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Introduction to GJ-integral Method

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1 Summary

The principal object of this paper is to give the summary of GJ-integral method proposed first in the paper of 3D fracture problem [1], which is a generalization of J-integral. In 2D fracture problem, the ***path-independent integral***

$$J_C(\vec{u},\vec{e}_1)=\int_C \{E(\varepsilon_{ij})(\vec{e}_1,\vec{n})-\sigma_{ij}n_j(\vec{e}_1\cdot\nabla u_i)\}\,dl \tag{1}$$

is introduced to represent the variation of the potential energies with respect to crack extension, where C is a closed curve surrounding the crack tip, described in contra-clockwise, $\vec{n}$ the unit outward normal of C, dl the line element of C, σ_{ij} the stress tensor, ε_{ij} the strain tensor, $\vec{u}$ the displacement vector and $E(\varepsilon_{ij})=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}$.

The formula (1) has three parameters $\vec{u}$, C and $\vec{e}_1$. The first $\vec{u}$ expresses the solution of elastic problem, the crack tip (singular point) is inside C and $\vec{e}_1$ the direction of moving crack tip.

Focusing attention on three parameters, GJ-integral is reconstructed in [3] for general variational problems as follows:

Let us consider a domain Ω in N-dimensional space $\mathbb{R}^N$ and the following variational problem.

Problem $P(f,V(\Omega))$: Let $m\geq 1$ be the integer and E be a given function in $C^2(\mathbb{R}^N\times\mathbb{R}^m\times\mathbb{R}^{Nm})$. For a given $f\in L^2(\Omega)^m$, find an element $u\in V_\tau(\Omega)$ such that

$$\mathcal{E}(u;f,\Omega)\leq\mathcal{E}(v;f,\Omega)\ \text{for all}\ v\in V(\Omega). \tag{2}$$

Here

$$\mathcal{E}(v;f,\Omega)=\int_\Omega\{E(x,v,\nabla v)-f\cdot v\}\,dx,\qquad v\in V(\Omega),$$

and $V(\Omega)$ is the closed subspace of Sobolev space $H^1(\Omega)^m$ of order 1. By differentiation of $\eta\mapsto\mathcal{E}(u+\eta v;f,\Omega)$, we therefore obtain the equation

$$\int_\Omega\{A_{ij}(x,u,\nabla u)\,\partial_j v_i+B_i(x,u,\nabla u)\,v\}\,dx=\int_\Omega f\cdot v\,dx,\quad \text{for all}\ \ v\in V(\Omega),$$

where $A_{ij}(x,p,q) = \partial_{q_{ij}} E(x,p,q)$ for $q = (q_{ij})$, $i = 1, \cdots, m$ and $j = 1, \cdots, N$ and $B_i(x,p,q) = \partial_{p_i} E(x,p,q)$ for $p = (p_i)$, $i = 1, \cdots, m$.

The classical prototype of variational problem is Poisson's equation in $\mathbb{R}^N$ with Dirichlet condition

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

which is implied from the case $m = 1$, $E(x,p,q) = \frac{1}{2}|q|$, $q = (q_j)_{j=1,\dots,N}$, $V(\Omega) = H_0^1(\Omega)$. For linear elasticity, $m = N$,

$$E(x,z,p) = \frac{1}{2}\sigma_{ij}(x,p)e_{ij}(p), \quad \sigma_{ij}(x,p) := c_{ijkl}(x)e_{ij}(p), \quad e_{ij}(p) = (p_{i,j} + p_{j,i})/2 \tag{3}$$

where the soomth function $c_{ijkl}(x)$ is called Hooke's tensor satisfying the symmetry conditions $c_{ijkl} = c_{jikl} = c_{klij}$.

We assume that the existence u of $P(f, V(\Omega))$ is obtained uniquely, called ***weak solution***. The question of classical existence is transformed into the question of regularity of weak solutions under appropriately smooth boundary conditions, that is modification to ***strong solution***. So a ***singular point*** p of u is characterized by $u|_{U(p)\cap\Omega} \notin H^2(U(p)\cap\Omega)^m$ for an arbitrary open neighborhood $U(p)$ of p.

1.1 How to reconstruct the generalization of J-integral.

First, introduce a domain $\omega \subset \mathbb{R}^N$ to catch singular points of u inside ω. Next, bring the vector field $\vec{X}$ derived from moving singular points.

Now we define GJ-integral $J_\omega(u, \vec{X})$ by

$$J_\omega(u,\vec{X}) = P_\omega(u,\vec{X}) + R_\omega(u,\vec{X}), \tag{4}$$

$$\begin{aligned}
P_\omega(u,\vec{X}) &= \int_{\partial(\omega\cap\Omega)} \left\{ E(x,u,\nabla u)(\vec{X}\cdot\vec{n}) - T(u)\cdot(\vec{X}\cdot\nabla u) \right\} ds, \\
R_\omega(u,\vec{X}) &= -\int_{\omega\cap\Omega} \left\{ (\vec{X}\cdot\nabla_x)E(x,u,\nabla u) + f\cdot(\vec{X}\cdot\nabla u) \right\} \\
&\quad + \int_{\omega\cap\Omega} \left\{ A_{ij}(x,u,\nabla u)(D_j X_k)(D_k u_i) - E(x,u,\nabla u)(\operatorname{div}\vec{X}) \right\} dx, \\
T_i(u) &= A_{ij}(x,u,\nabla u)n_j, \quad T = (T_i)_{i=1,\cdots,m}.
\end{aligned}$$

The integral $P_\omega(u,\vec{X})$ is derived from the original J-integral. However, $P_\omega(u,\vec{X})$ does not satisfy the important property of the original J-integral, that is, ***GJ-integral take zero when there is no singular point inside*** ω. So the volume integral $R_\omega(u,\vec{X})$ is added to satisfy the following

$$J_\omega(u,\vec{X})=0 \quad \text{for all } \vec{X} \text{ if } u|_{\omega\cap\Omega}\in H^2(\omega\cap\Omega)^m.$$

We should notice that $R_\omega(u,\vec{X})$ is well-defined for $u\in H^1(\Omega)^m$. For Poisson's equation, we have

$$\begin{aligned} P_\omega(u,\vec{X}) &= \int_{\partial(\omega\cap\Omega)}\left\{\frac{1}{2}|\nabla u|^2(\vec{X}\cdot\vec{n})-\frac{\partial u}{\partial n}(\vec{X}\cdot\nabla u)\right\}ds,\\ R_\omega(u,\vec{X}) &= -\int_{\omega\cap\Omega}\left\{f(X\cdot\nabla u)-(\nabla u\cdot\nabla X_k)D_k u+\frac{1}{2}|\nabla u|^2\mathrm{div}X\right\}dx. \end{aligned}$$

We call the following the *fundamental formula in GJ-integral method.*

If the movement of singular points are expressed by the use of a family $\{\Phi_t\}_{0\le t\le T}$ of mappings, then the derivative of potential energies are expressed as

$$\frac{d}{dt}\mathcal{E}(u(t);f,\Omega(t))\bigg|_{t=0}=-R_\Omega(u;\vec{X}_\Phi)+\int_\Omega\left\{\vec{X}_\Phi\cdot\nabla(f\cdot u)+(f\cdot u)\mathrm{div}\,\vec{X}_\Phi\right\}dx, \tag{5}$$

where $\Omega(t)=\Phi_t(\Omega))$, $u(t)$ the solution of $P(f,V_t(\Omega(t)))$, $\vec{X}_\Phi=d\Phi_t/dt|_{t=0}$. Here $P(f,V_t(\Omega(t)))$ denotes the variational problem finding the minimizer $u(t)$ of

$$\mathcal{E}(v;f,\Omega\,(t))=\int_{\Omega(t)}\{E(x,v,\nabla v)-f\cdot v\}\,dx, \qquad v\in V_t(\Omega\,(t))$$

over the closed subspace $V_t(\Omega\,(t))$ of $H^1(\Omega\,(t))^m$.

To prove the fundamental formula, we need the hypotheses for $\{\Phi_t\}_{0\le t\le T}$ and $P(f,V_t(\Omega(t)))$, whose necessary conditions are given in next section. Perhaps, we can prove the fundamental formula mathematically for general perturbation of singular points in general variational problems containing nonlinear case (first proof is given in [3] and also see [7, 8] for resent proof).

1.2 What kind of singular points can we catch by GJ-integral?

The fundamental formula indicate that GJ-integral can catch the perturbation of the boundary in following manner.

Consider the case non-existence of singular point, that is, $u\in H^2(\Omega)^m$. Then it is hold that $J_\Omega(u,\vec{X}_\Phi)=0$, which implies

$$\frac{d}{dt}\mathcal{E}(u(t);f,\Omega(t))\bigg|_{t=0}=P_\Omega(u;\vec{X}_\Phi)+\int_{\partial\Omega}fu\left(\vec{X}\cdot\vec{n}\right)ds. \tag{6}$$

Here it should be notice that $P_\Omega(u;\vec{X}_\Phi)$ is the integral over $\partial\Omega$. If $\partial\Omega$ is smooth, then there is an open neighborhood $U(\partial\Omega)$ of $\partial\Omega$ such that the projection $\mathcal{P}(x)$ of $x\in U(\partial\Omega)$ onto the surface $\partial\Omega$ is uniqely determined. Now consider perturbation of the boundary

$$\Phi_t(x)=x+th(\mathcal{P}(x))\vec{n}(\mathcal{P}(x))\chi(x),$$

where h is a smooth function defined on $\partial\Omega$ and $\chi \in C_0^\infty(U(\partial\Omega))$ the cut-off function such that $\chi \equiv 1$ near $\partial\Omega$. In this case, we have

$$P_\Omega(u, \vec{X}) = \int_{\partial\Omega} \{E(x, u, \nabla u) - T(u) \cdot (\partial_n u)\} h \, ds, \quad \partial_n u = \partial u / \partial n.$$

For Poisson's equation with Dirichlet condition,

$$\frac{d}{dt} \mathcal{E}(u(t); f, \Omega(t)) \Big|_{t=0} = \int_{\partial\Omega} \frac{1}{2} \left(\frac{\partial u}{\partial n}\right)^2 h \, ds. \tag{7}$$

Consider the perturbation $\Phi_t = x + t\vec{X}$, $\vec{X} \in C^2(\mathbb{R}^N)$, then (6) holds for Poisson's equation with Dirichlet condition, if Ω is convex. However, (7) is not true, if $\partial\Omega$ is non-smooth.

Now, we consider the singular points arising from mixed boundary conditions. Assume Dirichlet condition is given on $\Gamma_D \subset \partial\Omega$, and Neumann condition on $\Gamma_N = \partial\Omega - \overline{\Gamma_D}$ for Poisson's equation. Then the interface $\gamma = \overline{\Gamma_D} \cap \overline{\Gamma_N}$ become the set of singular points. Let us consider the tangential smooth perturbation Φ_t, that is, Φ_t is C^2-diffeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$, $\Phi_t(\partial\Omega) = \partial\Omega$, and

$$V_t(\Omega(t)) = \{v \in H^1(\Omega);\ v = 0 \quad \text{on } \Gamma_D(t) = \Phi_t(\Gamma_D)\}.$$

We also obtain the fundamental formula in this perturbation. In 2D case, if $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ is the set of two points $\{\gamma_1, \gamma_2\}$, we then have

$$\frac{d}{dt} \mathcal{E}(u(t); f, \Omega) \Big|_{t=0} = \sum_{k=1}^{2} \frac{\pi}{8} c(\gamma_k)^2 h_s(\gamma_k) \tag{8}$$

where $h_s(\gamma_k)$ is the speed of γ_k from Neumann part to Dirichlet part, that is, if the direction of the vector $d\Phi_t(\gamma_k)/dt$ is Neaumann part to Dirichlet part, then $|d\Phi_t(\gamma_k)/dt|$ otherwise $-|d\Phi_t(\gamma_k)/dt|$. Here $c(\gamma_k)$ is the coefficient of singular term at γ_k. Next we consider the normal perturbation Φ_t of $\delta\Omega$ but the interface is fixed, $\Phi_t(x) \equiv x$ near γ. Then from the fundamental formula, we can derive

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(u(t); f, \Omega(\tau)) \Big|_{t=0} &= \left\{ \int_{\Gamma_N(\delta)} (\partial u/\partial s)^2 h_n ds - \int_{\Gamma_D(\delta)} (\partial u/\partial n)^2 h_n ds \right\} \\ &\quad - \int_{\Gamma_N} f u \, h_n ds \end{aligned} \tag{9}$$

where $h_n = \vec{X}_\Phi \cdot \vec{n}$ and $\partial u/\partial s$ stands for the tangential derivative of u along $\partial\Omega$. The general perturbation in mixed boundary condition will mix the perturbation of boundary and interface. So we have interesting question: "Can we separate the quantities (8) and (9) from the fundamental formula (5) in general perturbation?". The partial answer is given in [5].

1.3 Shape derivative of solutions and Green's kernel

Let φ be an arbitrary function in $C_0^\infty(\Omega)$ and $w(t)$ be the solution of $P(\varphi, H_0^1(\Omega(t)))$, we then have

$$\begin{aligned}\mathcal{E}(u(t)+\epsilon w(t); f+\epsilon\varphi, \Omega(t)) - \mathcal{E}(u(t); f, \Omega(t)) \\ = -\epsilon\int_{\Omega(t)}\varphi\cdot u(t)dx + O(\epsilon^2)\end{aligned}$$

$$\mathcal{E}(u+\epsilon w; f+\epsilon\varphi, \Omega) - \mathcal{E}(\vec{u}; f, \Omega) = -\epsilon\int_\Omega \varphi\cdot u dx + O(\epsilon^2).$$

Combining the formulas just above, we can derive

$$\begin{aligned}\int_{\mathbb{R}^N}\varphi\cdot(u(t)-u)dx &= -\epsilon^{-1}[\mathcal{E}(u(t)+\epsilon w(t); f+\epsilon\varphi, \Omega(t)) - \mathcal{E}(u(t); f, \Omega(t))] \\ &\quad +\epsilon^{-1}[\mathcal{E}(u+\epsilon w; f+\epsilon\varphi, \Omega) - \mathcal{E}(u; f, \Omega)] + O(\epsilon^2).\end{aligned}$$

Dividing the both side of the relation just above by t and letting $t\to 0$, we have by (5)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N}\varphi\cdot u(t)dx &= \epsilon^{-1}\left[R_\Omega(u+\epsilon w, \vec{X}_\Phi) - R_\Omega(u, \vec{X}_\Phi)\right] \\ &\quad +O(\epsilon)\end{aligned}$$

From this we arrive the following.

$$\frac{d}{dt}\int_{\mathbb{R}^N}\varphi\cdot u(t)dx = \delta_u R_\Omega(u, w_\varphi; \vec{X}_\Phi) \quad \text{for all } \varphi\in C_0^\infty(\Omega), \tag{10}$$

where w_φ is the solution of $P(\varphi, H_0^1(\Omega))$ and

$$\delta_u R_\omega(u, w_\varphi; \vec{X}) = \lim_{\epsilon\to 0}\epsilon^{-1}\{R_\omega(u+\epsilon w_\varphi, \vec{X}) - R_\omega(u, \vec{X})\}. \tag{11}$$

The formula (10) gives the derivative of $u(t)$ with respect to the perturbation of the boundary in distribution sense.

Let δ be Dirac delta-function and $G(y,x)$, $x, y\in\Omega$ the Green's function, that is,

$$-\Delta_y G(y,x) = \delta(y-x)\ \forall y\in\Omega, \quad G(y,x) = 0\ \forall y\in\partial\Omega \ \text{ for a fixed } x\in\Omega.$$

Putting formally $\varphi(x) = \delta(x-x_0)$, $x_0\in\Omega$, we then have

$$\left.\frac{d}{dt}u(x_0,t)\right|_{t=0} = \delta_u R_\Omega(u, G(x,x_0); \vec{X}_\Phi). \tag{12}$$

We must check the validity on (12) by the singularity of G at $x_0 \in \Omega$.

In this simple case, $u \in H^2(\Omega)$ which leads

$$\delta_u P_\Omega(u, w_\varphi; \vec{X}) + \delta_u R_\Omega(u, w_\varphi; \vec{X}) = 0$$

where

$$\delta_u P_\omega(u, w_\varphi; \vec{X}) = \lim_{\epsilon \to 0} \epsilon^{-1}\{P_\omega(u + \epsilon w_\varphi, \vec{X}) - P_\omega(u, \vec{X})\}. \tag{13}$$

From this and (10), we then obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi_j \cdot u(t)dx = -\delta_u P_\Omega(u, w_{\varphi_j}; \vec{X}_\Phi) \tag{14}$$

for all $\varphi_j(x)$ which converges to $\delta(x - x_0)$ as $j \to \infty$. The sequence w_{φ_j} converges $G(x.x_0)$ as $j \to \infty$ and the right hand side of (14) has the limit because of $x_0 \notin \partial\Omega$.

$$\left.\frac{d}{dt}u(x_0, t)\right|_{t=0} = -\delta_u P_\Omega(u, G(x, x_0); \vec{X}_\Phi). \tag{15}$$

Next we derive Hadamard's variational formula from

$$\frac{d}{dt}\int_{\mathbb{R}^N} \varphi \cdot u(t)dx = -\delta_u P_\Omega(u, w_\varphi; \vec{X}_\Phi) \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{16}$$

Let $G_t(y, x)$ be the Green's function defined on $\Omega(t) \times \Omega(t)$, then

$$u(x, t) = \int_{\Omega(t)} G_t(y, x) f(y) dy.$$

Substituting this relation into (16) and using the relation

$$w_\varphi(z) = \int_\Omega G(y, z)\varphi(y)dy,$$

we then have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} \varphi(x)dx \int_{\Omega(t)} G_t(y, x) f(y) dy \\
&= \int_{\partial\Omega} dx \frac{\partial}{\partial n_z}\int_{\Omega_y} G(y, z) f(y) dy \frac{\partial}{\partial n_z}\int_{\Omega_z} G(x, z)\varphi(x) h(z) ds_z.
\end{aligned}$$

Then we arrive at Hadamard's variational formula by Fubini's theorem

$$\frac{d}{dt}G_t(y, x) = \int_{\Omega_z} \frac{\partial}{\partial n_z} G(y, z) \frac{\partial}{\partial n_z} G(x, z) h(z) ds_z. \tag{17}$$

The results (5) and (10) will be valid in general case. If the solution of $P(f, V(\Omega))$ belong to $H^2(\Omega)^m$, then (6) and (15) will also valid [2]. To derive Hadamard's variational formula, we need the smoothness of the boundary and the solution. However we can avoid such smoothness by Schwartz's theorem of kernels, and (10) will derive Hadamard's variational formula in general case by Fubini's theorem of distribution [9].

1.4 Numerical calculation by finite element method

Here we assume that Ω is polygonal or polyhedral domain. Let us denote by T_h a triangulation of Ω, where h denotes the maximum of diameter of triangle (tetrahedron) $K \in T_h$. By $P_h^1(\Omega)$ we denote the finite element space of degree 1, that is, $P_h^1(\Omega) = \{v : v|_K \in P^1, K \in T_h\}$. The finite element approximation u_h of the solution u of $P(f, V(\Omega$ is obtained by the system of equations

$$\int_\Omega \{A_{ij}(x, u_h, \nabla u_h) \partial_j v_{i,h} + B_i(x, u_h, \nabla u_h) v_h\} dx = \int_\Omega f \cdot v_h \, dx, \quad \text{for all } v_h \in V_h(\Omega),$$

where $V_h(\Omega) = V(\Omega) \cap P_h^1(\Omega)^m$ (here we consider Dirichlet boundary condition or mixed boundary condition). By Céa's lemma, we have

$$\|u - u_h\|_{1,\Omega} \leq C \inf_{v_h \in V_h(\Omega)} \|u - v_h\|.$$

with a constant $C > 0$ independent of h. For Poisson's equation with Dirichlet condition, if Ω is convex, then we have the estimation

$$\|u - u_h\|_{1,\Omega} \leq Ch^\alpha, \text{ with } \alpha = 1 \text{ and a constant } C > 0 \text{ independent of } h.$$

However, for non-convex domain, it will be that $\alpha < 1$.

Now we consider the perturbation $\Phi_t(x) = x + t\vec{X}$, then the fundamental formula (5) will implies approximation of $\frac{d}{dt}\mathcal{E}(u(t); f, \Omega)\big|_{t=0}$ by $-R_\Omega(u_h; \vec{X})$. The order of approximation is derived from the estimation

$$\left|R_\Omega(u; \vec{X}) - R_\Omega(u_h; \vec{X})\right| \leq C\left(\|u\|_{1,\Omega} + \|f\|_{1,\Omega}\right) \|u - u_h\|_{1,\Omega}$$

with a constant $C > 0$ independent of h. This finite element approximation is also applicable in fracture problems for finding ***energy release rate*** and ***stress intensity factors.***

2 Necessary conditions for fundamental formula

We write here some necessary conditions for fundamental formula (5). It is to be wished that they will be weakened. These conditions are given in [7].

We assume the fulfilment of the following hypotheses.

(H1) The map $\Phi_t : \mathbb{R}^N \to \mathbb{R}^N$ is one-to-one, $\Phi_t(\Omega) = \Omega(t)$, and Φ_t has the positive Jacobian, $\Phi_0(x) = x$ for all $x \in \mathbb{R}^N$.

(H2) $t \mapsto \Phi_t \in C^2([0, T], W^{2,\infty}(\mathbb{R}^N)^N)$. Here $C^2([0, T], W^{2,\infty}(\mathbb{R}^N)^N)$ stands for the space of two times continuously differentiable functions with respect to t, $0 \leq t \leq T$, with the values in $W^{2,\infty}(\mathbb{R}^N)^N$.

By (**H1**) and (**H2**), the map $v(y) \mapsto \Phi_t^* v(x) := v(\Phi_t(x))$ is one-to-one from $W^{1,2}(\Omega(t))^m$, $\Omega(t)$, $x \in \Omega$, onto $W^{1,2}(\Omega)^m$ and satisfies the estimate

$$C^{-1}\|v\|_{1,\Omega(t)} \le \|\Phi_t^* v\|_{1,\Omega} \le C\|v\|_{1,\Omega(t)} \qquad \text{for all } v \in H^1(\Omega(t))^m \tag{18}$$

with a constant C independent of t, v. Next assumption concerns the perturbation of boundary conditions given by $V_t(\Omega(t))$, namely,

(H3) The map $\Phi_t^* : H^1(\Omega\,(t))^m \to H^1(\Omega)^m$ is one-to-one from $V_t(\Omega\,(t))$ onto $V(\Omega)$.

In Introduction, we give the examples of Φ_t for boundary perturbation and Φ_t for the perturbation of $\overline{\Gamma_D} \cap \overline{\Gamma_N}$. For the crack extension, they can be found for instance in [1].

Now we state the conditions (19)-(23) for $E\,(x, v, \nabla v)$.

$$v \mapsto \int_{\Omega(\tau)} E(x, v, \nabla v)\,dx \text{ is Gâteaux differentiable on } V_t(\Omega(t)). \tag{19}$$

There is a constant $M_0 > 0$ such that

$$\begin{aligned} \|\delta E(x, v_1, \nabla v_1) - \delta E(x, v_2, \nabla v_2)\|_{0,\Omega(t)} &\le M_0\|v_1 - v_2\|_{1,\Omega(t)} \\ \text{for all } v_1,\, v_2 &\in V(\Omega(\tau)). \end{aligned} \tag{20}$$

There exists a constant $M_1 > 0$ such that

$$\begin{aligned} |\nabla_x A_{ij}(x,z,p)|,\ |\nabla_x B_i(x,z,p)| &\le M_1\left(|z|^2 + |p|^2\right)^{1/2} \\ |A_{ij}(x,z,p) - A_{ij}(x,\zeta,q)| &\le M_1\left(|z-\zeta|^2 + |p-q|^2\right)^{1/2} \end{aligned} \tag{21}$$

for i, j and for all $x \in \mathbb{R}^N$, $z, \zeta \in \mathbb{R}^m$, $p, q \in \mathbb{R}^{mN}$.

There is a constant $M_2 > 0$ such that

$$\begin{aligned} \nabla_p \nabla_z E(x,z,p)\xi\eta &\le M_2|\xi||\eta| \\ \nabla_p \nabla_p E(x,z,p)\eta\eta' &\le M_2|\eta||\eta'| \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}^m$, $\eta, \eta' \in \mathbb{R}^{mN}$.

There is a constant $\alpha_0 > 0$ independent of t such that

$$\begin{aligned} \int_{\Omega(t)} \{\delta E(v+w)[w] - \delta E(v)[w]\}\,dx &\ge \alpha_0\|w\|^2_{1,\Omega(t)} \\ &\text{for all } v, w \in V_t(\Omega(t)). \end{aligned} \tag{23}$$

Under the hypotheses (H1) - (H2) and (19)-(23), the fundamental formula holds, see [7] for the proof. Of course, the conditions (19)-(23) are valid in linear problem.

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