \{\pm 1\}$, we have the following formulas.*

(i)

$$(\Delta^\epsilon)^2 = e_n^\epsilon + \sum_{i=0}^{[n/2]} e_{n-2i}. \tag{2.1.1}$$

$$\Delta^+\Delta^- = \sum_{i=0}^{[(n-1)/2]} e_{n-1-2i}. \tag{2.1.2}$$

(ii) *For a partition δ with its length $\ell(\delta) < n$, we have*

(2.1.3)

$$(1/2+\delta)^\epsilon_{Spin(2n)}(1)_{SO(2n)} = (1/2+\delta)^{-\epsilon}_{Spin(2n)} + \sum_{\substack{\mu\supset\delta \\ |\mu/\delta|=1}} (1/2+\mu)^\epsilon_{Spin(2n)} + \sum_{\substack{\delta\supset\mu \\ |\delta/\mu|=1}} (1/2+\mu)^\epsilon_{Spin(2n)}.$$

If $\ell(\delta) = n$, we have

$$\begin{aligned}&(1/2+\delta)^\epsilon_{Spin(2n)}(1)_{SO(2n)} \\ &= \sum_{\substack{\mu\supset\delta \\ |\mu/\delta|=1, \\ \ell(\mu)\le n}} (1/2+\mu)^\epsilon_{Spin(2n)} + \sum_{\substack{\delta\supset\mu \\ |\delta/\mu|=1}} (1/2+\mu)^\epsilon_{Spin(2n)}.\end{aligned} \tag{2.1.4}$$

(iii) *For a partition μ with $\ell(\mu) < n$, we have*

(2.1.5)

$$\Delta^\epsilon \mu_{SO(2n)} = \sum_{\substack{\mu\supseteq\nu \\ \mu/\nu \text{ vertical strip} \\ |\mu/\nu|\equiv 0 \mod 2}} (1/2+\nu)^\epsilon_{Spin(2n)} + \sum_{\substack{\mu\supseteq\nu \\ \mu/\nu \text{ vertical strip} \\ |\mu/\nu|\equiv 1 \mod 2}} (1/2+\nu)^{-\epsilon}_{Spin(2n)}.$$

(iv) *For a partition $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ with $\ell(\mu) = n$, we write $\mu^+_{SO(2n)} = \mu_{SO(2n)}$ and $\mu^-_{SO(2n)} = (\mu_1, \dots, \mu_{n-1}, -\mu_n)_{SO(2n)}$. Then we have*

$$\Delta^{\epsilon_1} \mu^{\epsilon_2}_{SO(2n)} = \sum_{\substack{\mu\supseteq\nu \\ \mu/\nu \text{ vertical strip} \\ (-1)^{|\mu/\nu|}=\epsilon_1\epsilon_2}} (1/2+\nu)^{\epsilon_2}_{Spin(2n)}. \tag{2.1.6}$$

Particularly, for the exterior products e_i's if $i < n$, we have

$$\Delta^{\epsilon}(1^i)_{SO(2n)} = \sum_{0 \leqq 2s \leqq i} (1/2 + (1^{i-2s}))^{\epsilon}_{Spin(2n)} + \sum_{0 \leqq 2s \leqq i-1} (1/2 + (1^{i-1-2s}))^{-\epsilon}_{Spin(2n)}$$

and for e_n^{ϵ}, we have

$$\Delta^{\epsilon} e_n^{\epsilon} = \sum_{0 \leqq 2s \leqq n} (1/2 + (1^{n-2s}))^{\epsilon}_{Spin(2n)}$$

and

$$\Delta^{-\epsilon} e_n^{\epsilon} = \sum_{0 \leqq 2s \leqq n-1} (1/2 + (1^{n-1-2s}))^{\epsilon}_{Spin(2n)}.$$

(v)

$$(e_n^+)^2 = \sum_{\substack{s \equiv n \mod 2 \\ t \equiv 0 \mod 2 \\ 0 \leqq s+t \leqq n}} (2^s, 1^t)_{SO(2n)}, \quad (e_n^+)(e_n^-) = \sum_{\substack{s \equiv n-1 \mod 2 \\ t \equiv 0 \mod 2 \\ s+t \leqq n-1}} (2^s, 1^t)_{SO(2n)}.$$

$$(e_n^-)^2 = (2^{n-1}, -2)_{SO(2n)} + \sum_{\substack{s \equiv n \mod 2 \\ s \leqq n-2}} (2^s, 1^{n-s-1}, -1)_{SO(2n)} + \sum_{\substack{s \equiv n \mod 2 \\ t \equiv 0 \mod 2 \\ s+t \leqq n-2}} (2^s, 1^t)_{SO(2n)}.$$

From the above, we know all the irred spin representations occur in the space $\Delta \bigotimes \overset{k}{\bigotimes} V$. Also we have $\bigoplus_{i=0}^{2n} \overset{i}{\bigwedge} V \cong \Delta \bigotimes (\Delta)^*$, so we have

Lemma 2.2.
(2.2.1)

$$\mathbf{CP}_{\mathbf{l}}^{\mathbf{k}} = \mathrm{Hom}_{Pin(2n)}(\Delta \otimes \overset{k}{\bigotimes} V, \Delta \otimes \overset{l}{\bigotimes} V) \cong \bigoplus_{i=0}^{2n} ((\overset{i}{\bigwedge} V) \otimes \overset{k+l}{\bigotimes} V)^{Pin(2n)}$$

Here the superscript $Pin(2n)$ means the $Pin(2n)$-invariant space.

So we need to give an explicit $Pin(2n)$-equivariant isomorphism between $\bigoplus_{i=0}^{2n} \overset{i}{\bigwedge} V$ and $\Delta \bigotimes (\Delta)^*$.

First we give an action of $Lie(Pin(2n)) = \mathfrak{so}(2n, \mathbb{R})$ on Δ.As a basis of Δ, we take a basis parametrized by all the subsets of $[n] = \{1, 2, \ldots, n\}$ and denote the basis elements by $\{[\mathrm{I}]\}$, where $\mathrm{I} = \{i_1, i_2, \ldots, i_r\}$ $(1 \leqq i_1 < i_2 < \ldots < i_r \leqq n)$ just as before and we take a set of the simple root vectors as follows.

$$\begin{array}{ll} \mathrm{ad}(X_k) = E_{k,k+1} - E_{\overline{k+1},\overline{k}}, & \mathrm{ad}(X_n) = E_{n-1,\overline{n}} - E_{n,\overline{n-1}}, \\ \mathrm{ad}(Y_k) = E_{k+1,k} - E_{\overline{k},\overline{k+1}}, & \mathrm{ad}(Y_n) = E_{\overline{n},n-1} - E_{\overline{n-1},n}, \\ \mathrm{ad}(h_i) = E_{i,i} - E_{\overline{i},\overline{i}}. & \end{array}$$

, where $k \in \{1, 2, \ldots, n-1\}$ and $i \in \{1, 2, \ldots, n\}$.

The explicit action of Lie algebra $\mathfrak{so}(2n, S)$ on this base is given as

Lemma 2.3.

$$X_k[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, \dots, i_{s-1}, k+1, i_{s+1}, \dots, i_r] & \text{if } k = i_s \text{ and } k+1 < \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X_n[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, i_2, \dots, i_{r-2}] & \text{if } i_{r-1} = n-1 \text{ and } i_r = n \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_k[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, \dots, i_{s-1}, k, i_{s+1}, \dots, i_r] & \text{if } k+1 = i_s \text{ and } k > i_{s-1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Y_n[i_1, i_2, \dots, i_r] = \begin{cases} -[i_1, i_2, \dots, i_r, n-1, n] & \text{if } i_r < n-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$h_k[i_1, i_2, \dots, i_r] = \begin{cases} -\frac{1}{2}[i_1, i_2, \dots, i_r] & \text{if } k \in \{i_1, i_2, \dots, i_r\} \\ \frac{1}{2}[i_1, i_2, \dots, i_r] & \text{otherwise.} \end{cases}$$

, where $i_1, i_2, \dots, i_r$ are in the increasing order.

Moreover let A be a linear endomorphism of Δ defined by the degree operator $A[i_1, i_2, i_3, \dots, i_r] = (-1)^r[i_1, i_2, i_3, \dots, i_r]$. Then A becomes an associater of $(\Delta \bigotimes \det) \cong \Delta$ as $Pin(2n)$ modules and on $\Delta = \Delta^+ \bigoplus \Delta^-$, A is given by

$$A = \mathrm{id}_{\Delta^+} \bigoplus -\mathrm{id}_{\Delta^-}.$$

For any sequence $i_1, i_2, i_3, \dots, i_r$ of non-negative integers, we define the corresponding element $[i_1, i_2, i_3, \dots, i_r]$ in Δ such that it satisfys the alternating property on the indices as before.

A compact real form $\mathfrak{so}(2n)_{cpt}$ of $L(Spin(2n)) \bigotimes \mathbb{C} \cong \mathfrak{so}(2n, \mathbb{C})$ is generated by the elements $\sqrt{-1}h_i$ and $\sqrt{-1}(X_i + Y_i)$, $X_i - Y_i$, $(i = 1, 2, \dots, n)$ as a real Lie algebra.

Then the invariant hermitian forms of V and Δ under the action of $\mathfrak{so}(2n)_{cpt}$ are given such that the base $< u_1, u_2, \dots, u_n, u_{\overline{n}}, \dots, u_{\overline{1}} >$ of V and the base $[\mathrm{I}]_{\mathrm{I} \subset [n]}$ of Δ become orthonormal basis respectively.

Then an explicit embedding theorem of $\bigwedge^i V$ in the space $\Delta^* \otimes \Delta$ is given as follows.

We write the $\mathfrak{so}(2n)$-equivariant embedding ϕ_ℓ from $\bigwedge^\ell V$ to $\Delta^{\epsilon_1} \bigotimes (\Delta^{\epsilon_2})^*$ by $\phi_\ell^{\epsilon_1\epsilon_2}$. Here $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and we consider, as the notation, for example, $\phi_\ell^{\epsilon_1\epsilon_2} = \phi_\ell^{+-}$ if $\epsilon_1 = 1$ and $\epsilon_2 = -1$ and so on.

Theorem 2.4. *If n is an even positive integer, we have $(\Delta^+)^* \cong \Delta^+$ and $(\Delta^-)^* \cong \Delta^-$ and only the embeddings $\phi_\ell^{\epsilon_1\epsilon_2}$ corresponding to the case $\epsilon_1 = (-1)^{n-\ell}\epsilon_2$ occurs.*

If n is an odd positive integer, we have $(\Delta^+)^* \cong \Delta^-$ *and* $(\Delta^-)^* \cong \Delta^+$ *and only the embeddings* $\phi_\ell^{\epsilon_1\epsilon_2}$ *corresponding to the case* $\epsilon_1 = -(-1)^{n-\ell}\epsilon_2$ *occurs.*

$$\phi_\ell^{\epsilon_1\epsilon_2}(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>) = \sum_{\substack{[n]-J-I\supseteq K \\ (-1)^{i+k}=\epsilon_1 \\ (-1)^{j+k}=\epsilon_2}} \frac{(-1)^{|W-W\cap K|}}{2^{(n-|J|-|I|-1)/2}}[\underrightarrow{I}\,\underrightarrow{K}] \otimes [\underrightarrow{J}\,\underrightarrow{K}]^*$$

Here by small letters, we denote the number of the elements in the set indexed by their capital letters.

For e_n^+ *and* e_n^-, *we have the followings.*

$$\phi_n^{\epsilon_1\epsilon_2}(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>^{(\epsilon_1)}) = \sum_{\substack{[n]-J-I\supseteq K \\ (-1)^{i+k}=\epsilon_1 \\ (-1)^{j+k}=\epsilon_2}} \frac{(-1)^{|W-W\cap K|}}{2^{(n-|J|-|I|-1)/2}}[\underrightarrow{I}\,\underrightarrow{K}] \otimes [\underrightarrow{J}\,\underrightarrow{K}]^*$$

Moreover each of the above $\phi_\ell^{\epsilon_1\epsilon_2}$*'s becomes an isometric embedding with respect to the invariant hermitian inner products.*

Here for an exterior product $<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>$ of degree n, we denotes the basis elements of e_n^+ and e_n^- respectively by

$$<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>^{(+)} = \frac{1}{\sqrt{2}}(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}> +(-1)^{|I|} <\underrightarrow{J}, \underrightarrow{W^c}, \underleftarrow{\overline{W^c}}, \underleftarrow{\overline{I}}>)$$

and

$$<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>^{(-)} = \frac{1}{\sqrt{2}}(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}> -(-1)^{|I|} <\underrightarrow{J}, \underrightarrow{W^c}, \underleftarrow{\overline{W^c}}, \underleftarrow{\overline{I}}>)$$

,if the right-hand sides are non-zero.

Using the above, we obtain the following $Pin(2n)$ equivariant embedding.

Theorem 2.5. *As a* $Pin(2n)$ ($O(2n)$) *module, for an even* ℓ, *the equivariant isometric embedding* $\phi_\ell : \bigwedge^\ell V \longrightarrow \Delta \bigotimes \Delta^*$ *is given by*

$$\phi_\ell(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>) = \sum_{[n]-J-I\supseteq K} \frac{(-1)^{|W-W\cap K|}}{2^{(n-|J|-|I|)/2}}[\underrightarrow{I}\,\underrightarrow{K}] \otimes [\underrightarrow{J}\,\underrightarrow{K}]^*.$$

and for an odd ℓ, *the equivariant isometric embedding* $\phi_\ell : \bigwedge^\ell V \longrightarrow \Delta \bigotimes \Delta^*$ *is given by*

$$\phi_\ell(<\underrightarrow{J}, \underrightarrow{W}, \underleftarrow{\overline{W}}, \underleftarrow{\overline{I}}>) = \sum_{[n]-J-I\supseteq K} \frac{(-1)^{|W-W\cap K|+|I|+|K|}}{2^{(n-|J|-|I|)/2}}[\underrightarrow{I}\,\underrightarrow{K}] \otimes [\underrightarrow{J}\,\underrightarrow{K}]^*.$$

This theorem is crucial for us to consider the invariant theory for the group $Pin(2n)$ ($O(2n)$).

3. Invariant Theoretical Preliminaries

We consider the invariant theory for $Pin(2n)$ $(O(2n))$. As before, the First Main Theorem tells us that the invariant symmetric forms $(\mathbf{v}_i, \mathbf{v}_j)$'s generate the invariants. Since

$$\mathrm{End}_{Pin(2n)}(\Delta\otimes\bigotimes^{k} V) = (\Delta^*\otimes\Delta\otimes\bigotimes^{2k} V)^{Pin(2n)} = (\bigoplus_{i=0}^{n}\bigwedge^{2i} V\otimes\bigotimes^{2k} V)^{O(2n)},$$

it is enough to obtain an explicit base of the invariant polynomials in the space

$$(\bigwedge^{r} V^*\otimes\bigotimes^{s} V^*)^{O(2n)}(\subset (\bigotimes^{r+s} V^*)^{O(2n)} \subset P(\bigoplus^{r+s} V)^{O(2n)}).$$

We can assume $r+s\equiv 0 \mod 2$. Those basis elements are multilinear in each variable and has the alternating properties in the first r variables, regarded as the elements in $P(\bigoplus^{r+s} V)^{O(2n)}$.

As in the case of $Spin(2n+1)$, let $\mathbf{t}=\{t_1,t_2,\ldots,t_r\}$ $(t_1<t_2<\ldots<t_r)$ and $\mathbf{m}=\{m_1,\ldots,m_u\}$ and $\mathbf{l}=\{l_1,\ldots,l_u\}$ be ordered index sets (or sequences) such that as sets, they are mutually disjoint and satisfy the condition $[s]=\mathbf{t}\sqcup\mathbf{m}\sqcup\mathbf{l}$. By $\{\mathbf{m},\mathbf{l}\}$, we denote the $u=\dfrac{s-r}{2}$ pairs of indices $\{\mathbf{m},\mathbf{l}\}=\{\{m_1,l_1\},\{m_2,l_2\},\ldots,\{m_u,l_u\}\}$.

So the invariant polynomials can be written as sums of the following polynomials:

$$\mathbf{T}_{\mathbf{t},\{\mathbf{m},\mathbf{l}\}}=\frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r}\epsilon(\sigma)(\mathbf{x}_{\sigma^{-1}(1)},\mathbf{y}_{t_1})(\mathbf{x}_{\sigma^{-1}(2)},\mathbf{y}_{t_2})\ldots(\mathbf{x}_{\sigma^{-1}(r)},\mathbf{y}_{t_r})\times\prod_{j=1}^{u}(\mathbf{y}_{m_j},\mathbf{y}_{l_j}).$$

Here $\mathbf{x}_j$ $(j=1,2\ldots,r)$ are the components of the first r tensors in $\bigotimes^{r+s} V$ and $\mathbf{y}_j$ $(j=1,2\ldots,s)$ are the components of the latter s tensors in $\bigotimes^{r+s} V$.

Lemma 3.1. *Let* $\mathbf{CP}_{\mathbf{l}}^{\mathbf{k}}=\mathrm{Hom}_{Pin(2n)}(\Delta\otimes\bigotimes^k V,\Delta\otimes\bigotimes^l V)$. *If* $s=k+l\equiv 0 \mod 2$, *we take only* $\mathbf{t}$*'s such that* $|\mathbf{t}|\equiv 0 \mod 2$. *If* $s=k+l\equiv 1 \mod 2$, *we take only* $\mathbf{t}$*'s such that* $|\mathbf{t}|\equiv 1 \mod 2$.

Then the above $\mathbf{T}_{\mathbf{t},\{\mathbf{m},\mathbf{l}\}}$*'s span linearly the whole space of the* $\mathbf{CP}_{\mathbf{l}}^{\mathbf{k}}$.

Moreover if $s=k+l<2n+2$, $\mathbf{T}_{\mathbf{t},\{\mathbf{m},\mathbf{l}\}}$ $(\mathbf{t}\sqcup\mathbf{m}\sqcup\mathbf{l}=[s])$ *are linearly independent, i.e.,*

$$\mathbf{CP}_{\mathbf{l}}^{\mathbf{k}}=\bigoplus_{\mathbf{t}\sqcup\mathbf{m}\sqcup\mathbf{l}=[s]}\mathbb{C}\mathbf{T}_{\mathbf{t},\{\mathbf{m},\mathbf{l}\}}.$$

As before we have a natural correspondence between the generalized Brauer diagrams $\mathbf{GB}_{\mathbf{l}}^{\mathbf{k}}$ and the polynomials $\mathbf{T}_{\mathbf{t},\{\mathbf{m},\mathbf{l}\}}$.

We denotes these elements by adding the suffix 'inv' to the diagrams $\mathbf{GB}_{\mathbf{l}}^{\mathbf{k}}$

4. A Representation Theoretic parameterization

From Theorem 2.1, we can show that $\dim(\mathrm{Hom}_{Pin(2n)}(\Delta \bigotimes \bigwedge^k V, \Delta)) = 1$ and define a $Pin(2n)$-equivariant homomorphism from $\Delta \bigotimes \bigwedge^k V$ to Δ as follows.

Definition 4.1. *If k is even, we define*

$$\mathrm{pr}_k([\underrightarrow{\mathtt{T}}] \otimes < \underrightarrow{\mathtt{I}}, \underrightarrow{\mathtt{W}}, \overline{\underleftarrow{\mathtt{W}}}, \overline{\underleftarrow{\mathtt{J}}} >)$$
$$= \begin{cases} 0 & \textit{if } \mathtt{I} \not\subseteq \mathtt{T}, \\ \epsilon\begin{pmatrix} \underrightarrow{\mathtt{T}} & \\ \underrightarrow{\mathtt{I}} & \underrightarrow{\mathtt{K}} \end{pmatrix} (-1)^{|\mathtt{W}-\mathtt{W}\cap\mathtt{K}|} 2^{(|\mathtt{I}|+|\mathtt{J}|)/2} [\underrightarrow{\mathtt{J}}\, \underrightarrow{\mathtt{K}}] & \textit{if } \mathtt{I} \subseteq \mathtt{T} \end{cases}$$

If k is odd, we define

$$\mathrm{pr}_k([\underrightarrow{\mathtt{T}}] \otimes < \underrightarrow{\mathtt{I}}, \underrightarrow{\mathtt{W}}, \overline{\underleftarrow{\mathtt{W}}}, \overline{\underleftarrow{\mathtt{J}}} >)$$
$$= \begin{cases} 0 & \textit{if } \mathtt{I} \not\subseteq \mathtt{T}, \\ \epsilon\begin{pmatrix} \underrightarrow{\mathtt{T}} & \\ \underrightarrow{\mathtt{I}} & \underrightarrow{\mathtt{K}} \end{pmatrix} (-1)^{|\mathtt{W}-\mathtt{W}\cap\mathtt{K}|+|\mathtt{K}|+|\mathtt{J}|} 2^{(|\mathtt{I}|+|\mathtt{J}|)/2} [\underrightarrow{\mathtt{J}}\, \underrightarrow{\mathtt{K}}] & \textit{if } \mathtt{I} \subseteq \mathtt{T}. \end{cases}$$

Here we put $\mathtt{K} = \mathtt{T} - \mathtt{I}$ and $\epsilon\begin{pmatrix} \underrightarrow{\mathtt{T}} & \\ \underrightarrow{\mathtt{I}} & \underrightarrow{\mathtt{K}} \end{pmatrix}$ denotes the sign of the permutation obtained by arranging $\underrightarrow{\mathtt{T}}$ into $\underrightarrow{\mathtt{I}}\ \ \underrightarrow{\mathtt{K}}$ in this order.

Similarly we have $\dim(\mathrm{Hom}_{Pin(2n)}(\Delta, \Delta \bigotimes \bigwedge^k V)) = 1$ and and define a $Pin(2n)$-equivariant homomorphism from Δ to $\Delta \bigotimes \bigwedge^k V$ as follows.

Definition 4.2. *For an even k, we define*

$$\mathrm{inj}_k([\underrightarrow{\mathtt{I}}]) =$$
$$\sum_{\substack{\mathtt{I}\subset\mathtt{T} \\ \mathtt{K}=\mathtt{T}-\mathtt{I}}} \epsilon\begin{pmatrix} \underrightarrow{\mathtt{T}} & \\ \underrightarrow{\mathtt{I}} & \underrightarrow{\mathtt{K}} \end{pmatrix} \Big(\sum_{\substack{\mathtt{J}\subseteq([n]-\mathtt{T}) \\ \mathtt{W}\subseteq([n]-\mathtt{I}-\mathtt{J}) \\ |\mathtt{J}|+|\mathtt{I}|+2|\mathtt{W}|=k}} (-1)^{|\mathtt{W}-\mathtt{W}\cap\mathtt{K}|} 2^{(|\mathtt{J}|+|\mathtt{I}|)/2} [\underrightarrow{\mathtt{J}}\, \underrightarrow{\mathtt{K}}] \otimes k! < \underrightarrow{\mathtt{J}}, \underrightarrow{\mathtt{W}}, \overline{\underleftarrow{\mathtt{W}}}, \overline{\underleftarrow{\mathtt{I}}} >\Big)$$

For an odd k, we define

$$\mathrm{inj}_k([\underrightarrow{\mathtt{I}}]) = \sum_{\substack{\mathtt{I}\subset\mathtt{T} \\ \mathtt{K}=\mathtt{T}-\mathtt{I}}} \epsilon\begin{pmatrix} \underrightarrow{\mathtt{T}} & \\ \underrightarrow{\mathtt{I}} & \underrightarrow{\mathtt{K}} \end{pmatrix} \times$$
$$\Big(\sum_{\substack{\mathtt{J}\subseteq([n]-\mathtt{T}) \\ \mathtt{W}\subseteq([n]-\mathtt{I}-\mathtt{J}) \\ |\mathtt{J}|+|\mathtt{I}|+2|\mathtt{W}|=k}} (-1)^{|\mathtt{W}-\mathtt{W}\cap\mathtt{K}|+|\mathtt{J}|+|\mathtt{K}|} 2^{(|\mathtt{J}|+|\mathtt{I}|)/2} [\underrightarrow{\mathtt{J}}\, \underrightarrow{\mathtt{K}}] \otimes k! < \underrightarrow{\mathtt{J}}, \underrightarrow{\mathtt{W}}, \overline{\underleftarrow{\mathtt{W}}}, \overline{\underleftarrow{\mathtt{I}}} >\Big)$$

Using the above pr_p , we introduce a $Pin(2n)$-equivariant homomorphism $\mathrm{pr}_{\underrightarrow{T}}$ from $\Delta \bigotimes \bigotimes^k V$ to $\Delta \bigotimes \bigotimes^{k-p} V$ as before. Here $\mathrm{T} = \{t_1, t_2 \ldots , t_p\}$ $(t_1 < t_2 < \ldots < t_p)$is a subset of $[k]$ and denotes the positions in the tensor product $\bigotimes^k V$. Namely,

Definition 4.3. *Let* $\mathrm{pr}_{\underrightarrow{T}} : \Delta \bigotimes \bigotimes^k V \longrightarrow \Delta \bigotimes \bigotimes^{k-p} V$ *be the projection map obtained by the composition of the map* $\mathrm{Alt}_{\underrightarrow{T}}$ *and* pr_p*, i.e.,* $\mathrm{pr}_{\underrightarrow{T}} = \mathrm{pr}_p \circ \mathrm{Alt}_{\underrightarrow{T}}$. *Here* pr_p *acts on the alternating tensors sitting in the positions indexed by* $\underrightarrow{T}$ *in the space* $\Delta \bigotimes \bigotimes^k V$.

From the definition, $\mathrm{pr}_{\underrightarrow{T}}$ is the element in $\mathrm{Hom}_{Pin(2n)}(\Delta \bigotimes \bigotimes^k V, \Delta \bigotimes \bigotimes$
and has the alternating property with the index set T.

Similarly we define the $Pin(2n)$-equivariant embedding $\mathrm{inj}_{\underrightarrow{T}} \in \mathrm{Hom}_{Pin(2}$
as follows.

Definition 4.4. *Let* $\mathrm{inj}_{\underrightarrow{T}} : \Delta \bigotimes \overset{k-p}{\bigotimes} V \to \Delta \bigotimes \overset{k}{\bigotimes} V$ *be the immersion obtained by the composition of the map* $\mathrm{inj}_p : \Delta \longrightarrow \Delta \bigotimes \overset{p}{\bigwedge} V$ *and the linear embedding of the resulting tensors in the positions indexed by* $\underrightarrow{T}$.

From the definition, $\mathrm{inj}_{\underrightarrow{T}}$ has the alternating property on the index set T too.

Then we can define representaion-theoretic parameterization of the elements in $\mathbf{CP}_\mathrm{l}^\mathrm{k}$ by the generalized Brauer diagrams $\mathbf{GB}_\mathrm{l}^\mathrm{k}$ as before.

We fix an element of the diagrams $\mathbf{GB}_\mathrm{l}^\mathrm{k}$. and let T_u be its isolated points in the upper row and T_ℓ be its isolated points in the lower row.

Then the action represented by the isolated points in the upper row corresponds to the projection $\mathrm{pr}_{\underrightarrow{T_u}}$ and the action represented by the isolated points in the lower row corresponds to the immersion $\mathrm{inj}_{\underrightarrow{T_\ell}}$. Namely the total action represented by the isolated points corressponds to the composition map

$$\Delta \bigotimes \bigotimes^k V \xrightarrow{\mathrm{pr}_{\underrightarrow{T_u}}} \Delta \bigotimes \bigotimes^{k-p} V \xrightarrow{\mathrm{inj}_{\underrightarrow{T_\ell}}} \Delta \bigotimes \overset{k}{\bigotimes} V.$$

Finally we define the action corresponding to the points which are not isolated just in the same way as those of the ordinary Brauer diagrams.

We denote these elements by adding the suffix 'rt' to the diagrams of $\mathbf{GB}_\mathrm{l}^\mathrm{k}$.

Whether these elements span linearly the space $\mathbf{CS}_\mathrm{l}^\mathrm{k}$ or not, or whether these elements become a base or not is not clear at present. We show in the next section that if $k \leqq n$ and $l \leqq n$, we give the explicit relations between two parametrizations and that they become a base in this case.

5. Relation between two parameterization

Since the difference between two parametrizations are only in the actions corresponding to the isolated points, we give the relations between them. Let T_u ($|\mathrm{T}_u| = p$) be the isolated points in the upper row and let T_l ($|\mathrm{T}_l| = q$) be the isolated points in the lower row.

We denote the homomorphism $\Delta \otimes \otimes^p V \to \Delta \otimes \otimes^q V$, determined by the invariant polynomial by $\psi_{\mathrm{T}_l}^{\mathrm{T}_u}$, or simply by ψ_q^p if the isolated points are tacitly understood. Here the invariant polynomial which we consider in the above is given by

$$\sum_{\sigma \in \mathfrak{S}_r} \epsilon(\sigma)(\mathbf{x}_{\sigma^{-1}(1)}, \mathbf{y}_{t_1})(\mathbf{x}_{\sigma^{-1}(2)}, \mathbf{y}_{t_2}) \dots (\mathbf{x}_{\sigma^{-1}(r)}, \mathbf{y}_{t_r})$$

and we consider this element as the invariant polynomial in the space $(\bigwedge^{p+q} V)^* \otimes \bigwedge^{p+q} V$.

The relation of the above two actions are given as follows.

For any $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$, we have $\tau \circ \psi_{\mathrm{T}_l}^{\mathrm{T}_u} \circ \sigma = \psi_{\tau(\mathrm{T}_l)}^{\sigma^{-1}(\mathrm{T}_u)}$. So it is enough to give an explicit description for $\psi_{[q]}^{[p]}$ in terms of the representation theoretical operators, where $[p] = \{1, 2, \dots, p\}$ and $[q] = \{1, 2, \dots, q\}$.

Theorem 5.1. *Let $p \leqq n$ and $q \leqq n$. Then we have*

$$(5.1.1)\quad \psi_{[q]}^{[p]} = \sum_{i=0}^{\min(p,q)} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p}} \epsilon(\sigma)\, \epsilon(\tau) \frac{\mathrm{inj}_{\{\sigma([i+1,q])\}}}{(q-i)!} \frac{\begin{pmatrix} \tau([1,i]) \\ \sigma([1,i]) \end{pmatrix}}{i!} \frac{\mathrm{pr}_{\{\tau([i+1,p])\}}}{(p-i)!}$$

and
(5.1.2)

$$\mathrm{inj}_{[q]} \circ \mathrm{pr}_{[p]} = \sum_{i=0}^{\min(p,q)} (-1)^i \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p}} \epsilon(\sigma)\, \epsilon(\tau) \frac{\psi_{\{\sigma([i+1,q])\}}^{\{\tau([i+1,p])\}}}{(q-i)!(p-i)!} \otimes \frac{\begin{pmatrix} \tau([1,i]) \\ \sigma([1,i]) \end{pmatrix}}{i!}$$

Here we denote $\sigma([i+1,q]) = \{\sigma(i+1), \sigma(i+2), \dots, \sigma(q)\}$ and $\tau([i+1,p]) = \{\tau(i+1), \tau(i+2), \dots, \tau(p)\}$ and $\begin{pmatrix} \tau([1,i]) \\ \sigma([1,i]) \end{pmatrix}$ denotes the partial permutation of the tensor components obtained by sending the $\tau(1)$-th component in $\Delta \otimes \otimes^p V$ in the upper row to the $\sigma(1)$-th position in $\Delta \otimes \otimes^q V$ in the lower row and so on. Also we note that all the indices occurring in ψ, pr *and* inj *have the alternating property respectively.*

So if $k \leqq n$ and $l \leqq n$, the generalized Brauer diagrams under the representation theoretic parameterization also span the whole space $\mathbf{CP}_\mathrm{l}^\mathrm{k}$ and become a linear base of $\mathbf{CP}_\mathrm{l}^\mathrm{k}$.

Remark 5.2. *The right-hand side of the first formula 5.1.1 can be considered as the composition of the homomorphisms in this order, but the formula 5.1.2 cannot. So we put $\otimes$ in the right-hand side to show that each summand becomes a homomorphism as a whole.*

Example 5.3. *We show a few example of the relations of two parameterization. Let us assume that $n \geqq 2$.*

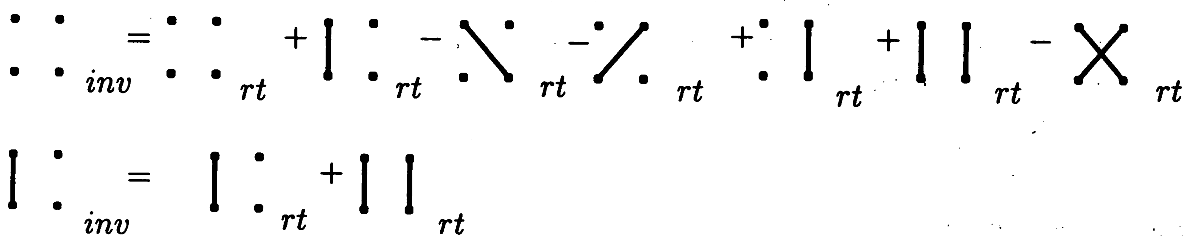


FIGURE 1. The relations of two parameterization

If $p > n$, we have $\Delta \bigotimes \bigwedge^p V \cong \Delta \bigotimes \bigwedge^{2n-p} V \bigotimes \det = \sum_{i=0}^{2n-p} [\Delta, (1^i)]$. So we need the $Pin(2n)$ isomorphism $A \bigotimes r_p : \Delta \bigotimes \bigwedge^p V \longrightarrow \Delta \bigotimes \bigwedge^{2n-p} V$, where A is the associater in Lemma 2.3 which is the isomorphism from $\Delta \bigotimes \det$ to Δ and r_p is a $Pin(2n)$ isomorphism from $\bigotimes \bigwedge^p V$ to $\det \bigotimes \bigwedge^{2n-p} V$ given as follows.

Lemma 5.4. *The $\mathfrak{so}(2n)$-isomorphism*

$$r_\ell : \bigwedge^{\ell} V \longrightarrow \bigwedge^{2n-\ell} V$$

is given by follows.

$$r_\ell(< \underrightarrow{\mathtt{J}}, \underrightarrow{\mathtt{W}}, \underleftarrow{\overline{\mathtt{W}}}, \underleftarrow{\overline{\mathtt{I}}} >) = (-1)^{|\mathtt{I}|} < \underrightarrow{\mathtt{J}}, \underrightarrow{\mathtt{W}^c}, \underleftarrow{\overline{\mathtt{W}^c}}, \underleftarrow{\overline{\mathtt{I}}} >$$

, where $\mathtt{W} \sqcup \mathtt{W}^c = [n] - \mathtt{J} - \mathtt{I}$ and $|\mathtt{J}| + |\mathtt{I}| + 2|\mathtt{W}| = \ell$.

The formulas are as follows. As before we define the homomorphism $r^{\boxed{\mathtt{p}}}_{\boxed{\mathtt{2n-p}}}$ by the composition of the alternating operator $\mathrm{Alt}_{\boxed{\mathtt{p}}}$ and r_p and the immersion of the alternating tensor to the positions $\boxed{\mathtt{2n-p}}$.

Theorem 5.5. *Let $p > n$ and $q < n$. Then as a homomorphism from $\Delta \bigotimes \bigotimes^p V$ to $\Delta \bigotimes \bigotimes^q V$, we have*

(5.5.1)

$$\psi^{\boxed{\mathtt{p}}}_{\boxed{\mathtt{q}}} = (-1)^{q+n} \sum_{\substack{\sigma \in \mathfrak{S}_{2n-p} \\ \tau \in \mathfrak{S}_q}} \epsilon(\sigma)\, \epsilon(\tau) \frac{\begin{pmatrix} \sigma([1,q]) \\ \tau([1,q]) \end{pmatrix}}{q!} \frac{\mathrm{pr}_{\{\sigma([q+1,2n-p])\}}}{(2n-p-q)!} (A \otimes r^{\boxed{\mathtt{p}}}_{\boxed{\mathtt{2n-p}}}).$$

We note that from the definition of ψ^p_q, $p + q$ must be less than or equal to $2n$, i.e., $p + q \leqq 2n$.

Let $p < n$ and $q > n$. Then as a homomorphism from $\Delta \bigotimes \bigotimes^p V$ to $\Delta \bigotimes \bigotimes^q V$, we have

(5.5.2)

$$\psi_{\boxed{q}}^{\boxed{p}} = (-1)^{p+n} \frac{q!}{(2n-q)!} (A \otimes r_{\boxed{q}}^{\boxed{2n-q}}) \sum_{\substack{\sigma \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{2n-q}}} \epsilon(\sigma)\,\epsilon(\tau) \frac{\mathrm{inj}_{\{\tau([p+1,2n-q])\}}}{(2n-p-q)!} \frac{\begin{pmatrix} \sigma([1, \\ \tau([1, \end{pmatrix}}{p!}$$

6. Relations between $Pin(2n)$- equivariant homomorphisms

In this section we give the relations between the compositions of pr and inj and the contractions and the immersion of the invariant forms, from which we can deduce the multiplication rules of the generalized Brauer diagrams.

Theorem 6.1. *The following formulas holds for* pr, inj, *and the contraction operators* $\mathrm{C}_{\{i,j\}}$ *and the linear immersion of the invariant element* $\mathrm{id}_{V\{i,j\}}$. *Here* $\mathrm{C}_{\{i,j\}}$ *denotes the contraction (with respect to S) of the i-th and j-th tensor components and* $\mathrm{id}_V = \sum_{i=1}^n (u_i \otimes u_{\bar{i}} + u_{\bar{i}} \otimes u_i)$ *and the subindices of* $\mathrm{id}_{V\{i,j\}}$ *denote the immersed positions, namely the first tensor goes to the i-th position and the second goes to the j-th position.*

(i) *If $p \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^p V$ to Δ (here we consider the tensor $\bigotimes^p V$ sits in the positions $\{q+1, q+2, \ldots, p+q\}$), we have*

$$\mathrm{pr}_{\{[1,q+p]\}} \circ \mathrm{inj}_{\{[1,q]\}} = (-1)^{(p+q)q} (2n-p)_q \, \mathrm{pr}_{\{[q+1,q+p]\}} \,. \tag{6.1.1}$$

Here $(2n-p)_q$ denotes the lower factorial, i.e., for any x and non-negative integer i, we define $(x)_i = x(x-1)(x-2)\cdots(x-(i-1))$. The above also holds for $p = 0$ and in that case we regards pr *in the right-hand side as the identity map of Δ.*

(ii) *If $p \leqq n$, as the homomorphisms from Δ to $\Delta \bigotimes \bigotimes^p V$ (here we consider the tensor $\bigotimes^p V$ sits in the positions $\{q+1, q+2, \ldots, p+q\}$), we have*

$$\mathrm{pr}_{\{[1,q]\}} \circ \mathrm{inj}_{\{[1,q+p]\}} = (-1)^{(p+q)q} (2n-p)_q \, \mathrm{inj}_{\{[q+1,q+p]\}} \,. \tag{6.1.2}$$

(iii) *If $p \leqq n$ and $q \leqq n$, as the homomorphisms from Δ to $\Delta \bigotimes \bigotimes^{p+q} V$, we have*

(6.1.3)

$$\mathrm{inj}_{\{[1,q]\}} \circ \mathrm{inj}_{\{[q+1,q+p]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{qi + \binom{i+1}{2} + pq} \sum_{\substack{\sigma \in \mathfrak{S}_q \\ \tau \in \mathfrak{S}_p[q]}} \epsilon(\sigma)\,\epsilon(\tau) \times$$

$$\frac{\prod_{u=1}^{i} \mathrm{id}_{V\{\sigma(u),\tau(q+u)\}}}{i!} \; \frac{\mathrm{inj}_{\{\sigma([i+1,q]),\tau([q+i+1,q+p])\}}}{(q-i)!(p-i)!}.$$

Here $\mathfrak{S}_p[q]$ denotes the symmetric groups acting on the p characters $\{q+1, q+2, \dots, q+p\}$.

(iv) *If $p \leqq n$ and $q \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^{p+q} V$ to Δ, we have*

(6.1.4)

$$\mathrm{pr}_{\{[1,q]\}} \circ \mathrm{pr}_{\{[q+1,q+p]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{qi+\binom{i}{2}} \sum_{\substack{\sigma\in\mathfrak{S}_q \\ \tau\in\mathfrak{S}_p[q]}} \epsilon(\sigma)\,\epsilon(\tau) \frac{\mathrm{pr}_{\{\sigma([i+1,q]),\tau([q+i+1,q+p])\}}}{(q-i)!(p-i)!} \frac{\prod_{u=1}^{i} \mathrm{C}_{\{\sigma(u),\tau(q+u)\}}}{i!}.$$

(v) *If $p \leqq n$ and $q \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^{p} V$ to $\Delta \bigotimes \bigotimes^{q} V$, (here we consider the tensor $\bigotimes^{p} V$ sits in the positions $\{q+1, q+2, \dots, q+p\}$.) we have*

(6.1.5)

$$\mathrm{pr}_{\{[q+1,q+p]\}} \circ \mathrm{inj}_{\{[1,q]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{pq} 2^{i} \sum_{\substack{\sigma\in\mathfrak{S}_q \\ \tau\in\mathfrak{S}_p[q]}} \epsilon(\sigma)\,\epsilon(\tau) \frac{\mathrm{inj}_{\{\sigma([i+1,q])\}}}{(q-i)!} \times \frac{\begin{pmatrix} \tau([q+1,q+i]) \\ \sigma([1,i]) \end{pmatrix}}{i!} \frac{\mathrm{pr}_{\{\tau([q+i+1,q+p])\}}}{(p-i)!}.$$

Here $\begin{pmatrix} \tau([q+1,q+i]) \\ \sigma([1,i]) \end{pmatrix}$ denotes the (partial) permutation which sends the $\tau(q+u)$-th component to $\sigma(u)$-th position.

(vi) *If $p \geqq t$ and $p-t \leqq n$ and $q \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^{p-t} V$ to $\Delta \bigotimes \bigotimes^{q} V$, (here we consider the tensor $\bigotimes^{p-t} V$ sits in the positions $\{q+t+1, q+t+2, \dots, q+p\}$.) we have*

(6.1.6)

$$\mathrm{pr}_{\{[q+1,q+p]\}} \circ \mathrm{inj}_{\{[1,q+t]\}} = \sum_{i=0}^{\min(p-t,q)} (-1)^{pq+t(p+q)} \Big(\sum_{u=0}^{i} \binom{i}{u} (2n-p-q+t+i-u)_t\Big) \times \sum_{\substack{\sigma\in\mathfrak{S}_q \\ \tau\in\mathfrak{S}_{p-t}[q+t]}} \epsilon(\sigma)\,\epsilon(\tau) \frac{\mathrm{inj}_{\{\sigma([i+1,q])\}}}{(q-i)!} \frac{\begin{pmatrix} \tau([q+t+1,q+t+i]) \\ \sigma([1,i]) \end{pmatrix}}{i!} \frac{\mathrm{pr}_{\{\tau([q+t+i+1,q+p])\}}}{(p-t-i)!}.$$

Here $\binom{i}{u}$ in the parentheses denotes the ordinary binomial coefficient. If $t=0$, $(2n-p-q+0+i-u)_0 = 1$ and we obtain 6.1.5.

(vii) *If $p \leqq n$ and $q \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^{q} V$ to $\Delta \bigotimes \otimes^{p} V$, (here we consider the tensor $\otimes^{q} V$ sits in the positions*

$\{p+q+1, p+q+2, \ldots, p+2q\}$.) *we have*

(6.1.7)

$$\prod_{i=1}^{q} \mathrm{C}_{\{i,p+q+i\}} \mathrm{inj}_{\{[1,q+p]\}} = \sum_{i=0}^{\min(p,q)} (-1)^{\binom{q}{2}} \sum_{\substack{\sigma\in\mathfrak{S}_q[q+p] \\ \tau\in\mathfrak{S}_p[q]}} \epsilon(\sigma)\,\epsilon(\tau) \frac{\mathrm{inj}_{\{\tau([q+i+1,q+p])\}}}{(p-i)!}$$

$$\frac{\begin{pmatrix} \sigma([p+q+1,p+q+i]) \\ \tau([q+1,q+i]) \end{pmatrix}}{i!} \frac{\mathrm{pr}_{\{\sigma([q+p+i+1,2q+p])\}}}{(q-i)!}.$$

(viii) *If $p \leqq n$ and $q \leqq n$, as the homomorphisms from $\Delta \bigotimes \bigotimes^q V$ to $\Delta \bigotimes \bigotimes^p V$, (here we consider the tensor $\bigotimes^q V$ sits in the positions $[q]$ and $\otimes^p V$ sits in the positions $\{p+q+1, p+q+2, \ldots, 2p+q\}$.) we have*

(6.1.8)

$$\mathrm{pr}_{\{[1,q+p]\}} \prod_{i=1}^{p} \mathrm{id}_{V\{q+i,p+q+i\}} = \sum_{i=0}^{\min(p,q)} (-1)^{\binom{p}{2}+pq} \sum_{\substack{\sigma\in\mathfrak{S}_q \\ \tau\in\mathfrak{S}_p[q+p]}} \epsilon(\sigma)\,\epsilon(\tau) \times$$

$$\frac{\mathrm{inj}_{\{\tau([p+q+i+1,q+2p])\}}}{(p-i)!} \frac{\begin{pmatrix} \sigma([1,i]) \\ \tau([p+q+1,p+q+i]) \end{pmatrix}}{i!} \frac{\mathrm{pr}_{\{\sigma([i+1,q])\}}}{(q-i)!}.$$

Remark 6.2. *If we exchange $2n$ in the above formulas into an indeterminate X simultaneously, we can define the 'generic' centralizer algebra of* $\mathbf{CP_k}$ *just as in the ordinary Brauer algebras.*

7. Examples of products of the generalized Brauer diagrams

From the result of the previous section, we can calculate the product of the generalized Brauer diagrams.

As we remarked after the statement of Theorem 6.1, we change $2n$ in the formulas of the theorem into an indeterminate X simultaneously and we write down the relations of the 'generic' centralizer algebra of $\mathbf{CP_k}$.

We summarize other relations between the contractions and the immersions and pr and inj which follow easily from the definitions. We fix the index set of the tensor positions of $\bigotimes^k V$ from 1 to k and after the contraction $\mathrm{C}_{\{i,j\}}$ we consider the i-th and the j-th components are occupied by the empty set and $\mathrm{id}_{V\{i,j\}}$ is allowed if the positions i and j are occupied by the empty.

Then we have the followings.

Lemma 7.1. (i)

$$\mathrm{C}_{\{s,t\}} \mathrm{id}_{V\{i,j\}} = \mathrm{id}_{V\{i,j\}} \mathrm{C}_{\{s,t\}} \qquad \textit{if } \{s,t\} \cap \{i,j\} = \emptyset.$$

(ii)

$$\mathrm{C}_{\{s,t\}}\,\mathrm{id}_{V\{t,j\}} = \binom{s}{j} \qquad \textit{if } s \neq j.$$

Here $\binom{s}{j}$ *denotes the (partial) permutation which sends s-th component to the j-th position.*

(iii)

$$\mathrm{C}_{\{s,t\}}\,\mathrm{id}_{V\{s,t\}} = (2n)\,\mathrm{id}\,.$$

(iv) *If* $s, t \in \mathrm{T}$, *we have*

$$\mathrm{C}_{\{s,t\}}\,\mathrm{inj}_{\mathrm{T}} = 0, \quad \mathrm{pr}_{\mathrm{T}}\,\mathrm{id}_{V\{s,t\}} = 0.$$

(v) *If* $s, t_1, t_2, \dots, t_r \in [n]$ *are different from each other, we have*

$$\binom{t_i}{s}\,\mathrm{inj}_{\{t_1,\dots,t_i,\dots,t_r\}} = \mathrm{inj}_{\{t_1,\dots,\overset{i}{s},\dots,t_r\}}, \quad \binom{t_1}{s}\,\mathrm{id}_{V\{t_1,t_2\}} = \mathrm{id}_{V\{s,t_2\}}.$$

(vi) *If* $s, t_1, t_2, \dots, t_r \in [n]$ *are different from each other, we have*

$$\mathrm{pr}_{\{t_1,\dots,t_i,\dots,t_r\}}\binom{s}{t_i} = \mathrm{pr}_{\{t_1,\dots,\overset{i}{s},\dots,t_r\}}, \quad \mathrm{C}_{\{t_1,t_2\}}\binom{s}{t_1} = \mathrm{C}_{\{s,t_2\}}\,.$$

(vii) *If* $s, t, t_1, t_2, \dots, t_r \in [n]$ *are different from each other, we have*

$$\mathrm{id}_{V\{s,t\}}\,\mathrm{inj}_{\{t_1,\dots,t_i,\dots,t_r\}} = \mathrm{inj}_{\{t_1,\dots,t_i,\dots,t_r\}}\,\mathrm{id}_{V\{s,t\}}$$

and

$$\mathrm{C}_{\{s,t\}}\,\mathrm{inj}_{\{t_1,\dots,t_i,\dots,t_r\}} = \mathrm{inj}_{\{t_1,\dots,t_i,\dots,t_r\}}\,\mathrm{C}_{\{s,t\}}\,.$$

Also we have

$$\mathrm{id}_{V\{s,t\}}\,\mathrm{pr}_{\{t_1,\dots,t_i,\dots,t_r\}} = \mathrm{pr}_{\{t_1,\dots,t_i,\dots,t_r\}}\,\mathrm{id}_{V\{s,t\}}$$

and

$$\mathrm{C}_{\{s,t\}}\,\mathrm{pr}_{\{t_1,\dots,t_i,\dots,t_r\}} = \mathrm{pr}_{\{t_1,\dots,t_i,\dots,t_r\}}\,\mathrm{C}_{\{s,t\}}\,.$$

Proof. The first three formulas are easy and the fourth and the fifth formulas follow directly from the alternating property with the indices in T of pr and inj. The rest are obvious from the definition. □

Remark 7.2. *If we put* $e_i = \mathrm{id}_{V\{i,i+1\}}\,\mathrm{C}_{\{i,i+1\}}$, *we can deduce easily* $e_i e_{i+1} e_i = e_i$ *from the above lemma. These relations were used to define the Birman-Wenzl algebras (q-analog of the Brauer centralizer algebras).*

Let us show some examples.

Example 7.3. *In this example we always assume* $n \geq k$ *and exploit the representation theoretical parameterization of the generalized Brauer diagrams, so we omit the subscript rt here. We calculate the product of* $y_5 y_8$ *in the* $k = 2$ *generalized Brauer diagrams of the introduction.*

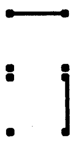
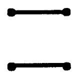
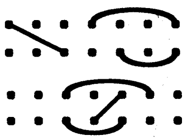
$y_5 y_8 =$  $= \quad -(X-1)$  $- \quad (X-1)$ 

FIGURE 2. The result of the product of y_5y_8

Here $y_8 = \mathrm{inj}_{\{1,2\}} \mathrm{C}_{\{1,2\}}$ *and* $y_5 = \mathrm{inj}_{\{1\}} \binom{2}{2} \mathrm{pr}_{\{1\}}$. *From the formula (6.1.2), we have* $\mathrm{pr}_{\{1\}} \mathrm{inj}_{\{1,2\}} = (X-1)_1 \mathrm{inj}_{\{2\}}$ *(we put* X *for* $2n$*.) and the targeting homomorphism is* $\mathrm{inj}_{\{1\}} \binom{2}{2} (X-1) \mathrm{inj}_{\{2\}} \mathrm{C}_{\{1,2\}} = (X-1) \mathrm{inj}_{\{1\}} \mathrm{inj}_{\{2\}} \mathrm{C}_{\{1,2\}}$. *From the formula (6.1.3), we have* $\mathrm{inj}_{\{1\}} \mathrm{inj}_{\{2\}} = -\mathrm{inj}_{\{1,2\}} - \mathrm{id}_{V\{1,2\}}$ *and the final formula is given in Figure 2.*

We calculate more complicated general case of Figure 3.

 $= 3(X-2)(X-3)$ 

$$-(X-2)(X-3)(X-4)\sum_{i=1}^{5}\sum_{j=1}^{5}(-1)^{i+j} z_i y_j$$

Here y_j *denotes the upper row and* z_i *denotes the lower row given as follows.*

 y_1  y_2  y_3  y_4

 y_5  z_1  z_2  z_3

 z_4  z_5

FIGURE 3. The result of the product of a more complicated example

In this case we must calculate

$$\mathrm{id}_{V\{3,5\}} \mathrm{inj}_{\{1,2,6,7\}} \binom{5}{4} \mathrm{pr}_{\{1,2,4,7\}} \mathrm{C}_{\{3,6\}} \mathrm{id}_{V\{5,7\}} \mathrm{inj}_{\{1,2,4,6\}} \binom{1}{3} \mathrm{pr}_{\{2,3,5,6\}} \mathrm{C}_{\{4,7\}}.$$

We note that the inside homomorphism $\binom{5}{4} \mathrm{pr}_{\{1,2,4,7\}} \mathrm{C}_{\{3,6\}} \mathrm{id}_{V\{5,7\}} \mathrm{inj}_{\{1,2,4,6\}} \binom{1}{3}$ *is an element in* $\mathbf{CP}^{\mathbf{1}}_{\mathbf{1}} = \mathrm{Hom}_{Pin(2n)}(\Delta \bigotimes V, \Delta \bigotimes V)$. *(*$V$ *sits in the first place in the upper row and sits in the forth place in the lower row.*

In general, the inside homomorphism is an element in $\mathbf{CP}^{\mathbf{s}}_{\mathbf{t}}$. *Here* s *is the number of the vertical lines (the edges from the upper row to the lower row) in the upper diagram and* t *is the number of the vertical* *lines in the lower diagram. Each of them corresponds to the indices in the upper row of the upper diagram and in the lower row of the lower diagram jointed with edges respectively. From Theorem 5.1, this element must be a linear combination of* $\mathbf{GB}_{\mathbf{t}}^{\mathbf{s}}$ *and using the formulas in Theorem 6.1, we can calculate this homomorphism explicitly.*

From the formula (6.1.8), we have

$$\mathrm{pr}_{\{1,2,3,4\}}\,\mathrm{id}_{V\{4,5\}} = -\,\mathrm{inj}_{\{5\}}\,\mathrm{pr}_{\{1,2,3\}} - \begin{pmatrix}1\\5\end{pmatrix}\mathrm{pr}_{\{2,3\}} + \begin{pmatrix}2\\5\end{pmatrix}\mathrm{pr}_{\{1,3\}} - \begin{pmatrix}3\\5\end{pmatrix}\mathrm{pr}_{\{1,2\}}$$

If we apply the conjugation of the permutation $\eta = \begin{pmatrix}3 & 4\\4 & 7\end{pmatrix}$*, we have*

$$\mathrm{pr}_{\{1,2,4,7\}}\,\mathrm{id}_{V\{5,7\}} = -\,\mathrm{inj}_{\{5\}}\,\mathrm{pr}_{\{1,2,4\}} - \begin{pmatrix}1\\5\end{pmatrix}\mathrm{pr}_{\{2,4\}} + \begin{pmatrix}2\\5\end{pmatrix}\mathrm{pr}_{\{1,4\}} - \begin{pmatrix}4\\5\end{pmatrix}\mathrm{pr}_{\{1,2\}}$$

From the formula (6.1.7), we have

$$\mathrm{C}_{\{1,5\}}\,\mathrm{inj}_{\{1,2,3,4\}} = \mathrm{inj}_{\{2,3,4\}}\,\mathrm{pr}_{\{5\}} + \mathrm{inj}_{\{3,4\}}\begin{pmatrix}5\\2\end{pmatrix} - \mathrm{inj}_{\{2,4\}}\begin{pmatrix}5\\3\end{pmatrix} + \mathrm{inj}_{\{2,3\}}\begin{pmatrix}5\\4\end{pmatrix}.$$

If we apply the conjugation of the permutation $\eta = \begin{pmatrix}1 & 2 & 3 & 4 & 5\\6 & 2 & 4 & 1 & 3\end{pmatrix}$*, we have*

$$\mathrm{C}_{\{3,6\}}\,\mathrm{inj}_{\{1,2,4,6\}} = -\,\mathrm{inj}_{\{1,2,4\}}\,\mathrm{pr}_{\{3\}} + \mathrm{inj}_{\{1,4\}}\begin{pmatrix}3\\2\end{pmatrix} - \mathrm{inj}_{\{1,2\}}\begin{pmatrix}3\\4\end{pmatrix} - \mathrm{inj}_{\{2,4\}}\begin{pmatrix}3\\1\end{pmatrix}.$$

Since $\mathrm{pr}_{\{2,3\}}\,\mathrm{inj}_{\{1,2\}} = -(X-2)\,\mathrm{inj}_{\{1\}}\,\mathrm{pr}_{\{3\}} - \{(X-1-2+1+1-0)+(X-1-2+1+1-1)\}\begin{pmatrix}3\\1\end{pmatrix}$ *from the formula (6.1.6), we have* $\mathrm{pr}_{\{2,4\}}\,\mathrm{inj}_{\{1,4\}} = (X-2)\,\mathrm{inj}_{\{1\}}\,\mathrm{pr}_{\{2\}} + (2X-3)\begin{pmatrix}2\\1\end{pmatrix}$. *The calculation goes on in almost similar way and the final result consists of* 26 *terms in which only one term contains a vertical line. It is given in the figure 3.*

8. Dual Pair and $Pin(2n)$ and $Spin(2n)$ representations

In this section we first define the subspace in $\Delta \bigotimes \bigotimes^k V$ on which the symmetric group of degree k and $Pin(2n)$ act as a dual pair. Next we determine the centralizer algebra $\mathbf{CS_k}$ of $Spin(2n)$ on the space $\Delta \bigotimes \bigotimes^k V$ and define the subspace on which the symmetric group of degree k and $Spin(2n)$ act as a dual pair.

For the time being, we always assume $n \geqq k$ and we consider the generalized Brauer diagrams under the representation theoretic parameterization.

We define the ideals of $\mathbf{CP_k}$ as follows.

For the generalized Brauer diagrams y and z, we consider the inside homomorphism of the product yz. (See the example 7.3 in the previous section.) Namely the inside homomorphism is the middle part between the contractions and pr of z and the immersion and inj of y.

For a generalized Brauer diagram z we denote by $v(z)$ the number of the vertical line (the edges from the upper row to the lower row) in z. Then the inside homomorphism is an element in $\mathbf{CP}^{\mathbf{v(z)}}_{\mathbf{v(y)}}$ and the vertical lines of the corresponding element in $\mathbf{GB}^{\mathbf{v(z)}}_{\mathbf{v(y)}}$ are at most $\min(v(z), v(y))$. From Theorem 5.1, the final result consists of the elements with at most $\min(v(z), v(y))$ vertical lines in $\mathbf{GB_k}$.

Hence if we denote by $\mathfrak{J}_s$ the linear subspace spanned by all the generalized Brauer diagrams z with $v(z) \leqq s$, $\mathfrak{J}_s$ becomes a two sided ideal of $\mathbf{CP_k}$. Also for a fixed index set $\mathtt{T} \subseteq [k]$, the linear subspace spanned by all the generalized Brauer diagrams z in which no vertical lines start from the $\mathtt{T}$ in the upper row becomes a left ideal and we denote this left ideal by $L_{\mathtt{T}}$. Similarly the linear subspace spanned by all the generalized Brauer diagrams z in which no vertical lines end with the $\mathtt{T}$ in the lower row becomes a right ideal and we denote this right ideal by $R_{\mathtt{T}}$.

At present we don't know what are the factor algebras of the chains of the two sided ideal $\mathfrak{J}_s$. Only we can say is that the top factor is isomorphic to the group algebra $\mathbb{R}[\mathfrak{S}_k]$. Here $\mathbb{R}[\mathfrak{S}_k]$ is the subalgebra of $\mathbf{CP_k}$ consisting of the diagrams with k vertical lines, since $\mathbf{CP_k}$ contains the ordinary Brauer centralizer algebra and the symmetric group of degree k in natural way. Namely we have

$$\mathbf{CP_k} = \mathbb{R}[\mathfrak{S}_k] \bigoplus \mathfrak{J}_{k-1}.$$

Let us define the subspace T^0_k of $\Delta \bigotimes \bigotimes^k V$.

Definition 8.1. *Let T^0_k denotes the intersection of all the kernels of the contractions $\mathrm{C}_{\{i,j\}}$ ($1 \leqq i < j \leqq k$) and the projections $\mathrm{pr}_{\{i_1,i_2,\ldots,i_r\}}$ ($r > 0$ and $1 \leqq i_1 < i_2 < \ldots < i_r \leqq k$).*

We note that any element of the two sided ideal $\mathfrak{J}_{k-1}$ acts on this space by 0 and T^0_k is a $\mathbf{CP_k} \times Pin(2n)$ subspace. (Actually it becomes an $\mathfrak{S}_k \times Pin(2n)$ module.) Then we have the following theorem.

Theorem 8.2. *Let us assume $n \geqq k$. Then the symmetric group of degree k and $Pin(2n)$ becomes a dual pair on the subspace T^0_k. Namely T^0_k is decomposed into the direct sum of the tensor products of the irreducible representations with multiplicity free as follows.*

$$T^0_k = \sum_{\lambda:partitions\ of\ size\ k} \lambda_{\mathfrak{S}_k} \bigotimes [\Delta, \lambda]_{Spin(2n)}. \tag{8.2.1}$$

Next we move to the case $k > n$.

Let us define the subspace $T^0_{k,s}$ $(s = 1, 2, \ldots, n)$ of $\Delta \bigotimes \bigotimes^k V$. We note that $\mathbf{CP_k}$ contains the groups algebra $\mathbb{R}[\mathfrak{S}_k]$.

Definition 8.3. *Let $T^0_{k,s}$ denote the intersection of all the kernels of the contractions* $C_{\{i,j\}}$ ($1 \leqq i < j \leqq k$) *and the projections* $\mathrm{pr}_{\{i_1,i_2,\ldots,i_r\}}$ ($r > 0$ *and* $1 \leqq i_1 < i_2 < \ldots < i_r \leqq k$) *and the alternating operators* $\mathrm{Alt}_{\{i_1,i_2,\ldots,i_r\}}$ ($r > s$) *of degree greater than s.*

Then the space $T^0_{k,s}$ become a $\mathbf{CP_k} \times Pin(2n)$ module. For, from Theorem 5.1 and Theorem 5.5, on this space $T^0_{k,s}$, the homomorphism $\psi^{\mathrm{T_u}}_{\mathrm{T_\ell}}$ corresponding to the isolated points in a generalized Brauer diagrams is 0 if $|T_u| > |T_\ell|$. If $|T_u| < |T_\ell|$, then contractions must appear in the upper row, since the upper and the lower row consist of the same number of dots. Therefore only in the case that the number of the isolated points in the upper row is equal to that in the lower row, ψ acts on this space non-trivially and in that case, as a homomorphism, ψ is the composition of the alternation and the immersion up to a scalar. From the definition, $T^0_{k,s}$ is stable under the action of $\mathfrak{S}_k$.

Since $\mathbf{CP_k}$ is spanned linearly by the generalized Brauer diagrams under the invariant theoretic parameterization, the space $T^0_{k,s}$ is invariant under the action of $\mathbf{CP_k}$. Only the diagrams under the invariant theoretic parameterization with the same number of the isolated points in their upper and lower rows and without contractions in their upper rows act on this space non-trivially.

We have the following theorem.

Theorem 8.4. *Let us assume $k > n$. Then the symmetric group of degree k and $Pin(2n)$ becomes a dual pair on the subspace $T^0_{k,s}$. Exactly speaking, $T^0_{k,s}$ is decomposed into the direct sum of the tensor products of the irreducible representations with multiplicity free as follows.*

$$T^0_{k,s} = \sum_{\substack{\lambda:\text{partitions of size } k \\ \ell(\lambda) \leqq s}} \lambda_{\mathfrak{S}_k} \bigotimes [\Delta, \lambda]_{Spin(2n)}. \tag{8.4.1}$$

We move to the case for $Spin(2n)$.

For $Spin(2n)$ representations, we consider the endomorphism $(A \otimes \mathrm{id}) \in \mathrm{End}(\Delta \bigotimes \bigotimes^k V)$. Here A is the associater for Δ and is given by the degree operator $A[i_1, i_2, i_3, \ldots, i_r] = (-1)^r [i_1, i_2, i_3, \ldots, i_r]$.

Lemma 8.5. *The endomorphism $A \otimes \mathrm{id}$ commutes with both of the actions of $\mathbf{CP_k}$ and $Spin(2n)$ on the space $\Delta \bigotimes \bigotimes^k V$.*

Moreover we have

$$\mathrm{pr}_p \circ (A \otimes \mathrm{id}) = (-1)^p A \circ \mathrm{pr}_p$$

and

$$\mathrm{inj}_q \circ A = (-1)^q (A \otimes \mathrm{id})\, \mathrm{inj}_q\,.$$

Then we first decompose the space $\Delta \bigotimes \bigotimes^k V$ into the irreducible constituents under the action of $Pin(2n) \times \mathbf{CP_k}$. Since $Pin(2n)$ and

$\mathbf{CP_k}$ are semisimple and act on the space as a dual pair, from the Wedderburn's theorem, we have

$$\Delta \bigotimes \overset{k}{\bigotimes} V = \bigoplus_{\substack{\ell(\lambda) \leqq n \\ |\lambda| \leqq k}} [\Delta, \lambda]_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}.$$

Here $\lambda_{\mathbf{CP_k}}$ is the irreducible representation of $\mathbf{CP_k}$. We consider the subspace $[\Delta, \lambda]_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}$ as $Spin(2n) \times \mathbf{CP_k}$ module. Then we have

$$[\Delta, \lambda]_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}} = (1/2 + \lambda)^{+}_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}} \bigoplus (1/2 + \delta)^{-}_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}.$$

Since $A \otimes \mathrm{id}$ commutes with the action of $\mathbf{CP_k}$ and the space $[\Delta, \lambda]_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}$ is the homogeneous component of the irreducible representation $\lambda_{\mathbf{CP_k}}$ of $\mathbf{CP_k}$, we have

$$(A \otimes \mathrm{id})([\Delta, \lambda] \bigotimes \lambda_{\mathbf{CP_k}}) = [\Delta, \lambda] \bigotimes \lambda_{\mathbf{CP_k}}.$$

Since $(A \otimes \mathrm{id})^2 = \mathrm{id}$, ± 1 eigenspaces of $A \otimes \mathrm{id}$ in $[\Delta, \lambda] \bigotimes \lambda_{\mathbf{CP_k}}$ become $Spin(2n) \times \mathbf{CP_k}$ modules.

Here $\Delta^{+} \bigotimes \bigotimes^{k} V$ is $+1$ eigenspace and $\Delta^{-} \bigotimes \bigotimes^{k} V$ is -1 eigenspace of $A \otimes \mathrm{id}$ in $\Delta \bigotimes \bigotimes^{k} V$.

From Theorem 2.1, $(1/2 + \lambda)^{\pm}_{Spin(2n)}$ appears in $\Delta^{\pm} \bigotimes \bigotimes^{k} V$ if and only if $k - |\lambda| \equiv 0 \mod 2$. Also $(1/2 + \lambda)^{\mp}_{Spin(2n)}$ appears in $\Delta^{\pm} \bigotimes \bigotimes^{k} V$ if and only if $k - |\lambda| \equiv 1 \mod 2$.

So $(-1)^{k-|\lambda|}$ eigenspace of $(A \otimes \mathrm{id})$ in $[\Delta, \lambda] \bigotimes \lambda_{\mathbf{CP_k}}$ is the irreducible representation $(1/2 + \lambda)^{+}_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}$ and $(-1)^{k-|\lambda|+1}$ eigenspace of $(A \otimes \mathrm{id})$ in $[\Delta, \lambda] \bigotimes \lambda_{\mathbf{CP_k}}$ is the irreducible representation $(1/2+\lambda)^{-}_{Spin(2n)} \bigotimes \lambda_{\mathbf{CP_k}}$.

We define the extension of the algebra $\mathbf{CP_k}$ by $(A \otimes \mathrm{id})$.

Definition 8.6. *Let* $\mathbf{CS_k}$ *be the subalgebra of* $\mathrm{Hom}(\Delta \bigotimes \bigotimes^{k} V, \Delta \bigotimes \bigotimes^{k} V)$ *generated by* $\mathbf{CP_k}$ *and* $(A \otimes \mathrm{id})$.

Then we have the following theorem,

Theorem 8.7. $\mathbf{CS_k}$ *and* $Spin(2n)$ *act on the space* $\Delta \bigotimes \bigotimes^{k} V$ *as a dual pair. Namely the space* $\Delta \bigotimes \bigotimes^{k} V$ *is decomposed into a direct sum of the irreducible modules of* $\mathbf{CS_k} \times Spin(2n)$ *with multiplicity free.*

Then Theorem 8.2 for $Spin(2n)$ is given as follows. Since we have Lemma 8.5, T^0_k in Definition 8.1 is $(A \otimes \mathrm{id})$ stable and We decompose T^0_k into the ± 1 eigenspaces of $(A \otimes \mathrm{id})$ and denote them by $T^{0,\pm}_k$. Then from Theorem 8.2, if we note that $|\lambda| = k$ in this case, we have the following.

Theorem 8.8. *Let us assume* $n \geqq k$. *Then the symmetric group of degree* k *and* $Spin(2n)$ *becomes a dual pair on the subspaces* $T^{0,+}_k$ *and* $T^{0,-}_k$ *respectively. Namely* $T^{0,\pm}_k$ *are decomposed into the direct sum of* *the tensor products of the irreducible representations with multiplicity free as follows.*

(8.8.1) $$T_k^{0,+} = \sum_{\lambda:\text{partitions of size } k} \lambda_{\mathfrak{S}_k} \bigotimes (1/2+\lambda)^+_{Spin(2n)}$$

and

(8.8.1) $$T_k^{0,-} = \sum_{\lambda:\text{partitions of size } k} \lambda_{\mathfrak{S}_k} \bigotimes (1/2+\lambda)^-_{Spin(2n)}.$$

Theorem 8.4 for $Spin(2n)$ is given as follows.

We assume that $k > n$.

The subspace $T^0_{k,s}$ $(s = 1, 2, \dots, n)$ of $\Delta \bigotimes \bigotimes^k V$ in Definition 8.3 is also $(A \otimes \mathrm{id})$ stable. We decompose $T^0_{k,s}$ into the ± 1 eigenspaces of $(A \otimes \mathrm{id})$ and denote them by $T^{0,\pm}_{k,s}$.

From the same reason as before, we have the following theorem.

Theorem 8.9. *Let us assume $k > n$. Then the symmetric group of degree k and $Spin(2n)$ becomes a dual pair on the subspaces $T^{0,+}_{k,s}$ and $T^{0,-}_{k,s}$ respectively. Exactly speaking, $T^{0,\pm}_{k,s}$ are decomposed into the direct sum of the tensor products of the irreducible representations with multiplicity free as follows.*

(8.9.1) $$T_{k,s}^{0,+} = \sum_{\substack{\lambda:\text{partitions of size } k\\ \ell(\lambda)\leqq s}} \lambda_{\mathfrak{S}_k} \bigotimes (1/2+\lambda)^+_{Spin(2n)}.$$

(8.9.1) $$T_{k,s}^{0,-} = \sum_{\substack{\lambda:\text{partitions of size } k\\ \ell(\lambda)\leqq s}} \lambda_{\mathfrak{S}_k} \bigotimes (1/2+\lambda)^-_{Spin(2n)}.$$

REFERENCES

[1] J. Birman and H. Wenzl, Braids, link polynomials and a new algebra, *Trans. Amer. Math. Soc.* **313**, (1989) pp. 249–273

[2] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Annals of Mathematics* **38**, No 4, (1937) pp. 857–872.

[3] R. Brauer and H. Weyl, Spinors in n dimensions, *Amer. J. Math.* **57**, (1935) pp. 425–449

[4] K. Koike, Representations of Spinor Groups and the Difference Characters of SO(2n), *Adv. Math.* **128**, (1997) pp. 40–81.

[5] K. Koike, On Representation of the Classical Groups, *Amer. Math. Soc. Trans.* **183**, (1998) pp. 79–100.

[6] K. Koike, Spin representations and centralizer algebras for $\mathrm{Spin}(2n+1)$, *preprint*

[7] K. Koike, Spin representations and centralizer algebras for $\mathrm{Spin}(2n)$, *preprint*

[8] K. Koike and I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n, *J. Algebra* **107**, (1987) pp. 466–511.

[9] H. Weyl, The Classical Groups, their Invariants and Representations, *2nd edition, Princeton Univ. Press*, Princeton, N.J., (1946)