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Monte Carlo Method for pricing of Bermuda type derivatives

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ be a filtered space with the usual condition, and $\{B_t\}_{t\in[0,\infty)}$ be a d-dimensinal Brownian motion. Let $T > 0$, and let $\sigma : [0, T] \times \mathbf{R}^D \to \mathbf{R}^D \times \mathbf{R}^d$ and $b : [0, T] \times \mathbf{R}^D \to \mathbf{R}^D$ be continuous functions. For each $s \in [0, T]$ and $x \in \mathbf{R}^D$, let $X(t; s, x)$, $t \in [s, T]$ be a solution of the following SDE.

$$X(t; s, x) = x + \int_s^t \sigma(r, X(r; s, x))dB_r + \int_s^t b(r, X(r; t, x))dr, \qquad t \in [s, T]. \quad (1)$$

We assume that the above SDE 1 has a path-wise unique solution for every $(s, x) \in [0, T] \times \mathbf{R}^D$.

Let $\tilde{\mathcal{S}}_s^t$, $0 \leq s \leq t \leq T$, be the set of $\mathcal{F}_t$-stopping times τ with $s \leq \tau \leq t$. Let $g : [0, T] \times \mathbf{R}^D \to \mathbf{R}$ be a continuous function with suitable conditions. Then, concerning the pricing of American derivatives, we are interested in computing the following value function,

$$u(s, x) = \sup\{E[g(\tau, X(\tau; s, x))]; \tau \in \tilde{\mathcal{S}}_s^T\}, (s, x) \in [0, T] \times \mathbf{R}^D.$$

There are several attempts to compute the value function u numerically. However, it seems that there are not so good method if D is not small. Let $N \geq 2$ and let T_n, $n = 0, 1, \ldots, N$, be positive numbers such that $0 = T_0 < T_1 < \ldots < T_N = T$. Let $\mathcal{S}_n$, $n = 0, 1, \ldots, N$, be the set of $\mathcal{F}_t$-stopping times taking value in $\{T_n, T_{n+1}, \ldots, T_N\}$. Concerning the pricing of Bermuda type derivatives, we are interested in computing the following value functions.

$$v_n(x) = \sup\{E[g(\tau, X(\tau; s, x))]; \tau \in \mathcal{S}_n\}, \qquad n = 0, 1, \ldots, N.$$

Let us define a probability measure $p_n(x, \cdot)$ over $\mathbf{R}^D$ for each $n = 0, 1, \ldots, N-1$, and $x \in \mathbf{R}^D$ by

$$p_n(x, A) = P(X(T_{n+1}; T_n, x) \in A), \qquad \text{for a Borel set } A \text{ in } \mathbf{R}^D,$$

and define a operator P_n, $n = 0, 1, \ldots, N-1$, by

$$P_n f(x) = \int_{\mathbf{R}^D} f(y) p_n(x, dy) = E[f(X(T_{n+1}; T_n, x))]$$

for a measurable function f on $\mathbf{R}^D$. Then v_n, $n = N, N-1, \dots, 0$, are given inductively by the following.

$$v_N(x) = g(T_N, x),$$
$$v_{n-1}(x) = (P_{n-1}v_n)(x) \vee g(T_{n-1}, x).$$

So the value function $v_0(x)$ is easily given mathematically. However, if D is not small, it is not easy to memorize a function on $\mathbf{R}^D$, and so it is not easy to compute $v_0(x)$.

Several people suggest a Monte-Carlo method to compute the value function. In this paper, we discuss the method given by [?]. We assume the following assumption (A).
(A1) D_n, $n = 0, 1, \dots, N-1$, are measurable sets in $\mathbf{R}^N$ such that $(P_n v_{n+1})(x) \geq g(T_n, x)$ for any $x \in \mathbf{R}^D \setminus D_n$.

Remark 1 (1) $D_n = \mathbf{R}^D$ *satisfies the assumption* (A1).
(2) *If* $g(t, x) \geq 0$, *for any* $(t, x) \in [0, T] \times \mathbf{R}$, *then* $D_n = \{x \in \mathbf{R}^D; g(T_n, x) > 0\}$ *satisfies the assumption* (A1).

Now let $L_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\vec{X}_{n,\ell} = \{X_{n,\ell}(m)\}_{m=0}^N$, $\ell = 1, \dots, L_n$, $n = 0, 1, \dots, N-1$, are identically independent random vectors whose distribution is the same as the distribution of $\{X(T_m; 0, x)\}_{m=0}^N$. Let $K_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\psi_{n,k}$, $k = 1, \dots, K_n$, $n = 0, 1, \dots, N-1$, are functions on $\mathbf{R}^D$. Then we define functions H_n, $n = N, N-1, \dots, 1, 0$, on $\mathbf{R}^D$ inductively by the following.

$$H_N(x) = 1.$$

When $\vec{H}_{n+1} = \{H_m\}_{m=n+1}^N$, are given we let

$$\sigma_{n,\ell} = \min\{m \geq n+1; H_m(X_{n,\ell}(m)) > 0\}, \qquad \ell = 1, \dots L_n.$$

Then we let $\{\tilde{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{\ell=1}^{L_n} |g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n,\ell}(n))|^2 1_{D_n}(X_{n,\ell}(n)).$$

Finally we define H_n by

$$H_n(x) = \begin{cases} g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbf{R}^D \setminus D_n. \end{cases}$$

Then we let

$$\tilde{v}_0 = \frac{1}{L_0} \sum_{\ell=1}^{L_0} g(\sigma_{0,\ell}, X_{0,\ell}(\sigma_{0,\ell}))$$

and

$$\tilde{\sigma} = \min\{T_n; H_n(X(T_n; 0, x)) > 0\}.$$

We think that $\tilde{v}_0$ is an approximation of the value function $v_0(x)$ and the stopping time $\tilde{\sigma}$ as a candidate of the optimal stopping time.

2 Preliminary Results

Let $W_n = \mathbf{R}^{(N+1-n)D}$, $n = 0, 1, \ldots, N$, and let $P_x^{(n)}$, $x \in \mathbf{R}^D$, be the distribution of $\{X(T_m; T_n, x)\}_{m=n}^{N}$ on W_n. Then $P_x^{(n)}$, $n = 0, 1, \ldots, N$, $x \in \mathbf{R}^D$, is a Markov chain on $\mathbf{R}^D$.

For any measurable function h on $\mathbf{R}^D$ and $n, m = 0, 1, \ldots, N$ with $n \leq m$, let $\tau_m(\cdot; h)$: $W_n \to \{m, N\}$ by

$$\tau_m(w; h) = \begin{cases} m, & h(w(m)) > 0, \\ N, & h(w(m)) \leq 0. \end{cases}$$

Lemma 2 *Let $h_n : \mathbf{R}^N \to \mathbf{R}$, $n = 0, 1, \ldots, N$, be given, and assume that $h_n(x) \leq 0$, $x \in \mathbf{R}^N \setminus D_n$, and that $h_N(x) = 1$. Let $\sigma_n : W_n \to \{n, n+1, \ldots, N\}$ be given by*

$$\sigma_n(w) = \sigma_n(w; \{h_m\}_{m=n}^{N-1}) = \bigwedge_{m=n}^{N-1} \tau_m(w; h_m), \qquad w \in W_n.$$

Moreover, let $u_n : \mathbf{R}^D \to \mathbf{R}$ be given by

$$u_n(x) = u_n(x; \{h_m\}_{m=n}^{N}) = E^{P_x^{(n)}}[g(T_{\sigma_n}, w(\sigma_n))], \qquad x \in \mathbf{R}^D,$$

Then we have the following.
(1) $|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|P_n u_{n+1}(x) - (g(T_n, x) - h_n(x))|$ *for any* $n = 0, 1, \ldots, N-1$, *and* $x \in \mathbf{R}^D$.
(2) $|u_n(x) - v_n(x)|$

$$\leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x) 1_{\{1\}}(sgn(P_n u_{n+1}(x) - g(T_n, x)) sgn(h_n(x))) |P_n u_{n+1}(x) - g(n, x)|.$$

Here

$$sgn(a) = \begin{cases} 1, & a > 0, \\ 0, & a = 0, \\ -1, & a < 0. \end{cases}$$

Proof. Note that $u_n(x) \leq v_n(x)$, for all $n = 0, 1, \ldots, N-1$, and $x \in \mathbf{R}^D$. Let $\tilde{u}_n(x) = g(T_n, x) - h_n(x)$, $x \in \mathbf{R}^D$.

Let $n = 0, 1, \ldots, N-1$, and $x \in \mathbf{R}^D$, and fix them for a while.
Case 1. Suppose that $h_n(x) > 0$.
Then we see that $x \in D_n$ and $g(T_n, x) > \tilde{u}_n(x)$. So we have

$$v_n(x) = g(T_n, x) + (P_n v_{n+1}(x) - g(T_n, x)) \vee 0 \leq g(T_n, x) + |P_n v_{n+1}(x) - \tilde{u}_n(x)|.$$

This implies

$$g(T_n, x) \geq v_n(x) - |P_n(v_{n+1} - u_{n+1})(x)| - |P_n u_{n+1}(x) - \tilde{u}_n(x)|.$$

Case 2. Suppose that $h_n(x) \leq 0$, and $x \in D_n$.
Then we see that $g(T_n, x) \leq \tilde{u}_n(x)$. So we see that

$$v_n(x) \leq P_n v_{n+1}(x) \vee \tilde{u}_n(x) \leq P_n u_{n+1}(x) + |P_n(v_{n+1} - u_{n+1})(x)| + |P_n u_{n+1}(x) - \tilde{u}_n(x)|.$$

Case 3. Suppose that $h_n(x) \le 0$, and $x \in \mathbf{R}^D \setminus D_n$.
Then we see that $g(T_n, x) \le (P_n v_{n+1})(x)$. So we have

$$v_n(x) = P_n v_{n+1}(x) \le P_n u_{n+1}(x) + |P_n(u_{n+1} - v_{n+1})(x)|.$$

So we see thatfor any $n = 0, 1, \dots, N-1$,

$$\begin{aligned} u_n &= 1_{\{h_n>0\}} g(T_n, \cdot) + 1_{\{h_n \le 0\}}(P_n u_{n+1}) \\ &\ge 1_{\{h_n>0\}}(v_n - |P_n(v_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|) \\ &\quad + 1_{\{h_n \le 0\}} 1_{D_n}(v_n - |P_n(v_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|) \\ &\quad\quad + 1_{\{h_n \le 0\}} 1_{\mathbf{R}^D \setminus D_n}(v_n - |P_n(v_{n+1} - u_{n+1})|). \end{aligned}$$

Thus we see that

$$0 \le v_n - u_n \le |P_n(v_{n+1} - u_{n+1})| + |1_{D_n} P_n u_{n+1} - \tilde{u}_n|.$$

This implies the assertion (1).

Now let us prove the assetion (2). Let ξ is a positive measurable function on $\mathbf{R}^D$. Since $\tau_n(w; \xi h_n) = \tau_n(w; h_n)$, we see from the assertion (1) that

$$|u_n(x) - v_n(x)| \le |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|P_n u_{n+1}(x) - g(T_n, x) + \xi(x) h_n(x)|.$$

Noting that

$$\inf\{a + tb;\ t > 0\} = 1_{\{1\}}(sgn(a)sgn(b))|a|, \qquad a, b \in \mathbf{R},$$

we have the assertion (2).

This completes the proof. ∎

Let ν_0 be a probability measure on $\mathbf{R}^D$ and define probability measures ν_n, $n = 1, \dots, N$, inductively by

$$\nu_{n+1}(dx) = \int_{\mathbf{R}^D} p_n(y; dx) \nu_n(dy), \qquad n = 0, 1, \dots, N-1.$$

Then we have the following as an easy consequence of Lemma 2.

Corollary 3 *Let h_n and u_n be the same as the previous lemma. Then we have the following.*

$$\begin{aligned} &(\int_{\mathbf{R}^D} |u_n(x) - v_n(x)|^2 \nu_n(dx))^{1/2} \\ \le\, &|(\int_{\mathbf{R}^D} |u_{n+1} - v_{n+1}(x)|^2 \nu_{n+1}(dx))^{1/2} + \int_{D_n} |P_n u_{n+1}(x) - (g(T_n, x) - h_n(x))|^2)^{1/2} \end{aligned}$$

for any $n = 0, 1, \dots, N-1$.

3 Main Result

Let ν_0 be a probability measure over $\mathbf{R}^D$. Let $L_n \geq 1$, $n = 0,1,\dots,N-1$, and $\vec{X}_{n,\ell} = \{X_{n,\ell}(m)\}_{m=0}^N$, $\ell = 1,\dots,L_n$, $n = 0,1,\dots,N-1$, are identically independent random vectors defined on the probability measure $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ whose distribution is $P_{\nu_0}^{(0)} = \int_{\mathbf{R}^D} P_x^{(0)}\nu_0(dx)$. Let $K_n \geq 1$, $n = 0,1,\dots,N-1$, and $\psi_{n,k}$, $k = 1,\dots,K_n$, $n = 0,1,\dots,N-1$, are functions on $\mathbf{R}^D$.

Then we define functions $H_n : \mathbf{R}^D \times \tilde{\Omega} \to \mathbf{R}$, $n = N, N-1,\dots,1,0$, on $\mathbf{R}^D$ inductively by the following procedure.

$$H_N(x) = 1.$$

When $\vec{H}_{n+1} = \{H_m\}_{m=n+1}^N$, are given we let

$$\sigma_{n,\ell} = \min\{m \geq n+1; H_m(X_{n,\ell}(m)) > 0\}, \qquad \ell = 1,\dots L_n.$$

Then we let $\tilde{a}_n = \{\tilde{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n}\sum_{\ell=1}^{L_n} |g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n\psi_{n,k}(X_{n,\ell}(n))|^2 1_{D_n}(X_{n,\ell}(n)).$$

Finally we define H_n by

$$H_n(x) = \begin{cases} g(T_n,x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k}\psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbf{R}^D \setminus D_n. \end{cases}$$

Let $U_n(x) = u_n(\cdot;\{H_m\}_{m=n}^N))(x)$. Here u_n is as in Lemma 2. Let $\bar{a}_n = \{\bar{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$\bar{F}_n(\{a_k\}_{k=1}^{K_n}) = \int_{D_n} |(P_nU_{n+1})(x) - \sum_{k=1}^{K_n} a_n\psi_{n,k}(x))|^2\nu_n(dx).$$

We assume the following.

(A2) $\psi_{n,k}$, $k = 1,\dots,K_n$, is linear ly independent in $L^2(D_n;d\nu_n)$, $n = 0,1,\dots,N-1$, where ν_n is the probability law of $w(n)$ under $P_{\nu_0}^{(0)}(dw)$.

(A3) $\int_{D_n} \psi_{n,k}(x)^4\nu_n(dx) < \infty$ $k = 1,\dots,K_n$, $n = 0,1,\dots,N-1$. and

$$\int_{\mathbf{R}^D} E^{P_x^{(0)}}[(\sum_{m=1}^N g(T_n,w(T_n))^4]\nu_0(dx) < \infty, \qquad n = 0,1,\dots,N..$$

Then we have the following.

Theorem 4 (1) *There is a constant $C > 0$ such that*

$$E^{\tilde{P}}[1 \wedge (\sum_{k=1}^{K_n} |\tilde{a}_{n,k} - \bar{a}_{n,k}|^2)] \leq \frac{C}{L_n}.$$

(2) $(\int_{\mathbf{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx))^{1/2}$

$$\leq (\int_{\mathbf{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx))^{1/2} + (\int_{\mathbf{R}^D} (\sum_{k=1}^{K_n} (\tilde{a}_{n,k} - \bar{a}_{n,k})\psi_{n,k}(x))^2 \nu_n(dx))^{1/2}$$

$$+ \inf\{(\int_{D_n} |(P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_k \psi_{n,k}(x)|^2 \nu_n(dx))^{1/2};\ a_k \in \mathbf{R},\ k = 1, \dots, K_n\}$$

Proof. Let $\mathcal{I}_n$, $n = 0, 1, \dots, N-1$, be the σ-algebra generated by $\vec{X}_{n,\ell}$, $\ell = 1, \dots, L_n$, and let $\mathcal{B}_n$, $n = 0, 1, \dots, N-1$, be the σ-algebra generated by $\cup_{m=n}^{N-1} \mathcal{I}_m$. Inductively, we see that H_n is $\mathcal{B}_n$-measurable, $n = N-1, N-2, \dots, 0$. Also, we have

$$F_n(\{a_k\}_{k=1}^{K_n}) = \sum_{k,k'=1}^{K_n} C_{n,k,k'}^{(2)} a_k a_{k'} - 2\sum_{k=1}^{K_n} c_{n,k}^{(1)} a_k + C_n^{(0)},$$

where

$$C_{n,k,k'}^{(2)} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k} \psi_{n,k'})(X_{n,\ell})$$

$$c_{n,k}^{(1)} = \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k})(X_{n,\ell}) g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})).$$

Note that $\sigma_{n,\ell} = \sigma_{n+1}(\vec{X}_{n,\ell}(\cdot); \{H_m\}_{m=n+1}^N)$. Therefore we have

$$\bar{C}_{n,k,k'}^{(2)} = E^{\tilde{P}}[C_{n,k,k'}^{(2)} | \mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x)\psi_{n,k'}(x)\nu_n(dx), \tag{2}$$

and

$$\bar{c}_{n,k}^{(1)} = E^{\tilde{P}}[c_{n,k}^{(1)} | \mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x)(P_n U_{n+1})(x)\nu_n(dx). \tag{3}$$

Let $R_{n,k,k'}^{(2)} = C_{n,k,k'}^{(2)} - \bar{C}_{n,k,k'}^{(2)}$, and $r_{n,k}^{(1)} = c_{n,k}^{(1)} - \bar{r}_{n,k}^{(1)}$. Let $C_n^{(2)} = \{C_{n,k,k'}^{(2)}\}_{k,k'=1}^D$, $\bar{C}_n^{(2)} = \{\bar{C}_{n,k,k'}^{(2)}\}_{k,k'=1}^D$, and $R_n^{(2)} = \{R_{n,k,k'}^{(2)}\}_{k,k'=1}^D$ be $D \times D$ random matrices, and let Let $c_n^{(1)} = \{c_{n,k}^{(1)}\}_{k=1}^D$, $\bar{c}_n^{(1)} = \{\bar{c}_{n,k}^{(1)}\}_{k=1}^D$, and $r_n^{(1)} = \{r_{n,k}^{(1)}\}_{k=1}^D$, be D-dimesional random vectors. Then we see that

$$\tilde{a}_n = C_n^{(2)-1} c_n^{(1)}, \qquad \bar{a}_n = \bar{C}_n^{(2)-1} \bar{c}_n^{(1)}., \qquad n = 0, \dots, N-1.$$

Also, we see that

$$E^{\tilde{P}}[(R_{n,k,k'}^{(2)})^2]$$

$$= \frac{1}{L_n} E[Var[1_{D_n}(X_{n,1}(n))\psi_{n,k}(X_{n,1}(n))\psi_{n,k'}(X_{n,1}(n)) | \mathcal{B}_{n+1}]]$$

$$\leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 \psi_{n,k'}(x)^2 \nu_n(dx)$$

$$E^{\tilde P}[(r^{(1)}_{n,k})^2] = \frac{1}{L_n}E[Var[1_{D_n}(X_{n,1}(n))\psi_{n,k}(X_{n,1}(n))g(\sigma_{n,1}, X_{n,1}(\sigma_{n,1}))|\mathcal{B}_{n+1}]]$$

$$\le \frac{1}{L_n}\int_{D_n}\psi_{n,k}(x)^2 E^{P^{(n)}_x}[g(T_{\sigma_{n+1}(w;\{H_m\}^N_{m=n+1})})w(\sigma_{n+1}(w;\{H_m\}^N_{m=n+1})))^2]\nu_n(dx).$$

If $\| \bar C^{(2)-1}_n R^{(2)}_n \| \le 1/2$, we have

$$\| (\bar C^{(2)}_n + R^{(2)}_n)^{-1}\bar C^{(2)-1}_n \| = \| ((I + \bar C^{(2)-1}_n R^{(2)}_n)^{-1} - I)\bar C^{(2)-1}_n \| \le 2 \| \bar C^{(2)-1}_n \| \| R^{(2)}_n \| .$$

Here $\| \cdot \|$ is the operator norm of a matrix. So if $\| \bar C^{(2)-1}_n \| \| R^{(2)}_n \| \le 1/2$ and $|\bar c^{(1)}_n| \le 1$, we have

$$\begin{aligned}|\tilde a_n - \bar a_n| &= |((\bar C^{(2)}_n + R^{(2)}_n)^{-1} - \bar C^{(2)-1}_n)(\bar c^{(1)}_n + r^{(1)}_n) + \bar C^{(2)-1}_n r^{(1)}_n| \\ &\le 2 \| \bar C^{(2)-1}_n \|^2 \| R^{(2)}_n \| (|\bar c^{(1)}_n| + 1) \| \bar C^{(2)-1}_n \| |r^{(1)}_n|\end{aligned}$$

So we have

$$\begin{aligned}&E^{\tilde P}[|\tilde a_n - \bar a_n|^2 \wedge 1] \\ &\le E^{\tilde P}[|\tilde a_n - \bar a_n|^2, \| \bar C^{(2)-1}_n \| \| R^{(2)}_n \| \le 1/2, |\bar c^{(1)}_n| \le 1] \\ &\quad + \tilde P(\| \bar C^{(2)-1}_n \| \| R^{(2)}_n \| > 1/2) + \tilde P(|\bar c^{(1)}_n| > 1)\end{aligned}$$

$$\le (4(|\bar c^{(1)}_n| + 1)^2 \| \bar C^{(2)-1}_n \|^4 + 4 \| \bar C^{(2)-1}_n \|^{-2})E^{\tilde P}[\| R^{(2)}_n \|^2] + (\| \bar C^{(2)-1}_n \|^2 + 1)E^{\tilde P}[|r^{(1)}_n|^2].$$

Also we have

$$|\bar c^{(1)}_n| \le (\int_{D_n}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)\nu_n(dx))^{1/2}(\int_{\mathbf{R}^D} E^{P^{(0)}_x}[(\sum_{m=1}^N g(T_m, w(T_m))^2]\nu_0(dx))^{1/2},$$

$$E^{\tilde P}[\| R^{(2)}_n \|^2] \le \frac{1}{L_n}\int_{D_n}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)^2\nu_n(dx),$$

and

$$E^{\tilde P}[|r^{(1)}_n|^2] \le \frac{1}{L_n}(\int_{D_n}(\sum_{k=1}^{K_n}\psi_{n,k}(x)^2)^2\nu_n(dx))^{1/2}(\int_{\mathbf{R}^D} E^{P^{(0)}_x}[(\sum_{m=1}^N g(T_m, w(T_m))^4]\nu_0(dx))^{1/2}.$$

This implies the assertion (1).

The assertion (2) is an easy consequence of Lemma 2. ∎

Let $V_n = \sum_{k=1}^{K_n}\mathbf{R}\psi_{n,k} \subset L^2(\mathbf{R}^D; d\nu_n)$, $n = 0, 1, \dots, N-1$. Then it is easy to see that U_n's are determined by $\vec X_{n,\ell}$, $\ell = 1, \dots, L_n$, $n = 0, \dots, N$ and V_n's and are independent of a choice of bases $\{\psi_{n,k}\}_{k=1}^{K_n}$. Let

$$d_n = \inf\{(\int_{\mathbf{R}^D}(\sum\psi_k(x)^2)^2\nu_n(dx))^{1/2}; \{\psi_k\}_{k=1}^{K_n} \text{ is a orhogonal basis of } V_n\},$$

and

$$c_0 = (\sum\int_{\mathbf{R}^D} E^{P^{(0)}_x}[(\sum_{m=1}^N g(T_m, w(T_m))^4]\nu_0(dx))^{1/4}.$$

Then we have the following from the proof of Theorem 4.

Corollary 5 $E[(\int_{\mathbf{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx)) \wedge 1]^{1/2}$

$$\leq E[(\int_{\mathbf{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx)) \wedge 1]^{1/2} + 4(L_n)^{-1/2} d_n (K_n^{1/2} c_0^{1/2} + 1)$$

$$+ E[\inf\{(\int_{D_n} |(P_n U_{n+1})(x) - \psi(x)|^2 \nu_n(dx))\ \psi \in V_n\} \wedge 1]^{1/2}.$$

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