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Semigroup Approach to a Pair Formation Model in Human Demography

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1 Introduction

Since the well known work by Sharpe and Lotka [20], modern mathematical demography has been mainly developed based on the one-sex theory of stable populations. However from the beginning the one-sex theory has been confronted with fundamental difficulties. For example, to calculate the net reproduction rate (the basic reproduction number) R_0 in the one-sex theory, the development of the other sex is assumed to be consistent with the assumption of constant fertility and constant mortality for the sex under consideration. As was pointed out by Kuczynski, the male and female net reproduction rates do not necessarily coincide each other, and so they would provide contradictory predictions for the future trend of the population considered [16]. That is, we have not yet fully answered to the fundamental question what is the true measure for the reproductivity of human population. This is the origin of what we call *two-sex problem* in demography.

It would be clear that the two-sex problem is caused by neglecting *pair formation phenomena* in human reproduction, which is essentially nonlinear interaction between sexes. It would be Kendall [10] that first formulated the two-sex population model by using ODE system. Though the Kendall's model is simple and it neglected age structure, about 40 years have passed until its mathematical implications are fully cleared by Hadeler *et al.* [3] and Waldstätter [22]. Castillo-Chavez and Huang [1] extended the Kendall's pair formation model to take into account the logistic effect.

On the other hand, an age-structured pair formation model (monogamous marriage model) was first formulated by Fredrickson [2]. Next Staroverov [21] introduced more general age-duration dependent pair formation, which was first introduced by Hadeler [4] among Western researchers. In contrast with the ODE model without age structure, up to now, very little is known about the age-structured nonlinear two-sex models. However through 1980's, HIV/AIDS epidemic has stimulated many studies for sexually transmitted diseases, and revived people's concern about pair formation phenomena in human populations [8].

In the early studies, Keyfitz [11] tried to determine the mathematical form of marriage function empirically. Waldstätter [22] first proved global existence for solutions of the Staroverov's model. Martcheva and Milner [12] discussed well posedness for the Fredrickson model. Inaba [7] formulated an age-duration dependent model for two-sex population reproduced by first marriage, and proved its well-posedness and conditions for existence of persistent solutions. Prüss and Schappacher [17] have shown conditions on the vital rates which imply the existence or nonexistence of exponentially growing persistent age-distributions for the Staroverov's age-duration dependent two-sex model with the marriage function of harmonic mean type. Recently Inaba [9] proved the existence of persistent solution for the age-duration dependent pair formation model with general homogeneous marriage function. Zacher [27] have also provided the existence result for persistent solutions in the Staroverov model.

Since the pioneering work by Webb [23] and Metz and Diekmann [15], the semigroup approach to structured population models has been widely studied, and it has been proved that the semigroup setting could be very powerful to examine mathematical implications of population models. In this short note, we are mainly concerned with the Fredrickson's age-dependent pair formation model. Our main purpose is to develop a semigroup approach to the Fredrickson model, which would be more simple than the method by Iannelli and Martcheva [6], and to show an existence result for persistent solutions.

2 The Fredrickson's monogamous marriage model

In the following we consider a bisexual closed human population with monogamous marriage system. Let $p_m(t,a)$ ($p_f(t,b)$) be the density of single male (female) population aged a (b) at time t and let $p_c(t,a,b)$ be the density of couples of which husband is aged a and wife is aged b at time t. Let $\mu_m(a)$ and $\mu_f(b)$ be the male and female natural death rate, $\beta(a,b)$ and $\sigma(a,b)$ are the fertility rate and the divorce rate respectively for the the couple of which husband is aged a and wife is aged b. Let γ is the ratio of male newborns to the total newborns. Then the basic model, which is first formulated by Fredrickson [2], is formulated as follows:

$$\begin{cases} \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)p_m(t,a) = -\mu_m(a)p_m(t,a) + \int_0^\infty p_c(t,a,b)[\sigma(a,b)+\mu_f(b)]db - \int_0^\infty \rho(t,a,b)db \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial b}\right)p_f(t,b) = -\mu_f(b)p_f(t,b) + \int_0^\infty p_c(t,a,b)[\sigma(a,b)+\mu_m(a)]da - \int_0^\infty \rho(t,a,b)da \\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right)p_c(t,a,b) = -(\sigma(a,b)+\mu_m(a)+\mu_f(b))p_c(t,a,b)+\rho(t,a,b) \\ p_m(t,0) = \gamma\int_0^\infty\int_0^\infty \beta(a,b)p_c(t,a,b)dadb, \\ p_f(t,0) = (1-\gamma)\int_0^\infty\int_0^\infty \beta(a,b)p_c(t,a,b)dadb, \\ p_c(t,0,b) = p_c(t,a,0) = 0. \end{cases} \tag{2.1}$$

The source term $\rho(t,a,b)$ denotes the number of new born couples of male aged a and female aged b per unit time, which is given by a nonlinear operator Ψ, which is called *marriage function* in demographic terminology.

$$\rho(t,a,b) = \Psi(p_m(t,\cdot),p_f(t,\cdot))(a,b).$$

From its biological meaning, we suppose that the marriage function satisfies the following basic axioms:

1. $\Psi(u,v)\geq 0$ if $(u,v)\geq 0$
2. $\Psi(u,0)=\Psi(0,v)=0$,
3. $\Psi(u,v)\leq\Psi(u',v')$ if $(u,v)\leq(u',v')$,
4. $\Psi(ku,kv)=k\Psi(u,v)$ for $k>0$

Though axiom (1)-(3) are self-evident, axiom (4), called *homogeneity condition*, reflects the saturation effect in a large population, that is, individuals have only a limited number of social contacts with other individuals. Hence the homogeneity condition is not necessarily essential to the marriage function, sometime it could be replaced by other condition (see [1]).

Furthermore in order to apply semigroup approach, we adopt the following technical assumption:

Assumption 2. 1 *(1) $\mu_m,\mu_f\in L^\infty(0,\infty)$, $\beta,\sigma\in L^\infty(\Omega)$, and there is a number $\omega>0$ such that $\beta(a,b)=0$ for $a>\omega$ or $b>\omega$. That is, ω is an upper bound of reproductive age.*

(2) The marriage function Ψ is a bounded operator from $Y_+ := L^1_+ \times L^1_+$ to $L^1_+(\Omega)$, and it is locally Lipschitz continuous in Y_+ with L^1 norm; there exists an increasing function $L(r)$ such that $\|\Psi(f) - \Psi(g)\|_{L^1} \le L(r)\|f-g\|_Y$ for all $f, g \in \{f \in Y_+ : \|f\|_Y \le r\}$.

(3) There exists a number $\eta > 0$ such that for $(u,v) \in Y_+$

$$\int_0^\infty \Psi(u,v)(a,b)db \le \eta u(a), \quad \int_0^\infty \Psi(u,v)(a,b)da \le \eta v(b).$$

A simple example of marriage function which satisfies Assumption 3.5 is a proportional mixing function (see [22], Chap. 3) given by

$$\Psi(u,v)(a,b) = \frac{\theta(a,b)u(a)v(b)}{\int_0^\infty u(a)da + \int_0^\infty v(b)db}.$$

where $\theta(a,b) \in L^\infty(a,b)$ reflects the average number of social contacts and the age preference between sexes. If we assume that the random mating occurs in the total population, the proportionate mixing function could be replaced by the following marriage functions:

$$\Psi(u,v)(a,b) = \frac{\theta(a,b)u(a)v(b)}{\int_0^\infty u(a)da + \int_0^\infty v(b)db + \int_0^\infty \int_0^\infty w(a,b)dadb},$$

where $w(a,b)$ denotes the density of couples. Moreover if we neglect the axiom of competition between age classes, a simple harmonic mean function could be used as a marriage function:

$$\Psi(u,v)(a,b) = \frac{\theta(a,b)u(a)v(b)}{u(a)+v(b)}.$$

In the following, we adopt the notations as $\bar{\mu} := \text{ess.}\sup(\mu_m(a), \mu_f(b))$, $\underline{\mu} := \text{ess.}\inf(\mu_m(a), \mu_f(b))$, $\bar{\beta} := \text{ess.}\sup \beta(a,b)$, $\bar{\sigma} := \text{ess.}\sup \sigma(a,b)$ and $\underline{\sigma} := \text{ess.}\inf \sigma(a,b)$.

3 Population semigroup

Let us consider a real Banach lattice $X = L^1(0,\infty) \times L^1(0,\infty) \times L^1(\Omega)$ with norm $\|\cdot\|_X$ where $\Omega := (0,\infty)\times(0,\infty)$ and

$$\|f\|_X := \int_0^\infty |f_1(a)|da + \int_0^\infty |f_2(b)|db + 2\int_0^\infty \int_0^\infty |f_3(a,b)|dadb.$$

Then the state space of the population is given by the positive cone X_+ and the norm $\|f\|_X$, $f \in X_+$ denotes the total size of the population.

First define the population operator $A : \mathcal{D}(A) \subset X \to X$ as follows:

$$Af := \left(-\frac{df_1(a)}{da}, -\frac{df_2(b)}{db}, -Df_3(a,b)\right)^\tau,$$

where τ denotes the transpose of the vector and the operator D denotes the directional derivative along the vector $(1,1)$ defined by

$$Df(a,b) = \lim_{h\to 0}\frac{f(a+h,b+h)-f(a,b)}{h},$$

and the domain $\mathcal{D}(A)$ is given by

$$\mathcal{D}(A) := \{f \in X : f_1, f_2 \text{ are absolutely continuous}, f_1', f_2' \in L^1(0,\infty),$$

f_3 is absolutely continuous along the direction $(1,1)$ for almost every (a,b),

$$Df_3 \in L^1(\Omega),\ \ f_1(0) = \gamma \int_0^\infty \int_0^\infty \beta(a,b) f_3(a,b)dadb,$$

$$f_2(0) = (1-\gamma)\int_0^\infty \int_0^\infty \beta(a,b) f_3(a,b)dadb,\ \ f_3(a,0) = f_3(0,b) = 0\}$$

Next we define the perturbation terms $B: X \to X$ and $C: X \to X$ by

$$Bf = \begin{pmatrix} -\mu_m(a) f_1(a) + \int_0^\infty f_3(a,b)[\sigma(a,b) + \mu_f(b)]db \\ -\mu_f(b) f_2(b) + \int_0^\infty f_3(a,b)[\sigma(a,b) + \mu_m(a)]da \\ -(\mu_m(a) + \mu_f(b) + \sigma(a,b)) f_3(a,b) \end{pmatrix} \tag{3.1}$$

$$Cf = \begin{pmatrix} -\int_0^\infty \Psi(f_1, f_2)(a,b)db \\ -\int_0^\infty \Psi(f_1, f_2)(a,b)da \\ \Psi(f_1, f_2)(a,b) \end{pmatrix}. \tag{3.2}$$

Then the pair formation model (2.1) can be formulated as a Cauchy problem in X as

$$\frac{d}{dt} f(t) = (A+B)f(t) + Cf(t),\quad f(0) = f_0, \tag{3.3}$$

where $f_0 \in X_+$ is the initial data.

For analytical treatments of the above Cauchy problem, instead of X we will also use its complexification $\tilde{X} := \{x + iy; x, y \in X\}$ with norm $\|x+iy\|_{\tilde{X}} := \sup_{\theta \in [0,2\pi]} \|x\cos\theta + y\sin\theta\|_X$. On the complex Banach lattice $\tilde{X}$, a linear operator $L: X \to X$ has a natural extension $\tilde{L}$ as $\tilde{L}(x+iy) = Lx + iLy$ (see [19]).

In order to show that the linear operator $A+B$ generates a strongly continuous positive semigroup, let us confirm that $A+B$ satisfies Hille-Yosida condition on $\tilde{X}$.

Lemma 3. 1 *Let $\Lambda := \{\lambda \in \mathbf{C} : \Re\lambda > -\underline{\mu}\}$. Then $\Lambda \subset \rho(A+B)$. Moreover let α be a number such that $\alpha = \max(-\underline{\mu}, -\beta/2)$. Then the following estimate holds:*

$$\|(\lambda - (A+B))^{-1}\|_{\tilde{X}} \le \frac{1}{\lambda - \alpha},\quad \text{for } \lambda > \alpha \tag{3.4}$$

Proof: Let us consider the resolvent equation

$$(\lambda - (A+B))f = \phi,\ \ f \in \mathcal{D}(A+B),\ \ \phi \in \tilde{X},$$

That is, we can write

$$\lambda f_1(a) + f_1'(a) + \mu_m(a) f_1(a) - \int_0^\infty f_3(a,b)[\sigma(a,b) + \mu_f(b)]db = \phi_1(a),$$

$$\lambda f_2(b) + f_2'(b) + \mu_f(b) f_2(b) - \int_0^\infty f_3(a,b)[\sigma(a,b) + \mu_m(a)]da = \phi_2(b),$$

$$\lambda f_3(a,b) + Df_3(a,b) + (\mu_m(a) + \mu_f(b) + \sigma(a,b)) f_3(a,b) = \phi_3(a,b).$$

By formal integration, we obtain the following expression:

$$f_1(a) = f_1(0)e^{-\lambda a}\ell_m(a) + \int_0^a e^{-\lambda(a-s)}\frac{\ell_m(a)}{\ell_m(s)}\left[\phi_1(s) + \int_0^\infty f_3(s,b)[\sigma(s,b) + \mu_f(b)]db\right]ds, \tag{3.5}$$

$$f_2(b) = f_2(0)e^{-\lambda b}\ell_f(b) + \int_0^b e^{-\lambda(b-s)}\frac{\ell_f(b)}{\ell_f(s)}\left[\phi_2(s) + \int_0^\infty f_3(a,s)[\sigma(a,s) + \mu_m(a)]da\right]ds, \tag{3.6}$$

$$f_3(a,b) = \begin{cases} \int_0^b \phi_3(a-b+s,s)e^{-\lambda(b-s)}\pi(a-b+s,s;a,b)ds & (a>b), \\ \int_0^a \phi_3(s,b-a+s)e^{-\lambda(a-s)}\pi(s,b-a+s;a,b)ds & (a<b), \end{cases} \tag{3.7}$$

where π is the survival function for pairs defined by

$$\pi(a,b;a+h,b+h) := \exp\left(-\int_0^h [\mu_m(a+s)+\mu_f(b+s)+\sigma(a+s,b+s)]ds\right).$$

Since it is easy to see that for $\lambda \in \Lambda$ the right hand side of the above expressions (3.5)-(3.7) defines a bounded linear operator from $\tilde{X}$ to $\mathcal{D}(A+B)$ and it is no other than the resolvent operator $(\lambda-(A+B))^{-1}$. Moreover we know that for $\lambda \in \mathbf{R} \cap \Lambda$ the resolvent operator $(\lambda-(A+B))^{-1}$ is a positive operator, so first we estimate $\|(\lambda-(A+B))^{-1}\|_{\tilde{X}}$ by using the positivity on the cone $\tilde{X}_+$.

First note that for real function $\phi \in X \subset \tilde{X}$, it follows that $\|\phi\|_{\tilde{X}} = \|\phi\|_X$. Let us assume that $\phi \in \tilde{X}_+$ and $f = (\lambda-(A+B))^{-1}\phi \in \tilde{X}_+$. Then by integrating system (3.5)-(3.7) and using the positivity of ϕ and f, it follows that

$$\lambda\|f_1\|_X + \int_0^\infty f_1'(a)da + \int_0^\infty \mu_m(a)f_1(a)da - \int_0^\infty da\int_0^\infty f_3(a,b)[\sigma(a,b)+\mu_f(b)]db = \|\phi_1\|_X, \tag{3.8}$$

$$\lambda\|f_2\|_X + \int_0^\infty f_2'(b)db + \int_0^\infty \mu_f(b)f_2(b)db - \int_0^\infty db\int_0^\infty f_3(a,b)[\sigma(a,b)+\mu_m(a)]da = \|\phi_2\|_X, \tag{3.9}$$

$$\lambda\|f_3\|_X + \int_0^\infty\int_0^\infty Df_3(a,b)dadb + \int_0^\infty\int_0^\infty (\mu_m(a)+\mu_f(b)+\sigma(a,b))f_3(a,b)dadb = \|\phi_3\|_X. \tag{3.10}$$

From (3.5)-(3.7), we know that for $\lambda \in \mathbf{R}\cap\Lambda$, $f_1(\infty) = f_2(\infty) = f_3(\infty,b) = f_3(a,\infty) = 0$. By adding term to term in (3.8)-(3.10), we obtain

$$\lambda(\|f_1\|_X + \|f_2\|_X + 2\|f_3\|_X) = \|\phi_1\|_X + \|\phi_2\|_X + 2\|\phi_3\|_X$$
$$-\int_0^\infty \mu_m(a)f_1(a)da - \int_0^\infty \mu_f(b)f_2(b)db + \int_0^\infty\int_0^\infty (\beta(a,b)-\mu_m(a)-\mu_f(b))f_3(a,b)dadb.$$

Let us define a number α such that $\alpha := \max\{-\underline{\mu}, -\bar{\beta}/2\}$. Then it follows immediately that

$$\lambda(\|f_1\|_X + \|f_2\|_X + 2\|f_3\|_X) \le \|\phi_1\|_X + \|\phi_2\|_X + 2\|\phi_3\|_X + \alpha(\|f_1\|_X + \|f_2\|_X + 2\|f_3\|_X).$$

Therefore we conclude that if $\lambda > \alpha$ and ϕ is positive, then the estimate (3.4) holds.

Next consider the case that ϕ is real but not necessarily positive. ϕ can be decomposed as $\phi = \phi_1 - \phi_2$ where $\phi_1 = \max(\phi, 0)$ and $\phi_2 = \max(-\phi, 0)$, and we can write $\|\phi\|_X = \|\phi_1\|_X + \|\phi_2\|_X$. Moreover we obtain

$$f = (\lambda - A)^{-1}\phi = f_1 - f_2, \quad f_i := (\lambda - A)^{-1}\phi_i \in \tilde{X}_+.$$

From the above argument, we know that for $\lambda > \alpha$

$$\|f_i\|_X \le \frac{\|\phi_i\|_X}{\lambda - \alpha}$$

On the other hand, it is easily seen that

$$\max(f,0) \le f_1, \quad \max(-f,0) \le f_2$$

Thus we conclude that

$$\|f\|_X = \|\max(f,0)\|_X + \|\max(-f,0)\|_X \le \|f_1\|_X + \|f_2\|_X \le \frac{1}{\lambda-\alpha}(\|\phi_1\|_X + \|\phi_2\|_X) = \frac{\|\phi\|_X}{\lambda-\alpha}.$$

Then we again reach to the estimate (3.4).

Finally consider the case of ϕ is a general element in $\tilde{X}$. Observe that

$$\|f\|_{\tilde{X}} = \sup_{0\le\theta<2\pi} \|(\Re f)\cos\theta + (\Im f)\sin\theta\|_X$$

$$\le \sup_{0\le\theta<2\pi} \|[(\lambda-(A+B))^{-1}(\Re\phi)]\cos\theta + [(\lambda-(A+B))^{-1}(\Im\phi)]\sin\theta\|_X$$

$$\le \sup_{0\le\theta<2\pi} \|(\lambda-(A+B))^{-1}[(\Re\phi)\cos\theta + (\Im\phi)\sin\theta]\|_X$$

$$\le \frac{1}{\lambda-\alpha}\sup_{0\le\theta<2\pi} \|(\Re\phi)\cos\theta + (\Im\phi)\sin\theta\|_X = \frac{\|\phi\|_{\tilde{X}}}{\lambda-\alpha}.$$

Then we again arrive at the estimate (3.4). □

Lemma 3. 2 *The perturbed population operator $A+B$ is a densely defined closed linear operator in $\tilde{X}$.*

Proof: First it is easy to see that $A+B$ is a closed linear operator, though we omit the proof. Next we show that $\mathcal{D}(A+B)$ is dense. For $\lambda > \alpha$ and any $\phi \in X$, we define $f_\lambda = \lambda(\lambda-(A+B))^{-1}\phi$. Then if we can show that $\lim_{\lambda\to\infty} f_\lambda = \phi$, our proof completes since $f_\lambda \in \mathcal{D}(A+B)$. To this end, we write the resolvent as follows:

$$(\lambda-(A+B))^{-1}\phi = (\lambda-(A+B)_0)^{-1}\phi + M(\lambda)\phi,\ \lambda>\alpha,\ \phi\in\tilde{X},$$

where $(A+B)_0$ is the operator corresponding to the special case of the operator $A+B$ with zero boundary condition (that is, $\beta=0$: no birth), and $M(\lambda)$ is the linear operator defined by the difference between $(\lambda-(A+B))^{-1}$ and $(\lambda-(A+B)_0)^{-1}$. Since it is easily seen that $(A+B)_0$ is a generator of C_0 semigroup, it follows that

$$\lim_{\lambda\to\infty} \lambda(\lambda-(A+B)_0)^{-1}\phi = \phi.$$

Hence it is sufficient to show that $\lim_{\lambda\to\infty} M(\lambda)\phi = 0$ in order to complete our proof. Note that if we write $(g_1(\lambda), g_2(\lambda), g_3(\lambda))^\tau = M(\lambda)\phi$, then we have $g_3(\lambda)=0$ and

$$g_1(\lambda) = e^{-\lambda a}\ell_m(a)\gamma\int\int_\Omega \beta(a,b)U_\lambda(\phi_3)(a,b)dadb,$$

$$g_2(\lambda) = e^{-\lambda b}\ell_f(b)(1-\gamma)\int\int_\Omega \beta(a,b)U_\lambda(\phi_3)(a,b)dadb,$$

where $U_\lambda(\phi_3)$ is given by

$$U_\lambda(\phi_3)(a,b) := \begin{cases} \int_0^b \phi_3(a-b+s,s)e^{-\lambda(b-s)}\pi(a-b+s,s;a,b)ds & (a>b) \\ \int_0^a \phi_3(s,b-a+s)e^{-\lambda(a-s)}\pi(s,b-a+s;a,b)ds & (a<b) \end{cases}. \tag{3.11}$$

Let

$$J_i = \int\int_{\Omega_i} |U_\lambda(\phi_3)(a,b)|dadb, \quad (i=1,\ 2)$$

where $\Omega_1 = \{(a,b) : b\ge a,\ 0\le a,\ 0\le b\}$, $\Omega_2 = \{(a,b) : b\le a,\ 0\le a,\ 0\le b\}$. By change of variables, we can obtain

$$J_1 = \int\int_\Delta |U_\lambda(\phi_3)(x,x+y)|dxdy$$

where $\Delta := \{(x, y) : 0 \le x, 0 \le y\}$. From (3.11), we have

$$J_1 \le \int\int_\Delta dxdy \int_0^x |\phi_3(s, s+y)| e^{-\lambda(x-s)} \pi(s, s+y; x, x+y) ds$$

$$\le \int\int_\Delta dxdy \int_0^x |\phi_3(s, s+y)| e^{-(\lambda+2\underline{\mu}+\underline{\sigma})(x-s)} ds = \int_0^\infty dy \int_0^\infty dx \int_0^x |\phi_3(s, s+y)| e^{-(\lambda+2\underline{\mu}+\underline{\sigma})(x-s)} d$$

$$= \int_0^\infty dy \int_0^\infty ds |\phi_3(s, s+y)| \int_s^\infty e^{-(\lambda+2\underline{\mu}+\underline{\sigma})(x-s)} dx.$$

From the above inequality, it is not difficult to see that

$$J_1 \le \frac{1}{\lambda + 2\underline{\mu} + \underline{\sigma}} \int_0^\infty \int_0^\infty ds |\phi_3(s, s+y)| dsdy.$$

Applying the same kind of argument to J_2, we can arrive at the following estimate:

$$\|U_\lambda(\phi_3)\|_{L^1(\Omega)} = J_1 + J_2 \le \frac{1}{\lambda + 2\underline{\mu} + \underline{\sigma}} \|\phi_3\|_{L^1(\Omega)}. \tag{3.12}$$

Next from the concrete expression of $M(\lambda)$, we can observe that

$$\|M(\lambda)\phi\|_{\tilde{X}} \le |f_1(0)| \int_0^\infty e^{-\lambda a} da + |f_2(0)| \int_0^\infty e^{-\lambda a} da$$

$$\le \frac{\gamma\bar{\beta}}{\lambda} \|U_\lambda(\phi_3)\|_{L^1(\Omega)} + \frac{(1-\gamma)\bar{\beta}}{\lambda} \|U_\lambda(\phi_3)\|_{L^1(\Omega)} \le \frac{\bar{\beta}}{\lambda(\lambda + 2\underline{\mu} + \underline{\sigma})} \|\phi_3\|_{L^1(\Omega)}.$$

Then it follows immediately that

$$\lim_{\lambda \to \infty} \lambda \|M(\lambda)\phi\|_{\tilde{X}} = 0$$

Thus we can conclude that the operator $(A + B)$ is densely defined. □

Proposition 3. 3 *The operator $A + B$ is an infinitesimal generator of a strongly continuous positive semigroup $T(t)$ which satisfies*

$$\|T(t)\| \le e^{\alpha t}. \tag{3.13}$$

Proof: From Lemma 3.2 and 3.3, we know that $A + B$ satisfies the Hille-Yosida condition, so it generates a strongly continuous semigroup $T(t) = e^{t(A+B)}$ on X and from Lemma 3.2 we can obtain the estimate (3.13). From Hille's formula we obtain that

$$T(t) = \lim_{n \to \infty} \left(I - \frac{t}{n}(A + B) \right)^{-n} = \lim_{n \to \infty} \left(\frac{n}{t} \right) \left(\frac{n}{t} - (A + B) \right)^{-1},$$

where lim denotes strong convergence. Since the resolvent operator is a positive operator, we can conclude that $T(t)$ is also positive. □

Under the Assumption 3.1, if we choose a number ϵ such that $\epsilon < 1/\eta$, then for $u \in X_+$ we have

$$(I + \epsilon C)u = \begin{pmatrix} u_1(a) - \epsilon \int_0^\infty \Psi(u_1, u_2)(a, b) db \\ u_2(b) - \epsilon \int_0^\infty \Psi(u_1, u_2)(a, b) da \\ u_3(a, b) + \epsilon \Psi(u_1, u_2) \end{pmatrix} \ge \begin{pmatrix} u_1(a) - \frac{1}{\eta} \int_0^\infty \Psi(u_1, u_2)(a, b) db \\ u_2(b) - \frac{1}{\eta} \int_0^\infty \Psi(u_1, u_2)(a, b) da \\ u_3(a, b) + \frac{1}{\eta} \Psi(u_1, u_2) \end{pmatrix} \ge 0.$$

Then we obtain the following:

Lemma 3. 4 *Under the Assumption 3.1, there exists a constant $\epsilon > 0$ such that*

$$(I + \epsilon C)(X_+) \subset X_+. \tag{3.14}$$

Using the above observation, we can construct a positive solution (semiflow) of the Cauchy problem (3.3):

Proposition 3. 5 *Let $f_0 \in X_+$. Then the Cauchy problem (3.3) has a unique mild solution in X_+, which defines a semiflow $S(t)f_0$ such that $S(t)(X_+) \subset X_+$.*

Proof: Instead of the original equation (3.3), let us consider the following equivalent equation as

$$\frac{d}{dt}f(t) = \left(A + B - \frac{1}{\epsilon}\right) f + \frac{1}{\epsilon}(I + \epsilon C)f, \tag{3.15}$$

where ϵ is a positive number. It is well known that the mild solution of (3.15) is given as a solution of the following integral equation:

$$f(t) = e^{-\frac{1}{\epsilon}t}e^{(A+B)t}f_0 + \frac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-s)}e^{(A+B)(t-s)}[f(s) + \epsilon Cf(s)]ds, \tag{3.16}$$

where the constant ϵ is chosen so small that (3.14) holds. Since the nonlinear perturbation is assumed to be locally Lipschitz continuous, the local solution of (3.15) is constructed by the standard iterative procedure:

$$f^0(t) = f_0, \tag{3.17}$$

$$f^{n+1}(t) = e^{-\frac{1}{\epsilon}t}e^{(A+B)t}f_0 + \frac{1}{\epsilon}\int_0^t e^{-\frac{1}{\epsilon}(t-s)}e^{(A+B)(t-s)}[f^{n+1}(s) + \epsilon Cf^{n+1}(s)]ds,$$

Thanks to the positivity of $e^{(A+B)t}$ and $I + \alpha C$, we can prove $u^{n+1} \in X_+$ iteratively. In fact, if f_0, $f^n \in X_+$, the integral part of the variation of constants formula (3.17) is a convex linear combination of those positive terms. Since the operator C is locally Lipschitz continuous, the sequence $u^n(t)$ converges to a positive, mild local solution $f(t) \in X_+$. From Proposition 3.3, we have the following estimate

$$\|f(t)\| \le e^{(\alpha-\frac{1}{\epsilon})t}\|f_0\| + \frac{K}{\epsilon}\int_0^t e^{(\alpha-\frac{1}{\epsilon})(t-s)}\|f(s)\|ds,$$

where $K := \|I + \epsilon C\|$. Then it is easily seen that the following estimate holds:

$$\|f(t)\| \le \|f_0\|e^{(\alpha-\frac{1-K}{\epsilon})t}.$$

Since the norm of local solution grows at most exponentially as time evolves, so it can be extended to a global solution. Hence we can define a flow $S(t)$ by $S(t)f_0 = f(t)$. □

4 Malthusian population growth

In this section, we examine existence of exponentially growing persistent solutions (Malthusisan population growth) under the assumption that the marriage function is homogeneous of degree one.

Let us consider the nonlinear eigenvalue problem associated with (3.3) as

$$\lambda f = (A + B)f + Cf, \quad f \in \mathcal{D}(A + B). \tag{4.1}$$

Then it is clear that if there exists a real eigenvalue λ associated with positive eigenvector f, $e^{\lambda t}f(a)$ becomes a persistent solution. Conversely if there exists a persistent solution $e^{\lambda t}f(a)$ to (3.3), λ and f must satisfy the nonlinear eigenvalue problem (4.1).

In the following, to avoid technical difficulties, we assume that pair formation and dissolution occurs between *reproductive* sexes, that is, males and females under age ω. Though this assumption looks as restrictive, it would be not necessarily unrealistic assumption, since non-reproductive pair formation is rare case, and even though it exists, it gives no effect to the population growth.

Furthermore, in order to look for the solution of (4.1), let us introduce the following normalization:

$$g_1(t,a) = \frac{f_1(t,a)}{\|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1}}, \quad g_2(t,b) = \frac{f_2(t,b)}{\|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1}},$$

$$g_3(t,a,b) = \frac{f_3(t,a,b)}{\|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1}}.$$

Under the above assumptions, the system (2.1) can be rewritten as follows:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) g_1(t,a) = -(\mu_m(a) + \lambda(t))g_1(t,a) \\ \qquad + \int_0^\omega g_3(t,a,b)[\sigma(a,b) + \mu_f(b)]db - \int_0^\omega \Psi(g_1(t), g_2(t))(a,b)db, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial b}\right) g_2(t,b) = -(\mu_f(b) + \lambda(t))g_2(t,b) \\ \qquad + \int_0^\omega g_3(t,a,b)[\sigma(a,b) + \mu_m(a)]da - \int_0^\omega \Psi(g_1(t), g_2(t))(a,b)da, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right) g_3(t,a,b) \\ \quad = -(\lambda(t) + \sigma(a,b) + \mu_m(a) + \mu_f(b))g_3(t,a,b) + \Psi(g_1(t), g_2(t))(a,b), \\ g_1(t,0) = \gamma \int_0^\omega \int_0^\omega \beta(a,b) g_3(t,a,b)dadb, \\ g_2(t,0) = (1-\gamma) \int_0^\omega \int_0^\omega \beta(a,b) g_3(t,a,b)dadb, \\ g_3(t,0,b) = g_3(t,a,0) = 0, \\ \lambda(t) = \Lambda(g_1(t), g_2(t), g_3(t)), \end{cases} \tag{4.2}$$

where Λ is a functional given by

$$\Lambda(g_1, g_2, g_3) := \int_0^\omega \int_0^\omega (\beta(a,b) + \mu_m(a) + \mu_f(b) + 2\sigma(a,b))g_3(a,b)dadb$$

$$- \int_0^\infty [\mu_m(a)g_1(a) + \mu_f(a)g_2(a)]da - 2\int_0^\omega \int_0^\omega \Psi(g_1(t), g_2(t))(a,b)dadb.$$

Note that $\lambda(t)$ is the growth rate of single population size, that is,

$$\frac{d}{dt}(\|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1}) = \lambda(t)(\|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1}).$$

Moreover let us consider the corresponding time-independent problem of system (4.2) as follows:

$$\begin{cases} u'(a) = -(\mu_m(a) + \lambda)u(a) + \int_0^\omega w(a,b)[\sigma(a,b) + \mu_f(b)]db - \int_0^\omega \Psi(u,v)(a,b)db, \\ v'(b) = -(\mu_f(b) + \lambda)v(b) + \int_0^\omega w(a,b)[\sigma(a,b) + \mu_m(a)]da - \int_0^\omega \Psi(u,v)(a,b)da, \\ w_a(a,b) + w_b(a,b) = -(\lambda + \sigma(a,b) + \mu_m(a) + \mu_f(b))w(a,b) + \Psi(u,v)(a,b), \\ u(0) = \gamma \int_0^\omega \int_0^\omega \beta(a,b)w(a,b)dadb, \\ v(0) = (1-\gamma) \int_0^\omega \int_0^\omega \beta(a,b)w(a,b)dadb, \\ w(0,b) = w(a,0) = 0, \\ \lambda = \Lambda(u,v,w). \end{cases} \tag{4.3}$$

We can observe that if the time-independent problem (4.3) has a solution (u,v,w) with $\|u\|_{L^1} + \|v\|_{L^1} = 1, u \geq 0, v \geq 0, w \geq 0$, then $(u,v,w)e^{\lambda t}$, $\lambda = \Lambda(u,v,w)$ becomes a persistent solution of the original system (2.1).

If we use the expression $w(a,b) = U_\lambda(\Psi(u,v))(a,b)$, the system (4.3) can be reduced to the following (u,v,λ) system:

$$\begin{cases} u'(a) = -(\mu_m(a)+\lambda)u(a) - \int_0^\omega \Psi(u,v)(a,b)db + \int_0^\omega U_\lambda(\Psi(u,v))(a,b)[\sigma(a,b)+\mu_f(b)]db, \\ v'(b) = -(\mu_f(b)+\lambda)v(b) - \int_0^\omega \Psi(u,v)(a,b)da + \int_0^\omega U_\lambda(\Psi(u,v))(a,b)[\sigma(a,b)+\mu_m(a)]da, \\ u(0) = \gamma \int_0^\omega \int_0^\omega \beta(a,b)U_\lambda(\Psi(u,v))(a,b)dadb, \\ v(0) = (1-\gamma)\int_0^\omega \int_0^\omega \beta(a,b)U_\lambda(\Psi(u,v))(a,b)dadb, \\ \lambda = \Lambda(u,v,U_\lambda(\Psi(u,v))). \end{cases} \tag{4.4}$$

Therefore the nonlinear eigenvalue problem (4.1) can be reduced to the problem to seek a solution (u,v,λ) of (4.4) in $Z_+ \times \mathbf{R}$, where $Z_+ := \{(u,v) \in Y_+ : \|u\|_{L^1} + \|v\|_{L^1} = 1\}$.

Let $f := (u,v) \in Z_+$. Then the eigenvalue problem (4.4) can be formally written as follows:

$$\begin{cases} A_1 f + F(\lambda, f) = \lambda f, \\ \lambda = \Lambda(u,v,U_\lambda(\Psi(u,v))), \end{cases} \tag{4.5}$$

where A_1 and F are defined as follows:

$$(A_1 f)(a) := (-u'(a), -v'(a))^T,$$

$$\mathcal{D}(A_1) := \{f = (u,v) \in Y : u(0) = \gamma \int_0^\omega \int_0^\omega \beta(a,b)U_\lambda(\Psi(u,v))(a,b)dadb,$$

$$v(0) = (1-\gamma)\int_0^\omega \int_0^\omega \beta(a,b)U_\lambda(\Psi(u,v))(a,b)dadb\},$$

$$F(\lambda, f)(a) := \begin{pmatrix} -\mu_m(a)u(a) - \int_0^\omega \Psi(u,v)(a,x)dx + \int_0^\omega U_\lambda(\Psi(u,v))(a,x)[\sigma(a,x)+\mu_f(x)]dx \\ -\mu_f(a)v(a) - \int_0^\omega \Psi(u,v)(x,a)dx + \int_0^\omega U_\lambda(\Psi(u,v))(x,a)[\sigma(x,a)+\mu_m(x)]dx \end{pmatrix}.$$

Further, for a small number $\epsilon > 0$, the eigenvalue problem (4.5) can be transformed into a fixed point equation as

$$\begin{cases} f = \frac{1}{\epsilon}(\frac{1}{\epsilon} - A_1)^{-1}((1-\epsilon\lambda)f + \epsilon F(\lambda, f)), \\ \lambda = \Lambda(u,v,U_\lambda(\Psi(u,v))). \end{cases} \tag{4.6}$$

This fixed point equation can be expressed as

$$\phi = \Phi(\phi), \quad \phi = (u,v,\lambda) \in Z_+ \times \mathbf{R}, \tag{4.7}$$

where the operator Ψ is defined by

$$\Psi(\phi) := (\Phi_\epsilon^1(u,v,\lambda), \Phi_\epsilon^2(u,v,\lambda), \Lambda(u,v,U_\lambda(\Psi(u,v))),$$

for $\phi = (u,v,\lambda) \in Z_+ \times \mathbf{R}$, and mappings Φ_ϵ^1 and Φ_ϵ^2 are defined on $Z_+ \times \mathbf{R}$ as follows:

$$\begin{cases} \Phi_\epsilon^1(u,v,\lambda)(a) := \gamma B(u,v,\lambda)e^{-\frac{1}{\epsilon}a} + \frac{1}{\epsilon}\int_0^a e^{-\frac{1}{\epsilon}(a-s)}\{u(s) - \epsilon(\mu_m(s)+\lambda)u(s) \\ \quad -\epsilon\int_0^\infty \Psi(u,v)(s,x)dx + \epsilon\int_0^\infty [\mu_f(x)+\sigma(s,x)]U_\lambda(\Psi(u,v))(s,x)dx\}ds, \\ \Phi_\epsilon^2(u,v,\lambda)(a) := (1-\gamma)B(u,v,\lambda)e^{-\frac{1}{\epsilon}a} + \frac{1}{\epsilon}\int_0^a e^{-\frac{1}{\epsilon}(a-s)}\{v(s) - \epsilon(\mu_f(s)+\lambda)v(s) \\ \quad -\epsilon\int_0^\infty \Psi(u,v)(x,s)dx + \epsilon\int_0^\infty [\mu_m(x)+\sigma(x,s)]U_\lambda(\Psi(u,v))(x,s)dx\}ds, \end{cases}$$

where the functional B is defined by

$$B(u,v,\lambda) := \int_0^\infty \int_0^\infty \beta(a,b) U_\lambda(\Psi(u,v))(a,b)dadb.$$

It is clear that a fixed point of (4.7) in $Z_+ \times \mathbf{R}$ is no other than a solution to the problem (4.4).

Lemma 4. 1 *There exist numbers $\underline{\lambda} \le \bar{\lambda}$ such that for any $(u,v,\lambda) \in Z_+ \times [\underline{\lambda}, \bar{\lambda}]$, it follows that*

$$\underline{\lambda} \le \Lambda(u,v,U_\lambda(\Psi(u,v))) \le \bar{\lambda}. \tag{4.8}$$

Proof. For $(u,v) \in Z_+$, it follows from Assumption 2.1 that

$$\int_0^\omega \int_0^\omega \Psi(u,v)(a,b)dadb \le \eta.$$

Then we have for $(u,v) \in Z_+$ and any $\lambda \in \mathbf{R}$,

$$\Lambda(u,v,U_\lambda(\Psi(u,v))) \ge -\bar{\mu} - \eta.$$

On the hand, it is easily seen that for $(u,v) \in Z_+$ and any $\lambda \in \mathbf{R}$,

$$\Lambda(u,v,U_\lambda(\Psi(u,v))) \le (\bar{\beta} + 2\bar{\mu} + 2\bar{\sigma}) \int_0^\omega \int_0^\omega U_\lambda(\Psi(u,v))(a,b)dadb.$$

If we restrict the domain of λ as $\lambda \ge -\bar{\mu} - \eta$, we obtain that

$$\int_0^\omega \int_0^\omega U_\lambda(\Psi(u,v))(a,b)dadb \le \eta \sup_{0 \le s \le \omega} \int_s^\omega e^{-(\lambda + 2\underline{\mu} + \underline{\sigma})(x-s)} dx \le \frac{\eta}{\bar{\mu}+\eta}(e^{(\bar{\mu}+\eta)\omega} - 1).$$

Therefore if we choose $\underline{\lambda}$ and $\bar{\lambda}$ as

$$\underline{\lambda} = -\bar{\mu} - \eta, \ \bar{\lambda} = \frac{\eta(\bar{\beta} + 2\bar{\mu} + 2\bar{\sigma})}{\bar{\mu}+\eta}(e^{(\bar{\mu}+\eta)\omega} - 1),$$

then we conclude that (4.8) holds for any $(u,v,\lambda) \in Z_+ \times [\underline{\lambda}, \bar{\lambda}]$. □

Lemma 4. 2 *Let $(u,v,\lambda) \in Y_+ \times \mathbf{R}$. Then the following relation holds:*

$$\|\Phi^1_\epsilon(u,v,\lambda)\|_{L^1} + \|\Phi^2_\epsilon(u,v,\lambda)\|_{L^1} = (1 - \epsilon\lambda)(\|u\|_{L^1} + \|v\|_{L^1}) + \epsilon\Lambda(u,v,U_\lambda(\Psi(u,v))). \tag{4.9}$$

Proof: Since $\Phi^1_\epsilon(a) = \Phi^1_\epsilon(u,v,\lambda)(a)$ and $\Phi^2_\epsilon(b) = \Phi^2_\epsilon(u,v,\lambda)(b)$ are differentiable, we obtain

$$(d/da)\Phi^1_\epsilon(a) + \frac{1}{\epsilon}\Phi^1_\epsilon(a) = \frac{1}{\epsilon}\left[u(a) - \epsilon(\mu_m(a) + \lambda)u(a) - \epsilon\int_0^\infty \Psi(u,v)(a,b)db + \right.$$

$$\left. +\epsilon\int_0^\infty [\mu_f(b) + \sigma(a,b)]U_\lambda(\Psi(u,v))(a,b)db\right],$$

$$(d/db)\Phi^2_\epsilon(b) + \frac{1}{\epsilon}\Phi^2_\epsilon(b) = \frac{1}{\epsilon}\left[v(b) - \epsilon(\mu_f(b) + \lambda)v(b) - \epsilon\int_0^\infty \Psi(u,v)(a,b)da + \right.$$

$$\left. +\epsilon\int_0^\infty [\mu_m(a) + \sigma(a,b)]U_\lambda(\Psi(u,v))(a,b)da\right].$$

Integrating from zero to infinity in both sides and thanks to positivity, we have

$$-\Phi^1_\epsilon(0) + \frac{1}{\epsilon}\|\Phi^1_\epsilon\|_{L^1} = \left(\frac{1}{\epsilon} - \lambda\right)\|u\|_{L^1} - \int_0^\infty \mu_m(a)u(a)da - \int_0^\infty \int_0^\infty \Psi(u,v)(a,b)dadb$$

$$+ \int_0^\infty \int_0^\infty [\mu_f(b) + \sigma(a,b)]U_\lambda(\Psi(u,v))(a,b)dadb,$$

$$-\Phi_\epsilon^2(0) + \frac{1}{\epsilon}\|\Phi_\epsilon^2\|_{L^1} = \left(\frac{1}{\epsilon} - \lambda\right)\|v\|_{L^1} - \int_0^\infty \mu_f(b)v(b)db - \int_0^\infty \int_0^\infty \Psi(u,v)(a,b)dadb$$
$$+\int_0^\infty \int_0^\infty [\mu_m(a) + \sigma(a,b)]U_\lambda(\Psi(u,v))(a,b)dadb.$$

By adding therm to term and changing variables in integration, it follows that

$$-\int_0^\infty \int_0^\infty \beta(a,b)U_\lambda(\Psi(u,v))(a,b)dadb + \frac{1}{\epsilon}(\|\Phi_\epsilon^1\|_{L^1} + \|\Phi_\epsilon^2\|_{L^1}) =$$
$$= \left(\frac{1}{\epsilon} - \lambda\right)(\|u\|_{L^1} + \|v\|_{L^1}) - \int_0^\infty \mu_m(a)u(a)da - \int_0^\infty \mu_f(b)v(b)db$$
$$-2\int_0^\infty \int_0^\infty \Psi(u,v)(a,b)dadb$$
$$+\int_0^\infty \int_0^\infty (\mu_m(a) + \mu_f(b) + 2\sigma(a,b))U_\lambda(\Psi(u,v))(a,b)dadb.$$

It follows from definition of Λ that the relation (4.9) holds. □

Proposition 4. 3 *For system (2.1)-(2.6), there exists at least one persistent solution.*

Proof: To prove the existence of persistent solution, it is sufficient to show that the map Φ has a positive fixed point in $Z_+ \times [\underline{\lambda}, \bar{\lambda}]$. First, let us consider a new operator F defined by

$$F(\phi) := \left(\frac{\Phi_\epsilon^1(u,v,\lambda)}{\|\Phi_\epsilon^1(u,v,\lambda)\|_{L^1} + \|\Phi_\epsilon^2(u,v,\lambda)\|_{L^1}}, \frac{\Phi_\epsilon^2(u,v,\lambda)}{\|\Phi_\epsilon^1(u,v,\lambda)\|_{L^1} + \|\Phi_\epsilon^2(u,v,\lambda)\|_{L^1}}, \Lambda(u,v,U_\lambda(\Psi(u,v))\right),$$

where $\phi = (u,v,\lambda) \in Z_+ \times \mathbf{R}$. It follows from Lemma 4.2, we obtain that for $\phi \in Z_+ \times \mathbf{R}$

$$\|\Phi_\epsilon^1(u,v,\lambda)\|_{L^1} + \|\Phi_\epsilon^2(u,v,\lambda)\|_{L^1} \geq 1 - \epsilon(\bar{\lambda} - \underline{\lambda}).$$

Therefore if we choose ϵ in advance such that

$$0 < \epsilon < \min\{1/(\bar{\lambda} - \underline{\lambda}), 1/(\bar{\mu} + \bar{\lambda} + \eta)\},$$

then F is well defined as a completely continuous operator from $Z_+ \times [\underline{\lambda}, \bar{\lambda}]$ into itself. By Schauder's principle, we know that F has a fixed point in $Z_+ \times [\underline{\lambda}, \bar{\lambda}]$, denoted by (u^*, v^*, λ^*). Again from Lemma 4.2, we have

$$\|\Phi_\epsilon^1(u^*,v^*,\lambda^*)\|_{L^1} + \|\Phi_\epsilon^2(u^*,v^*,\lambda^*)\|_{L^1} = (1 - \epsilon\lambda^*) + \epsilon\lambda^* = 1.$$

That is, (u^*, v^*, λ^*) is a fixed point of Φ itself. This completes our proof. □

5 Discussion

In the real world, we are able to observe that a human population has been growing exponentially during some periods and its age-structure exhibits a stable distribution as given by the classical linear theory of Sharpe and Lotka. Therefore if the non-linear two-sex model has a homogeneous marriage function, it is reasonable to require that there exists exponentially growing persistent solution with stability in some sense. An idea of stability for persistent solutions of nonlinear models with homogeneous nonlinearity is introduced by Webb (1993).

Let us consider the following semilinear Cauchy problem on a Banach lattice X:

$$\frac{d}{dt}z(t) = Az(t) + F(z(t)),\ t \geq 0,\ z(0) = x, \tag{5.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of positive linear operators in X and F is a nonlinear operator in X_+ satisfying $F(kz) = kF(z), z \in X_+, k \geq 0$. If $z(t) = z^*$ is an equilibrium solution of (5.1), then so is $z(t) = kz^*$ for any $k > 0$, because the nonlinear term F is homogeneous. Thus there is no nontrivial attracting steady state. However the solution of (5.1) have *asynchronous exponential growth* (A.E.G.) on $U \subset X_+$ provides that there is a constant $r_0 \in \mathbf{R}$ (called the *intrinsic growth constant*) such that $Qx := \lim_{t\to\infty} e^{-r_0 t}z(t)$ exists, Qx is nonzero for all $x \in U$, and the range of Q is a one-dimensional subspace of X.

In order to state Webb's theorem, we assume the following additional hypotheses:

Assumption 5. 1 *(H.1) A is the infinitesimal generator of a strongly continuous semigroups of positive bounded linear operators $T(t)$, $t \geq 0$ in X;*

(H.2) The mild solution to the problem (5.1) uniquely exists for each $x \in X_+$ on some maximal interval of existence $[0, t_x)$;

(H.3) There exists $x_0 \in X_+$ with $\|x_0\| = 1$ and $r_0 \in R$ such that $Ax_0 + F(x_0) = r_0x_0$, F is Fréchet differentiable at x_0, $A_0 := A + F'[x_0]$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T_0(t), t \geq 0$ in X, and there exists a nonzero rank one projection P in X such that $\lim_{t\to\infty} e^{-r_0t}T_0(t) = P$.

Though we cannot show the proof, Webb's theorem can be stated as follows:

Proposition 5. 2 *Let (H.1)-(H.3) hold. Then there exists $\delta > 0$ such that if $x \in U := \{x \in X_+ \backslash \{0\} : \|(I-P)x\|/\|Px\| < \delta\}$, then $t_x = \infty$, $Qx := \lim_{t\to\infty} e^{-r_0t}z(t), z(0) = x$ exists, $Qx \in R(P)$ and $Qx \neq 0$.*

Note that the above theorem implies that x_0, the solution to the nonlinear eigenvalue problem $(A + F)x_0 = r_0 f_0$, is also a solution to the linear eigenvalue problem $(A + F'[x_0])x_0 = r_0x_0$.

If we apply the necessary and sufficient conditions for A.E.G. of a C_0 semigroup [?], it follows that $T_0(t) = \exp((A + F'[x_0])t)$ has A.E.G. with intrinsic growth constant r_0 if and only if $\omega_1(A + F'[x_0]) < r_0$ and r_0 is a strictly dominant simple eigenvalue of $A + F'[x_0]$ where $\omega_1(A)$ denotes the essential growth bound of the semigroup generated by A. In other words, we have to solve the stability problem for the two-sex model with linear marriage function (Fréchet derivative of homogeneous marriage function) in order to give an answer to stability problem for persistent solutions of nonlinear two-sex model with homogeneous marriage function. However, we have not yet known any answer to this question.

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